On the sets of homotopy classes of maps between triads

By

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Introduction

F. P. Peterson has generalized the Borsuk's cohomotopy groups to the sets of homotopy classes of maps of a CW-pair into a pair of spaces in [2].

In this paper, we intend to generalize them to the sets of the homotopy classes of maps of a CW-triad into a triad of spaces, and to study their two aspects, i.e., the aspect as a generalization of homotopy groups and that of cohomotopy groups.

We denote by $\pi(K; L, M \mid X; Y, Z)$ the set of homotopy classes of maps of a CW-triad $(K; L, M)$ with a base point k into a triad $(X; Y, Z)$ with a base point x_0.

We shall give a group structure to $\pi(K; L, M \mid X; Y, Z)$ under some conditions in §1, get two kinds of exact sequences in §2, and consider of fibring in §3.

In this paper the notations S and C are used as follows; the cone CX of X is the space obtained from $X \times I$ by shrinking $(X \times 1) \smile (x_0 \times I)$ to a point x_0, the suspension SX is that obtained by shrinking $(X \times 0) \smile (x_0 \times I) \smile (X \times 1)$ to a point x_0, and for a map $f: X \to Y$, $Cf: CX \to CY$ and $Sf: SX \to SY$ are naturally defined. We note that $C'CX$ and CSX are homeomorphic, where $C'CX$ is the cone of CX.

§1. Group structure

Let $f^{\#}: \pi(K'; L', M' \mid X; Y, Z) \to \pi(K; L, M \mid X; Y, Z)$ be induced by a map $f: (K; L, M) \to (K'; L', M')$, and $\varphi_{\#}: \pi(K; L, M \mid X; Y, Z) \to \pi(K; L, M \mid X'; Y', Z')$ by a map $\varphi: (X; Y, Z) \to (X'; Y', Z')$ as usual.

Let $S_{\#}: \pi(K; L, M \mid X; Y, Z) \to \pi(SK; SL, SM \mid SX; SY, SZ)$ be the function induced by the suspension as in [6]. Then by Theorem 5.1 of [6], we have

THEOREM 1.1. *Let X, Y and Z be $(n-1)$, $(l-1)$ and $(m-1)$-connected respectively, and assume that* $\dim K \leqq 2n-2$, $\dim L \leqq 2l-2$ *and* $\dim M \leqq 2m-2$. *Then $S_{\#}$ is one to one and natural with respect to maps f and φ.*

Let $(X; Y, Z)^{(K; L, M)}$ denote a function space of maps of $(K; L, M)$ into $(X; Y, Z)$ with the compact-open topology. Then by Theorem 6.1 of [1] we obtain

THEOREM 1.2. *There is a function $\lambda: \pi_r((X; Y, Z)^{(K; L, M)}) \to \pi(S^r K; S^r L, S^r M \mid X; Y, Z)$ which is one to one and natural with respect to maps f and φ, where $S^r = S(S^{r-1})$.*

Using these theorems we get

THEOREM 1.3. *Under the conditions given in Theorem* 1.1, *we can introduce into* $\pi(K; L, M \mid X; Y, Z)$ *the group structure, which is Abelian and natural with respect to maps* f *and* φ.

Proof. In the diagram

$$\pi(K; L, M \mid X; Y, Z) \xrightarrow{S_\sharp^2} \pi(S^2K; S^2L, S^2M \mid S^2X; S^2Y, S^2Z) \xleftarrow{\lambda} \pi_2((S^2X; S^2Y, S^2Z)^{(K;L,M)}),$$

$S_\sharp$ and λ are one to one and natural with respect to maps f and φ, and $\pi_2(S^2X; S^2Y, S^2Z)^{(K;L,M)})$ is an Abelian group. Therefore, using $\lambda^{-1}\circ S_\sharp^2$, we can define group structure of $\pi(K; L, M \mid X; Y, Z)$ which is Abelian and natural with respect to maps f and φ.

§2. Exact sequences

LEMMA 2.1. *Let* $(K; L, M)$ *be a CW-triad,* $f_0: K \to Y$ *be a map such that* $f_0(M) \subset Y_0 \subset Y$, *and suppose that* $g_0 = f_0 \mid L$ *admits the homotopy* $g_t: L \to Y$ *such that* $g_t(L \cap M) \subset Y_0$. *Then* f_0 *admits the homotopy* $f_t: K \to Y$ *such that* $f_t \mid L = g_t$ *and* $f_t(M) \subset Y_0$.

