On the equivalence of s. s. fibre bundles

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Barratt, Gugenheim and Moore have defined s. s. fibre bundles in [1]. In this paper we study some equivalence theorems between them which are analogous to those known in the case of topological spaces.

1. Preliminaries.

We shall use the terminology and notation of [1], for example, "complex" and "map" will mean "c. s. s. complex" and "c. s. s. map" respectively.

DEFINITION 1. The map (E, B, p) is called a fibre bundle if

(i) p is onto

(ii) for every map $b: \Delta^n \to B$, the induced map (E^b, Δ^n, p^b) is strongly equivalent to $(\Delta^n \times Y, \Delta^n, p^*)$, where $p^*(\varphi\delta^n, y) = \varphi\delta^n$ and Y is a given complex called the fibre of the bundle.

Let $\alpha(b)$ be an equivalence exhibiting the strong equivalence of (ii) for every $b \in B$, then we have the commutative diagram:

$$\begin{array}{ccccc} \Delta^n \times Y & \xrightarrow{\alpha(b)} & E^b & \xrightarrow{\tilde{b}} & E \\ \downarrow p^* & & \downarrow p^b & & \downarrow p \\ \Delta^n & \xrightarrow{1} & \Delta^n & \xrightarrow{b} & B \end{array}$$

The set $\{\alpha(b)\}$ for $b \in B$ is called an atlas for the bundle. We also define $\beta(b) = \tilde{b}\alpha(b)$, then $\{\alpha(b)\}$ and $\{\beta(b)\}$ determine each other. It is indicated in [1] that we can define $\{\beta(b)\}$ and hence $\{\alpha(b)\}$, so as

$$\beta(s_i b) = s_i \beta(b) \tag{1}$$

and such atlas is called the normalised atlas. From now on we shall invariably suppose atlases to be normalised.

In general $\alpha_i \neq \alpha(\partial_i b)$, where $\alpha_i(b)$ denote the isomoerphism $\partial_i \alpha(b) = \alpha(b)(\varepsilon^i \times 1)$. We define for $b \in B_n$

$$\xi^i_I(b) = [\alpha(\partial_i b)]^{-1} \alpha_i(b) \in A(Y)_{n-1} \tag{2}$$

and refer to $\{\xi^i(b)\}$ as the set of transformation elements associated with the atlas $\{\alpha(b)\}$.

For a subgroup-complex Γ of $A(Y)$, some definitions are given following to [1].

DEFINITION 2. An atlas $\{\alpha(b)\}$ all of whose transformation elements lie in Γ will be called a Γ-atlas; two Γ-atlases $\{\alpha(b)\}$, $\{\overline{\alpha}(b)\}$ will be called Γ-equivalent if $\overline{\alpha}(b)=\alpha(b)\gamma_I(b)$ where $\gamma(b)\in\Gamma$.

DEFINITION 3. A fibre bundle together with a given Γ-atlas will be called a coordinate Γ-bundle, and one with Γ-equivalent class of Γ-atlases will be called a Γ-bundle.

DEFINITION 4. Let $\mathfrak{B}=(E, B, p, Y, \Gamma, \{\alpha(b)\})$ and $\overline{\mathfrak{B}}=(\overline{E}, \overline{B}, \overline{p}, Y, \Gamma, \{\overline{\alpha}(\overline{b})\})$ be coordinate Γ-bundles. A map $(f, g):(E, B)\to(\overline{E}, \overline{B})$ will be called a Γ-map if

(i) $\overline{p}f=gp$

(ii) for every $b\in B$, $\xi^*(b)$ which is defined by $\xi_I^*(b)=[\overline{\alpha}(gb)]^{-1}(1\times f)\alpha(b)$ is an element of Γ.

And $\mathfrak{B}$ and $\overline{\mathfrak{B}}$ are called Γ-equivalent if there is a one-to-one onto Γ-map between them, and strongly Γ-equivalent additionally if $B=\overline{B}$ and $g=1$. These definitions may be naturally extended to those between Γ-atlases.

