On $\mathcal{C}$-excisive triads

By

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The principal purpose of this paper is to generalize well-known properties of excisive triads by means of the $\mathcal{C}$-notion of Abelian groups which was introduced by J.-P. Serre [5].

Namely, in §1 we shall define the Mayer-Vietoris sequence of $\mathcal{C}$-excisive triad and its $\mathcal{C}$-exactness will be shown (see (1.7) and (1.8)), in §2 the Blakers and Massey triad theorem given by J. C. Moore [2] will be extended for the case of $\mathcal{C}$-excisive triad (see (2.9)). And the Hurewicz isomorphism theorem for triad will be given in the last section (§3, (3.1)).

Throughout the present paper, all triads will be assumed to be those of arcwise connected topological spaces, and homology will always mean singular cubic homology. By $\mathcal{C}=\mathcal{C}(\mathrm{I}, \mathrm{II}_B)$ for example, we mean that $\mathcal{C}$ is a class of Abelian groups which satisfies the axioms (I) and (II_B) given in [5]. Im, Ker, Coker and $\lambda^{-1}(\ )$ mean image, kernel, cokernel and inverse image by λ, respectively.

A triad $(X; X_1, X_2)$ will be called $\mathcal{C}$-*excisive*, if $X=X_1\cup X_2$, X, X_1, X_2 and $X_1\cap X_2$ are arcwise connected and the inclusion map $k_2:(X_1, X_1\cap X_2)\to(X, X_2)$ induces the $\mathcal{C}$-isomorphism

$$k_{2*}: H_q(X_1, X_1\cap X_2)\to H_q(X, X_2) \qquad \text{for all } q.$$

A triad $(X; X_1, X_2)$ will be called $\mathcal{C}$-*proper*, if $(X_1\cup X_2; X_1, X_2)$ is $\mathcal{C}$-excisive.

Let A and B be two subgroups of a same Abelian group. A will be called $\mathcal{C}$-*equal* to B, if the inclusion maps $A\cap B\to A$ and $A\cap B\to B$ are $\mathcal{C}$-isomorphisms.

A sequence of groups $\{G_q, f_q\}$ will be called $\mathcal{C}$-*exact*, if the image of G_{q+1} by $f_{q+1}: G_{q+1}\to G_q$ is $\mathcal{C}$-equal to the kernel of $f_q: G_q\to G_{q-1}$, for each q.

Let $(X; X_1, X_2)$ be a triad. Let $x\in X$, and let X^* be the space of paths in X which start at x. Define $p: X^*\to X$ by $p(f)=f(1)$. Let $X_\alpha^*=p^{-1}(X_\alpha)$ $(\alpha=1, 2)$. The triad (X^*, X_1^*, X_2^*) will be called the *associated* triad of the triad $(X; X_1, X_2)$.

§1. The Mayer-Vietoris sequence of a $\mathcal{C}$-excisive triad

(1.1) PROPOSITION. Let G, H and K be Abelian groups and let $\lambda: G\to H$ and $\mu: H\to K$ be homomorphisms such that $\mu\circ\lambda$ is a $\mathcal{C}$-isomorphism of G with K, where $\mathcal{C}=\mathcal{C}(\mathrm{I})$. Then $\operatorname{Im}\lambda\cap\operatorname{Ker}\mu\in\mathcal{C}$, λ is $\mathcal{C}$-monomorphic, μ is $\mathcal{C}$-epimorphic, μ is a $\mathcal{C}$-isomorphism of $\operatorname{Im}\lambda$ with K and the inclusion map $\theta: \operatorname{Im}\lambda/H_1+\operatorname{Ker}\mu/H_1\to H/H_1$ is

$\mathcal{C}$-isomorphic (in detail, monomorphic and $\mathcal{C}$-epimorphic) where $H_1 = \operatorname{Im}\lambda \cap \operatorname{Ker}\mu$ is a subgroup of H.

Proof. Since $\mu\circ\lambda$ is a $\mathcal{C}$-isomorphism of G with K, $K/\mu\lambda(G)\in\mathcal{C}$ and $\lambda^{-1}(H_1)\in\mathcal{C}$. The first relation implies that $\mu\,|\,\lambda(G):\lambda(G)\to K$ is $\mathcal{C}$-epimorphic. It follows from the second relation that $H_1\in\mathcal{C}$, namely $\mu\,|\,\lambda(G):\lambda(G)\to K$ is $\mathcal{C}$-monomorphic. Therefore μ is a $\mathcal{C}$-isomorphism of $\lambda(G)$ with K. Since $K/\mu(H)$ is the image of the canonical epimorphism: $K/\mu\lambda(G)\to K/\mu(H)$, we have $K/\mu(H)\in\mathcal{C}$, namely μ is $\mathcal{C}$-epimorphic. Since $\operatorname{Ker}\lambda\subset\operatorname{Ker}(\mu\circ\lambda)\in\mathcal{C}$, λ is $\mathcal{C}$-monomorphic. And $\operatorname{Im}\lambda\cap\operatorname{Ker}\mu = \operatorname{Ker}(\mu\circ\lambda)\in\mathcal{C}$. Secondly it is clear that θ is a monomorphism, and