Proof. By (J) of [7] the homotopy $g_t \mid L \frown M$ admits an extension to $g_t'; M \to Y_0$ and then $g_0'(L\frown M) = g_0(L\frown M) = f_0(L\frown M)$. Defining $g_t'': L\smile M \to Y$ by $g_t'' \mid L = g_t$ and $g_t'' \mid M = g_t'$, we have $g_t''(M) \subset Y_0$. Now using (J) of [7] again, g_t'' may be extended to $f_t: K \to Y$. This is the required homotopy, for $f_t(M) = g_t''(M) \subset Y_0$ and $f \mid L = g_t'' \mid L = g_t$.

Similarly to Theorem 7.5 of [5], we can prove

LEMMA 2.2. *Let* $(K; L, M)$ *be a CW-triad, and* $(K_M; L_M, p_M)$ *be the CW-triad obtained by identifying* M *to a point* p_M *as in* [5]. *Then the canonical map* $f: (K; L, M) \to (K_M; L_M, p_M)$ *induces a* 1-1 *correspondence* $f^\sharp: \pi(K_M; L_M, p_M \mid X; Y, x_0) \to \pi(K; L, M \mid X; Y, x_0)$.

LEMMA 2.3. *Let* $(K; L, M)$ *be a CW-triad,* N *be a subcomplex of* $L\frown M$ *and* $(K_N; L_N, M_N)$ *be the CW-triad obtained by identifying* N *to a point. Then the canonical map* $f: (K; L, M) \to (K_N; L_N, M_N)$ *induces a* 1-1 *correspondence* $f^\sharp: \pi(K_N; L_N, M_N \mid X; Y, x_0) \to \pi(K; L, M \mid X; Y, x_0)$.

Proof. In the following sequence

$$\pi(K_M; L_M, p_M \mid X; Y, x_0) \xrightarrow{g^\sharp} \pi(K_N; L_N, M_N \mid X; Y, x_0) \xrightarrow{f^\sharp} \pi(K; L, M \mid X; Y, x_0)$$

induced by

$$(K; L, M) \xrightarrow{f} (K_N; L_N, M_N) \xrightarrow{g} (K_M; L_M, p_M),$$

we have $(g\circ f)^\sharp = f^\sharp \circ g^\sharp$.

On the other hand $g^\sharp$ and $(g\circ f)^\sharp$ are one to one by Lemma 2.2, hence $f^\sharp$ is also one to one.

Similarly to Theorem 7.6 of [5], we have

LEMMA 2.4. (EXCISION LEMMA) *Let $(K; L, M)$ be a CW-triad and N be a subset of $L\frown M$ such that $(K-N; L-N, M-N)$ is a CW-triad. Then the inclusion map $i:(K-N; L-N, M-N)\to(K; L, M)$ induces a 1-1 correspondence $i^\#:\pi(K; L, M \mid X; Y, x_0) \to \pi(K-N; L-N, M-N \mid X; Y, x_0)$.*

Using these lemmas the exact sequence of a pair given in [2] can be generalized to that of a triad as follows.

THEOREM 2.5. *Let $(K; L, M)$ be a CW-triad with a base point k such that $L\frown M$ is a CW-complex, and let $X\supset Y\ni y_0$. And suppose that X and Y are $(n-1)$- and $(m-1)$-connected respectively and that $\dim K\leqq 2n-2$, $\dim L\leqq 2m-2$. Then the following sequence (I) is exact and natural with respect to maps $f:(K; L, M)\to (K'; L', M')$ and $\varphi:(X; Y, x_0)\to(X', Y', x_0')$:*

$$\text{(I)}\quad \begin{array}{l} \pi(K; L, M \mid X; Y, x_0) \xrightarrow{j^\#} \pi(K; L, k \mid X; Y, x_0) \xrightarrow{i^\#} \pi(M; L\frown M, k \mid X; Y, x_0) \\ \qquad\qquad \xrightarrow{\Delta} \pi(K; L, M \mid SX; SY, x_0) \xrightarrow{j^\#} \cdots \end{array}$$