2. Equivalence lemma.

Similarly to 2.10. lemma of [2], we can prove

LEMMA 1. *Let $\mathfrak{B}=(E, B, p, Y, \Gamma, \{\alpha(b)\})$ and $\overline{\mathfrak{B}}=(\overline{E}, B, \overline{p}, Y, \Gamma, \{\overline{\alpha}(b)\})$ be coordinate Γ-bundles, $\{\xi^i(b)\}$ and $\{\overline{\xi}^i(b)\}$ be their transformation elements. Then $\mathfrak{B}$ and $\overline{\mathfrak{B}}$ are strongly Γ-equivalent if and only if there is a map $\lambda: B\to\Gamma$ such that for every $b\in B$ and $o\leqq i\leqq \dim b$*

$$\overline{\xi}_I^i(b)=[\lambda(\partial_i b)]_I^{-1}\,\xi_I^i(b)[\partial_i(\lambda b)]_I \tag{3}$$

and

$$\lambda_I(s_i b)=[s_i(\lambda b)]_I. \tag{4}$$

Proof. Let $(f, 1):(E, B)\to(\overline{E}, B)$ be a strong Γ-equivalence between $\mathfrak{B}$ and $\overline{\mathfrak{B}}$. We have for every $b\in B$

$$\overline{\xi}_I^i(b)=[\overline{\alpha}(\partial_i b)]^{-1}\,\overline{\alpha}_i(b)$$

$$\xi_I^i(b)=[\alpha(\partial_i b)]^{-1}\alpha_i(b)$$

$$\xi_I^*(b)=[\overline{\alpha}(b)]^{-1}(1\times f)\alpha(b),$$

consequently

$$\overline{\xi}^{i}_{I}(b)\,\partial_i \xi^{*}_{I}(b) = \xi^{*}_{I}(\partial_i b)\xi^{i}_{I}(b). \tag{5}$$

Here f, and hence $\overset{*}{\xi}_I(b)$, is one-to-one onto, so we may define $\lambda: B \to \Gamma$ by

$$\lambda_I(b) = [\overset{*}{\xi}_I(b)]^{-1}.$$

Then (3) follows to (5) and (4) follows to (1).

Conversely let $\lambda: B \to \Gamma$ be a map such that (3) and (4) hold. We define a map $f: E \to \overline{E}$ as follows.

Put $e \in E$, then $b = pe \in B$ and $\lambda b \in \Gamma$. In the diagrams

$$\begin{array}{ccccc} \Delta^n \times Y & \xrightarrow{\alpha(b)} & E^b & \xrightarrow{\widetilde{b}} & E \\ \downarrow p^* & 1 & \downarrow p^b & b & \downarrow p \\ \Delta^n & \longrightarrow & \Delta^n & \longrightarrow & B \end{array}, \qquad \begin{array}{ccccc} \Delta^n \times Y & \xrightarrow{\overline{\alpha}(b)} & E^b & \xrightarrow{\widetilde{\widetilde{b}}} & E \\ \downarrow \overline{p}^* & 1 & \downarrow \overline{p}^b & b & \downarrow \overline{p} \\ \Delta^n & \longrightarrow & \Delta^n & \longrightarrow & B \end{array}$$

$(\delta^n, e) \in E^b$ and $\alpha(b)$ is equivalence·

Therefore we may dafine f by

$$f(e) = \overline{\beta}(b)(\lambda b)^{-1}_I[\alpha(b)]^{-1}(\delta^n, e).$$

At first we see the c. s. s. property of f as follows,

$$\begin{aligned} f(\partial_i e) &= \overline{\beta}(\partial_i b)(\lambda \partial_i b)^{-1}_I[\alpha(\partial_i b)]^{-1}(\delta^{n-1}, \partial_i e) \\ &= \overline{\beta}(\partial_i b)(\lambda \partial_i b)^{-1}_I \xi^{i}_{i}(b)[\alpha_i(b)]^{-1}(\delta^{n-1}, \partial_i e) \\ &= \widetilde{\widetilde{\beta}}\overline{\alpha}(\partial_i b)\,\overline{\xi}^{i}_{I}(b)\,[\partial_i(\lambda b)]^{-1}_I[\alpha_i(b)]^{-1}(\delta^{n-1}, \partial_i e) \\ &= \overline{b}\alpha_i(b)[\partial_i(\lambda b)]^{-1}_I[\alpha_i(b)]^{-1}(\delta^{n-1}, \partial_i e) \\ &= \partial_i f(e) \end{aligned}$$

$$f(s_i e) = s_i f(e).$$

Next f is seen to be one-to-one onto because we can define f^{-1}.

$\overline{p}f = 1p$ is trivial.

At last we shall prove (ii) of definition 4. By the definition of f,

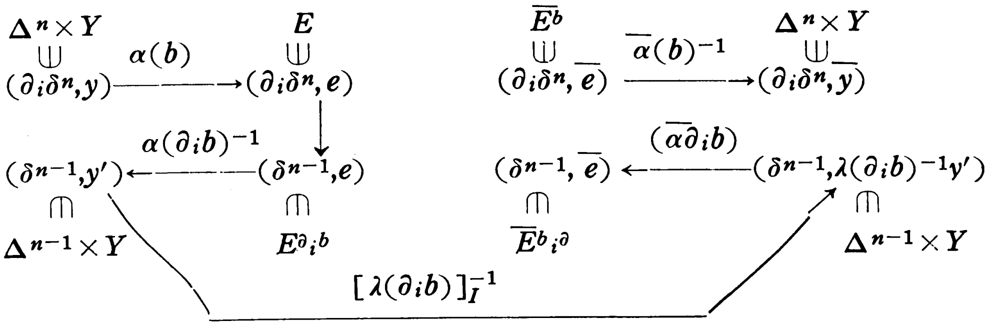
$$\overset{*}{\xi}_I(\delta^n, y) = (\lambda b)^{-1}_I(\delta^n, y).$$

Thus in order to prove that $\xi^*(b)$ belongs to Γ, it suffices to see that for any s. s. operator φ of height k-n and any element y of Y_k we have

$$\overset{*}{\xi}_I(b)(\varphi\delta^n, y)=(\varphi\delta^n, (\varphi\lambda b)^{-1}y). \tag{6}$$ (Cf [2, p. 647])

For $\varphi=s_{i_q}\cdots\cdots s_{i_1}\partial_{j_1}\cdots\cdots\partial_{j_p}$ we call $p+q$ the length of φ, and prove (6) by induction on it.

(i) For $\varphi=\partial_i$, considering the diagram



and using (3), we have

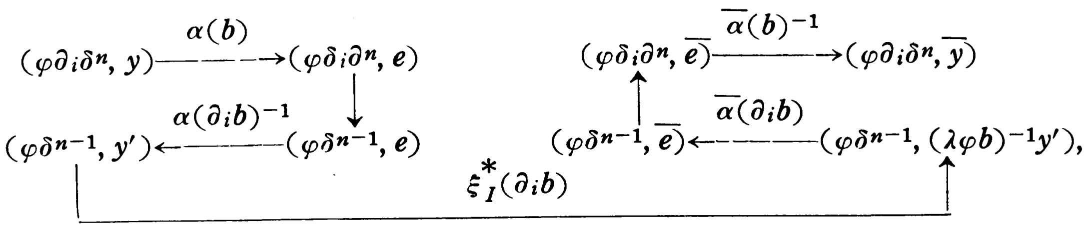
$$\overset{*}{\xi}_I(b)(\varphi\delta^n, y)= \partial_i\delta^n, \bar{y})$$

$$=[\bar{\xi}_I^i(b)]^{-1}[\lambda(\partial_i b)]_I^{-1}\xi_I^i(b)(\partial_i\delta^n, v)$$

$$=[\partial_i(\lambda b)]_I^{-1}(\partial_i\delta^n, y)=(\partial_i\delta^n, (\partial_i\lambda b)^{-1}y)$$

For, $\varphi=s_i$, (6) is easily proved using (4).

(ii) Now we assume that (6) holds for length of $\varphi\leqq r$. Let φ be an s. s. operator of height $k-n$ and length r, and $y\in Y_{k-1}$.
Considering the diagram analogous to that of (i),



we have

$$\overset{*}{\xi}_I(b)(\varphi\partial_i\delta^n, y)=[\bar{\xi}_I^i(b)]^{-1}\overset{*}{\xi}_I(\partial_i b)\,\xi_I^i(b)(\varphi\partial_i\delta^n, y)$$

$$=[\bar{\xi}_I^i(b)]^{-1}(\lambda\partial_i b)_I^{-1}\xi_I^i(b)(\varphi\partial_i\delta^n, y)$$

$$=(\varphi\partial_i\delta^n, (\varphi\partial_i\lambda b)^{-1}y).$$

Similarly we can prove for $y\in Y_{k+1}$

$$\overset{*}{\xi}_I(b)(\varphi s_i\delta^n, y)=(\varphi s_i\delta^n, (\varphi s_i\lambda b)^{-1}y).$$

3. The product bundle and the principal bundle.

We shall state two theorems (cf. 4.3 and 8.3 theorem in [2]) in this section.