where $i^\#$ and $j^\#$ are induced by inclusions $i:(M; L\frown M, k)\to(K; L, k)$ and $j:(K; L, k) \to(K; L, M)$ respectively, and Δ is defined as follows: let $a\in[a]\in\pi(M; L\frown M, k \mid X; Y, x_0)$ and extend a to a map $a':(K; L, M)\to(CX; CY, X)$ by Lemma 2.1, then the composition $h\circ a':(K; L, M)\to(SX; SY, x_0)$ represents $\Delta([a])$, where $h:(CX; CY, X) \to(SX; SY, x_0)$ is the canonical map collapsing X to x_0 as in [2].

Proof. The naturality and the relations $\operatorname{Im} j^\# = \operatorname{Ker} i^\#$ and $\operatorname{Im} i^\# \subset \operatorname{Ker}\Delta$ are obvious. We prove the relation $\operatorname{Im} i^\# \supset \operatorname{Ker}\Delta$ at first. In the following diagram

$$\begin{array}{ccccc}
\pi(K; L, k \mid X; Y, x_0) & \xrightarrow{i^\#} & \pi(M; L\frown M, k \mid X; Y, x_0) & \xrightarrow{\Delta} & \pi(K; L, M \mid SX; SY, x_0) \\
\downarrow S_\# & & & & \| \\
\pi(SK; SL, k \mid SX; SY, x_0) & & & & \\
\| & & & & \\
\pi(\psi_1^1) & \xrightarrow{\alpha} & \pi(\psi, \varphi)_1^1 & \xrightarrow{\beta} & \pi(\varphi_0^1) \\
 & & \| & & \\
 & & \pi(CK; CL, M \mid SX; SY, x_0) & &
\end{array}$$

the lower sequence is the exact sequence $(3.1)_1$ of [6], where ψ and φ are carriers from $(K, k; \{K, L, M\})$ to $(X, x_0; \{X, Y, \{x_0\}\})$ such that $\psi(K)=X$, $\psi(L)=Y$, $\psi(M)=Y$ or X according to $M\subset L$ or not, $\varphi(K)=X$, $\varphi(L)=\{x_0\}$ or Y according to $L\subset M$ or not, $\varphi(M)=\{x_0\}$. And α is defined by $\alpha([a])=[a\circ h]$ for $[a]\in\pi(\psi_1^1)$ and β is a restriction.

Let $[a]\in\operatorname{Ker}\Delta$, and extend $a:(M, L\frown M)\to(X, Y)$ to $a':(K; L, M)\to(CX; CY, X)$, then $h\circ a':(K; L, M)\to(SX; SY, x_0)$ represents $\Delta[a]$, i.e., $h\circ a'\simeq 0$. Denote $a'(p)$ by $(b(p), s(p))$ for $p\in K$, and define $Ta':(CK; CL, M)\to(CX; CY, X)$ by $Ta'(p, t)=(b(p), s(p)+t-s(p)\cdot t)$ where t and $s(p)$ are the parameters of a cone. Then $[h\circ Ta']\in\operatorname{Ker}\beta$ in the preceding exact sequence, and accordingly there is a map $f:(SK; SL, k)\to (SX; SY, x_0)$ such that $[f\circ h]=[h\circ Ta']$. There is a map $g:(K; L, k)\to(X; Y, x_0)$ such that $Sg\simeq f$, as $S_\sharp$ is one to one. Thus we have $(Sg)\circ h\simeq h\circ Ta'$ as maps of $(CK; CL, M)$ into $(SX; SY, x_0)$, hence $(Sg)\circ h\,|\,CM\simeq h\circ Ta'\,|\,CM$ as maps of $(CM; C(L\frown M), M)$ into $(SX; SY, x_0)$. Since $h\circ Ta'\,|\,CM=h\circ Ca=(Sa)\circ(h\,|\,CM)$, we have $S(g\,|\,M)\simeq Sa$ as maps of $(SM; S(L\frown M))$ into (SX, SY). Hence $g\,|\,M\simeq a$ as maps of $(M, L\frown M)$ into (X, Y), and accordingly $i^\sharp[g]=[a]$. Thus we have proved the relation $\operatorname{Im} i^\sharp\supset\operatorname{Ker}\Delta$ for the first part of (I), and the proof of the other parts are similar.

Now we can complete the proof by showing that $\operatorname{Im}\Delta=\operatorname{Ker} j^\sharp$. Denoting $N\times\varepsilon$ by N_ε, we consider the following diagram (*) analogously to [5]:

$$(*)\qquad\begin{array}{ccc}
\pi(M;\, L\frown M, k\,|\,X;\, Y, x_0) & \xrightarrow{\ \Delta\ } \pi(K; L, M\,|\,SX; SY, x_0) & \xrightarrow{\ j^\sharp\ } \pi(K; L, k\,|\,SX; SY, x_0)\\
\uparrow q_2^\sharp & \uparrow q_1^\sharp & \\
\pi(K_1\smile M_0\smile k_I; K_1\smile(L\frown M)_0\smile k_I, K_1\smile k_I\,|\,X; Y, x_0) & \xrightarrow{\ \Delta_1\ } \pi(K_I; K_1\smile L_I, K_1\smile M_0\smile k_I\,|\,SX; SY, x_0) & \\
\| & \downarrow i_1^\sharp & \\
\pi(K_1\smile M_0\smile k_I; K_1\smile(L\frown M)_0\smile k_I, K_1\smile k_I\,|\,X; Y, x_0) & \xrightarrow{\ \Delta_2\ } \pi(K_1\smile M_I; K_1\smile(L\frown M)_I, K_1\smile M_0\smile k_I\,|\,SX; SY, x_0) & \\
\uparrow r_2^\sharp & \uparrow r_1^\sharp & \\
\pi(M;\, L\frown M, k\,|\,X;\, Y, x_0) & \xrightarrow{\ \Delta_3\ } \pi(CM; C(L\frown M), M\,|\,SX; SY, x_0) &
\end{array}$$

where $q_1:(K; L, M)\to(K_I; K_1\smile L_I, K_1\smile M_0\smile k_I)$ and $q_2:(M; L\frown M, k)\to(K_1\smile M_0\smile k_I; K_1\smile(L\frown M)_0\smile k_I, K_1\smile k_I)$ are defined by $q_i(x)=x\times 0$ for $i=1,2$, $i_1:(K_1\smile M_I; K_1\smile(L\frown M)_I, K_1\smile M_0\smile k_I)\to(K_I; K_1\smile L_I, K_1\smile M_0\smile k_I)$ is inclusion, $r_1:(K_1\smile M_I; K_1\smile(L\frown M)_I, K_1\smile M_0\smile k_I)\to(CM; C(L\frown M), M)$ and $r_2:(K_1\smile M_0\smile k_I; K_1\smile(L\frown M)_0\smile k_I, K_1\smile k_I)\to(M; L\frown M, k)$ are given by identification of $K_1\smile k_I$ to k, and $\Delta_1, \Delta_2, \Delta_3$ are defined similarly to Δ. Then obviously $\operatorname{Ker} j^\sharp=\operatorname{Im} q_1^\sharp$, $q_2^\sharp$ is an isomorphism by Lemma 2.4, $i_1^\sharp$ is an isomorphism because $(K_1\smile M_I; K_1\smile(L\frown M)_I, K_1\smile M_0\smile k_I)$ is the deformation retract of $(K_I; K_1\smile L_I, K_1\smile M_0\smile k_I)$, $r_1^\sharp$ and $r_2^\sharp$ are isomorphisms by Lemma 2.3. In the following diagram

$$\begin{array}{ccccc}
\pi(M; L\frown M, k\,|\,X; Y, x_0) & & \xrightarrow{\ \Delta\ } & \pi(CM; C(L\frown M), M\,|\,SX; SY, x_0) & \\
\downarrow S_\sharp & & & \| & \\
\pi(SM; S(L\frown M), k\,|\,SX; SY, x_0) & & & & \\
\| & & & & \\
\pi(\varphi_1^1) \xrightarrow{\ \gamma\ } & \pi(\psi_1^1) & \xrightarrow{\ \alpha\ } & \pi(\psi, \varphi)_1^1 & \xrightarrow{\ \beta\ } \pi(\varphi_0^1)
\end{array}$$