A coordinate $\{1\}$ -bundle, where $\{1\}$ means the group complex consisting of the identity element 1 alone, is called a product bundle. When $\bar{\Gamma}\supset\Gamma$ and $\mathfrak{B}$ is a coordinate Γ-bundle, the coordinate $\bar{\Gamma}$-bundle $\bar{\mathfrak{B}}$ which is altered from $\mathfrak{B}$ only by regarding the group as $\bar{\Gamma}$ is called the $\bar{\Gamma}$-image of $\mathfrak{B}$.

THEOREM 1. *A coordinate Γ-bundle $\mathfrak{B}=(E, B, p, Y, \Gamma, \{\alpha(b)\})$ is strongly Γ-equivalent to the Γ-image of a product bundle if and only if there is a map $\lambda: B\to\Gamma$ such that*

$$\xi_I^i(b)[\partial_i(\lambda b)]_I=\lambda_I(\partial_i b)$$

and

$$\lambda_I(s_i b)=[s_i(\lambda b)]_I.$$

Proof. As every transformation element of a product bundle is 1, this theorem follows to lemma 1.

A coordinate Γ-bundle with a fibre Γ is ealled a principal coordinate bundle. For a coordinate Γ-bundle $\mathfrak{B}=(E, B, p, Y, \Gamma, \{\alpha(b)\})$, a map $f: B\to E$ such that pf is the identity is called a cross-section of $\mathfrak{B}$.

THEOREM 2. *Let $\mathfrak{B}=(E, B, p, \Gamma, \Gamma, \{\alpha(b)\})$ be a principal coordinate bundle. Then $\mathfrak{B}$ is strongly Γ-equivalent to the Γ-image of a product bundle if and only if it has a cross-section.*

Proof. We denote the transformation elements of $\mathfrak{B}$ by $\{\xi^i(b)\}$, and assume that a cross-section $f: B\to E$ is given. Define a map $\lambda: B\to\Gamma$ by $(\lambda b)_I=\alpha(b)^{-1}(\delta^n, f(b))$, then we have

$$\xi_I^i(b)[\partial_i(\lambda b)]_I=\lambda_I(\partial_i b)$$

and

$$\lambda_I(s_i b)=[s_i(\lambda b)]_I.$$

Thus $\mathfrak{B}$ is strongly Γ-equivalent to the Γ-image of a product bundle by theorem 1.

Conversely suppose that $\mathfrak{B}$ is strongly Γ-equivalent to the Γ-image of a product bundle. By theorem 1, there is a map $\lambda: B\to\Gamma$ such that

$$\xi_I^i(b)[\partial_i(\lambda b)]_I=\lambda_I(\partial_i b)$$

and

$$\lambda_I(s_i b) = [s_i(\lambda b)]_I.$$

We define $f: B \to E$ by

$$f(b) = \beta(b)(\delta^n, \lambda b),$$

then f is a cross-section of $\mathfrak{B}$ to be proved as follows.

(i)
$$\begin{aligned} f(\partial_i b) &= \beta(\partial_i b)(\delta^{n-1}, \lambda \partial_i b) \\ &= \partial_i \beta(b) \xi_I^i[(b)]^{-1} \lambda_I(\partial_i b)(\delta^{n-1}, 1) \\ &= \partial_i \beta(b)[\partial_i(\lambda b)]_I(\delta^{n-1}, 1) \\ &= \partial_i \beta(b)(\delta^{n-1}, \partial_i(\lambda b)) \\ &= \partial_i f(b), \end{aligned}$$

$$f(s_i b) = s_i f(b)$$

(ii)
$$p f(b) = p \beta(b)(\delta^n, \lambda b) = b$$

REMARK. Above theorems may be naturally extended to theorems about Γ-bundles.

References

[1] M. G. Barratt, V. K. A. M. Gugenheim and J. C. Moore: *On semisimplicial fibre-bundles*, American Journal of Mathematics, Vol. 81 (1959) p. 639.

[2] N. Steenrod: *The topology of fibre bundles*, Princeton, 1951.