the lower sequence is the exact sequence $(3.1)_1$ of [6], where ψ and φ are carriers from $(M, k; \{M, L\frown M\})$ to $(X, x_0; \{X, Y, \{x_0\}\})$ such that $\psi(M)=X, \psi(L\frown M)=Y$, $\varphi(M)=\varphi(L\frown M)=\{x_0\}$. Here α is an isomorphism as $\pi(\psi_1^1)=\pi(\psi_0^1)=0$, and $S^\sharp$ is an isomorphism also. On the other hand the diagram is commutative, because $\alpha S_\sharp[a]=[Sa\circ h]=[h\circ Ca]=\Delta_3[a]$. Therefore Δ_3 is an isomorphism. The commutativity of each square of the diagram (*) is easily seen.

In the diagram (*), for each $[a]\in\pi(M; L\frown M, k\,|\,X; Y, x_0)$ there is $[b]\in\pi(K_1\smile M_0\smile k_I; K_1\smile(L\frown M)_0\smile k_I, K_1\smile k_I\,|\,X; Y, x_0)$ such that $q_2^\sharp[b]=[a]$. Then $j^\sharp\Delta[a]=j^\sharp\Delta q_2^\sharp[b]=j^\sharp q_1^\sharp\Delta_1[b]=0$. Hence $\operatorname{Im}\Delta\subset\operatorname{Ker} j^\sharp$.

Conversely for each $[a]\in \operatorname{Ker} j^{\#}=\operatorname{Im} q_1{}^{\#}$ there is $[b]\in \pi(M; L\frown M, k \mid X; Y, x_0)$ such that $q_1{}^{\#}i_1{}^{\#-1}r_1{}^{\#}\Delta_3 r_2{}^{\#-1}q_2{}^{\#-1}[b]=[a]$. This formula may be reduced to $\Delta[b]=[a]$, and thus the last relation $\operatorname{Im}\Delta\supset\operatorname{Ker} j^{\#}$ is proved.

Now to generalize the homotopy sequence of a triad, we define a map $h:(CC'K; CK, C'K)\to(CSK; k, SK)$ for CW-complex K by the following formulae:

$$h(y, t)=k \text{ for } (y, t)\in CK, y\in K, t\in I,$$
$$h(y, s)=(y, s) \text{ for } (y, s)\in C'K, y\in K, s\in I$$

and $\quad h((y, s), t)=(h(y, s), t)$ for $(y, s)\in C'K$ and $((y, s), t)\in CC'K$.

Then the induced function

$$h^{\#}:\pi(CSK; k, SK \mid X; x_0, Z)\to\pi(CC'K; CK, C'K \mid X; x_0, Z)$$

is natural with respect to maps $f:K\to K'$ and $\varphi:(X, Z)\to(X', Z')$.

THEOREM 2.6. *Let $(X; Y, Z)$ be a triad with a base point x_0, K be a CW-complex with a base point k. And suppose that X, Y, Z and $Y\frown Z$ are $(n-1)$-connected and* $\dim K=N<2n-3$. *Then the following sequence* (II) *is exact and natural with respect to maps $f:K\to K'$ and $\varphi:(X; Y, Z)\to(X'; Y', Z')$:*

(II) $$\pi(CS^mK; S^mK, k \mid Y; Y\frown Z, x_0)\xrightarrow{i^{\#}}\pi(CS^mK; S^mK, k \mid X; Z, x_0)$$
$$\xrightarrow{\tilde{j}_{\#}}\pi(CC'S^{m-1}K; CS^{m-1}K, C'S^{m-1}K \mid X; Y, Z)\xrightarrow{\Delta}\pi(CS^{m-1}K; S^{m-1}K, k \mid Y; Y\frown Z, x_0)$$
$$\xrightarrow{i_{\#}}\cdots\xrightarrow{i_{\#}}\pi(CK; K, k \mid X; Z, x_0),$$

where $m=2n-3-N$, and $i:(Y, Y\frown Z)\to(X, Z)$ and $j:(X; x_0, Z)\to(X; Y, Z)$ are injections and $\tilde{j}_{\#}=j_{\#}\circ h^{\#}$, and Δ is a restriction.

Proof. The naturality is obvious. Let ψ, φ_1 and φ_2 be carriers from $(K, k; \{K\})$ to $(X, x; \{X, Y, Z\})$ such that $\psi(K)=X$, $\varphi_1(K)=Y$, $\varphi_2(K)=Z$. Let $\theta=\varphi_1\frown\varphi_2$, then $(\psi, \varphi_2)_1$ and $(\varphi_1, \theta)_1$ are carriers from $C(K, k; \{K\})$ to $(X, x; \{X, Y, Z\})$ such that $(\psi, \varphi_2)_1(CK)=X$, $(\psi, \varphi_2)_1(K)=Z$, $(\varphi_1, \theta)_1(CK)=Y$, $(\varphi_1, \theta)_1(K)=Y\frown Z$. Replacing ψ and φ by $(\psi, \varphi_2)_1$ and $(\varphi_1, \theta)_1$ in the exact sequence $(3.1)_0$ of [6], we have

$$\pi((\psi, \varphi_2)_1, (\varphi_1, \theta)_1)^0_{m+1}\xrightarrow{\beta}\pi(\varphi_1, \theta)^0_{m+1}\xrightarrow{\gamma}\pi(\psi, \varphi_2)^0_{m+1}\xrightarrow{\alpha}\pi((\psi, \varphi_2)_1, (\varphi_1, \theta)_1)^0_m$$
$$\xrightarrow{\beta}\cdots\xrightarrow{\gamma}\pi(\psi, \varphi_2)^0_1.$$

Here
$$\pi((\psi, \varphi_2)_1, (\varphi_1, \theta)_1)^0_{r+1}=\pi(CC'S^rK; CS^rK, C'S^rK \mid X; Y, Z),$$
$$\pi(\varphi_1, \theta)^0_{r+1}=\pi(CS^rK; S^rK, k \mid Y; Y\frown Z, x_0),$$
$$\pi(\psi, \varphi_2)^0_{r+1}=\pi(CS^rK; S^rK, k \mid X; Z, x_0)$$

for $r=0, 1, 2, \cdots$, and obviously γ and β consist with $i_{\#}$ and Δ respectively. We now show that α consists with $\tilde{j}_{\#}$. For a map $f\in[f]\in\pi(CS^rK; S^rK, k \mid X; Z, x_0)$, we have $\alpha[f]=[f\circ h]\in\pi(CC'S^{r-1}K; CS^{r-1}K, C'S^{r-1}K \mid X; Y, Z)$. On the other hand,

$$\tilde{j}_{\#}[f]=j_{\#}h^{\#}[f]=j_{\#}[f\circ h] \text{ where } [f\circ h]\in\pi(CC'S^{r-1}K; CS^{r-1}K, C'S^{r-1}K \mid X; x_0, Z)$$
$$=[j\circ f\circ h]\in\pi(CC'S^{r-1}K; CS^{r-1}K, C'S^{r-1}K \mid X; Y, Z),$$

hence we have $\alpha[f]=\tilde{j}_{\#}[f]$.

Thus the exactness is an immediate consequence of $(3.1)_0$ of [6].

REMARK. In Theorem 2.5 and Theorem 2.6, we can exclude the conditions about connectedness and dimension, if we weaken the meaning of exactness as in [6], i.e., a sequence is said to be exact when the inverse image of the distinguished element at any stage is the image of the preceding map.

§3. Fibring

In this section we generalize Proposition 1 in Chapter II of [4] and get an exact sequence.

THEOREM 3.1. *Let (X, p, Y) be a fibre space given in* [3], *and let $Y_0 \subset Y, X_0 = p^{-1}(Y_0)$, $p(x_0)=y_0$. Then for any contractible finite complex K and its subcomplex L, $p:(X; X_0, x_0) \to (Y; Y_0, y_0)$ induces a 1-1 correspondence $p_\sharp : \pi(K; L, k \mid X; X_0, x_0) \to \pi(K; L, k \mid Y; Y_0, y_0)$.*

Proof. 1°. Put $g \in [g] \in \pi(K; L, k \mid Y; Y_0, y_0)$. Define $f_0 : k \to X$ by $f_0(k)=x_0$. Then g is an extension of $p \circ f_0$. By Proposition 1 in Chapter II of [3], f_0 may be extended to $f: K \to X$ such that $p \circ f = g$. Then $f(L) \in X_0$ because $pf(L)=g(L) \subset Y_0$, hence $[f] \in \pi(K; L, k \mid X; X_0, x_0)$. We have $p_\sharp[f]=[g]$, and thus $p_\sharp$ is onto.

2°. Let $f' \in [f'] \in \pi(K; L, k \mid X; X_0, x_0)$, and suppose that $p \circ f' \simeq 0$ as maps of $(K; L, k)$ into $(Y; Y_0, y_0)$, and denote the homotopy by $g:(K \times I; L \times I, k \times I) \to (Y; Y_0, y_0)$. Define $f_0 : K \times 0 \smile k \times I \to X$ by $f_0(p, 0)=f'(p)$ for $p \in K$ and by $f_0(k, t)=x_0$ for $t \in I$, then g is the extension of $p \circ f_0 : K \times 0 \smile k \times I \to Y$. Hence, by Proposition 1 in Chapter II of [3], f_0 may be extended to $f: K \times I \to X$ such that $p \circ f = g$. Then $f(L \times I) \subset X_0$ because $pf(L \times I) = g(L \times I) \subset Y_0$, and $f(K \times 1) \subset X_0$ because $pf(K \times 1) = g(K \times 1) \subset Y_0$. Thus defining $f'': K \to X$ by $f''(x)=f(x, 1)$, we have $f''(K) \subset X_0$. Since K is contractible, we have $[f'']=0$ in $\pi(K; L, k \mid X; X_0, x_0)$. On the other hand $f' \simeq f''$, therefore $[f']=0$. Hence $p_\sharp$ is one to one.

REMARK. 1°. In this theorem if $\pi(K; L, k \mid X; X_0, x_0)$ and $\pi(K; L, k \mid Y; Y_0, y_0)$ have the group structure defined in Theorem 1.3, $p_\sharp$ is an isomorphism.

2°. Letting $Y_0=y_0$ in this theorem, $p_\sharp : \pi(K; L, k \mid X; F, x_0) \to \pi(K; L, k \mid Y; y_0, y_0)$ is one to one, where $F=p^{-1}(y_0)$.

THEOREM 3.2. *Let (X, p, Y) be a fibre space given in* [3], *and let $(X; X_1, X_2)$ and $(Y; Y_1, Y_2)$ be triads such that $p^{-1}(Y_1)=X_1$, $p^{-1}(Y_2)=X_2$ and $p(x_0)=y_0$. Suppose that $X, X_1, X_2, X_1 \frown X_2, Y, Y_1, Y_2$ and $Y_1 \frown Y_2$ are $(n-1)$-connected. Then the projection $p:(X; X_1, X_2) \to (Y; Y_1, Y_2)$ induces an isomorphism*

$$p_\sharp : \pi(CC'K; CK, C'K \mid X; X_1, X_2) \to \pi(CC'K; CK, C'K \mid Y; Y_1, Y_2)$$

for every finite complex K such that $\dim K \leqq 2n-4$.

Proof. In the following diagram

$$\left\{\begin{array}{ccccc}\pi(CSK;\, SK, k \mid X_1;\, X_1\frown X_2, x_0) & \xrightarrow{i_\#} & \pi(CSK;\, SK, k \mid X;\, X_2, x_0) & \xrightarrow{\tilde{j}_\#} \\ \downarrow p_\#^{(1)} & & \downarrow p_\#^{(2)} & \\ \pi(CSK;\, SK, k \mid Y_1;\, Y_1\frown Y_2, y_0) & \xrightarrow{i_\#} & \pi(CSK;\, SK, k \mid Y;\, Y_2, y_0) & \xrightarrow{\tilde{j}_\#} \\ \\ \pi(CC'K;\, CK, C'K \mid X;\, X_1, X_2) & \xrightarrow{\Delta} & \pi(CK;\, K, k \mid X_1;\, X_1\frown X_2, x_0) & \xrightarrow{i_\#} \\ \downarrow p_\# & & \downarrow p_\#^{(3)} & \\ \pi(CC'K;\, CK, C'K \mid Y;\, Y_1, Y_2) & \xrightarrow{\Delta} & \pi(CK;\, K, k \mid Y_1;\, Y_1\frown Y_2, y_0) & \xrightarrow{i_\#} \\ \pi(CK;\, K, k \mid X;\, X_2, x_0) & & & \\ \downarrow p_\#^{(4)} & & & \\ \pi(CK;\, K, k \mid Y;\, Y_2, y_0) & & & \end{array}\right\}$$

upper and lower sequences are exact and natural with respect to the projection p. Moreover $p_\#^{(1)}$, $p_\#^{(2)}$, $p_\#^{(3)}$ and $p_\#^{(4)}$ are isomorphisms by Remark of Theorem 3.1, thus $p_\#$ is also an isomorphism by the five lemma.

LEMMA 3.3. *$h^\#: \pi(CSK;\, k, SK \mid X;\, x_0, Z) \to \pi(CC'K;\, CK, C'K \mid X;\, x_0, Z)$ given in Theorem 2.6 is one to one.*

Proof. 1^0. For each $a \in [a] \in \pi(CC'K;\, CK, C'K \mid X;\, x_0, Z)$, define $b: (CSK;\, k, SK) \to (X;\, x_0, Z)$ by $b(k) = x_0$ and $b(p) = ah^{-1}(p)$ for $p \in CSK - k$. Then, if $p \in CK$ we have $bh(p) = b(k) = x_0 = a(p)$, if $p \notin CK$ we have $bh(p) = ah^{-1}h(p) = a(p)$, and thus in either case we have $b \circ h = a$. This means $h^\#[b] = [a]$, that is $h^\#$ is onto.

2^0. Let $b \in [b] \in \pi(CSK;\, k, SK \mid X;\, x_0, Z)$, and suppose that $b \circ h \simeq 0$ as maps of $(CC'K;\, CK, C'K)$ into $(X;\, x_0, Z)$. Let $F: (CC'K \times I;\, CK \times I, C'K \times I) \to (X;\, x_0, Z)$ be the homotopy between $b \circ h$ and 0. Defining $G: (CSK \times I;\, k \times I, SK \times I) \to (X;\, x_0, Z)$ by $G(k, t) = x_0$ and $G(p, t) = F(h^{-1}(p), t)$ for $p \in CSK - k$, we can see that G is a homotopy between b and 0 as follows:

$G(k, 0) = x_0 = b(k)$,

$G(p, 0) = F(h^{-1}(p), 0) = bh(h^{-1}(p)) = b(p)$ for $p \in CSK - k$,

$G(k, 1) = x_0$,

$G(p, 1) = F(h^{-1}(p), 1) = x_0$ for $p \in CSK - k$.

Thus $[b] = 0$, consequently $h^\#$ is one to one.

Using the 1-1 correspondences $p_\#$ and $h^\#$ given in Theorem 3.1 and Lemma 3.3, we can transform the exact sequence (II) given in Theorem 2.6, as follows:

THEOREM 3.4. *Let (X, p, Y) be a fibre space given in [3], F be its fibre, and K be a finite complex. And suppose that X, Y and F are $(n-1)$-connected and $\dim K = N < 2n-3$. Then the following sequence is exact, where $\bar{j}_\# = p_\# \circ h^{\#-1} \circ \tilde{j}_\#$, $\bar{\Delta} = \Delta \circ h^\# \circ p_\#^{-1}$ and $m = 2n-3-N$:*

$$\pi(CS^mK, S^mK \mid F, x_0) \xrightarrow{i_\#} \pi(CS^mK, S^mK \mid X, x_0) \xrightarrow{\bar{j}_\#} \pi(CS^mK, S^mK \mid Y, y_0)$$
$$\xrightarrow{\bar{\Delta}} \pi(CS^{m-1}K, S^{m-1}K \mid F, x_0) \xrightarrow{i_\#} \cdots \xrightarrow{\bar{j}_\#} \pi(CK, K \mid Y, y_0).$$

REMARK. In this theorem we can exclude the conditions about connectedness and dimension by weakening the meaning of exactness as in the Remark of §2.

An important special case is obtained by letting X be a path space on Y.

COROLLARY 3.5. *Denote by $\Omega(Y)$ the loop space on Y, and by x_0 the constant loop. Then we have a* 1-1 *correspondence*

$$\bar{\Delta}: \pi(CSK, SK \mid Y, y_0) \to \pi(CK, K \mid \Omega(Y), x_0).$$

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