$$\frac{H/H_1}{\operatorname{Im}\lambda/H_1+\operatorname{Ker}\mu/H_1} = \operatorname{Ext}\left(\mu(H)/\mu\lambda(G),\ \frac{\mu^{-1}\mu\lambda(G)/H_1}{\lambda(G)/H_1+\operatorname{Ker}\mu/H_1}\right)$$

where $\mu(H)/\mu\lambda(G)\subset K/\mu\lambda(G) = \operatorname{Coker}(\mu\circ\lambda)\in\mathcal{C}$.
On the other hand, put $G_1=\lambda^{-1}(H_1)$ and let $\bar\lambda: G/G_1\to\mu^{-1}\mu\lambda(G)/H_1$, $\bar\mu:\mu^{-1}\mu\lambda(G)/H_1\to\mu\lambda(G)$ be the homomorphisms induced by λ, μ respectively, then it is easy to see that $\bar\mu\circ\bar\lambda: G/G_1\to\mu\lambda(G)$ is an isomorphism. Therefore,

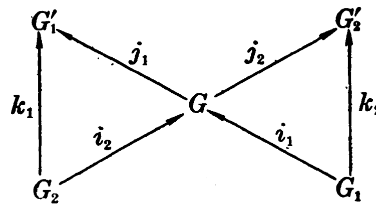
$$\mu^{-1}\mu\lambda(G)/H_1 = \bar\lambda(G/G_1)+\operatorname{Ker}\bar\mu = \lambda(G)/H_1+\operatorname{Ker}\mu/H_1.$$

Hence

$$\frac{H/H_1}{\operatorname{Im}\lambda/H_1+\operatorname{Ker}\mu/H_1}\approx\mu(H)/\mu\lambda(H)\in\mathcal{C}.$$

Consequently θ is $\mathcal{C}$-epimorphic. This completes the proof.

(1.2) LEMMA. In the following diagram of Abelian groups and homomorphisms, assume that commutativity holds in each triangle and $\operatorname{Im} i_\alpha = \operatorname{Ker} j_\alpha$ $(\alpha=1,2)$. If k_1 and k_2 are $\mathcal{C}$-isomorphisms, then the homomorphism $i: G_1+G_2\to G$ defined by $i(g_1,g_2)=i_1(g_1)+i_2(g_2)$ for $g_\alpha\in G_\alpha$ $(\alpha=1,2)$ is $\mathcal{C}$-isomorphic, where $\mathcal{C}=\mathcal{C}(\mathrm{I})$, and further we have that $\operatorname{Im} i_1\cap\operatorname{Im} i_2\in\mathcal{C}$.

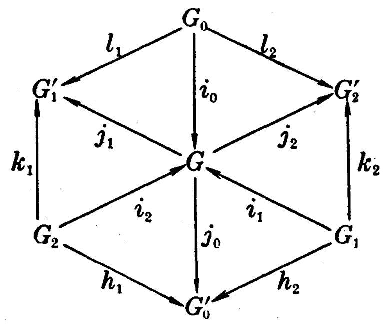


Proof. Let $g_\alpha\in G_\alpha$ $(\alpha=1,2)$ be elements such that $(g_1,g_2)\in\operatorname{Ker} i$, namely $i_1(g_1)+i_2(g_2)=0$. Applying j_1, we have $j_1i_2(g_2)=0$, i.e. $k_1(g_2)=0$. Similarly $k_2(g_1)=0$. Thus $\operatorname{Ker} i\subset\operatorname{Ker} k_1+\operatorname{Ker} k_2\in\mathcal{C}$. On the other hand, for each $x\in j_1^{-1}j_1i_2(G_2)$ there exists $y_2\in G_2$ such that $j_1(x)=j_1i_2(y_2)$. Therefore $x-i_2(y_2)\in\operatorname{Ker} j_1=\operatorname{Im} i_1$, hence there exists $y_1\in G_1$ such that $x-i_2(y_2)=i_1(y_1)$. Namely $x=i_1(y_1)+i_2(y_2)\in i_1(G_1)+i_2(G_2)$. Thus $j_1^{-1}j_1i_2(G_2)\subset i_1(G_1)+i_2(G_2)$. Since $j_1^{-1}j_1i_2(G_2)\supset i_1(G_1)+i_2(G_2)$, we have $j_1^{-1}j_1i_2(G_2) = i_1(G_1)+i_2(G_2)$, and

$$\operatorname{Coker} i=\frac{G}{i_1(G_1)+i_2(G_2)}\approx j_1(G)/j_1i_2(G_2)\subset G_1'/k_1(G_2)=\operatorname{Coker} k_1\in\mathcal{C}.$$

Thus i is a $\mathcal{C}$-isomorphism.
By (1.1), $i_1(G_1)\cap i_2(G_2)=i_1(G_1)\cap\operatorname{Ker} j_2\in\mathcal{C}$.

(1.3) LEMMA. (The generalization of the hexagonal lemma, cf. [1].) In the following diagram of groups and homomorphisms, assume that commutativity holds in each triangle, $\operatorname{Im} i_\alpha = \operatorname{Ker} j_\alpha$ $(\alpha=1, 2)$, $j_0 i_0 = 0$. Then for each $x \in G_0$ we have that



$$h_1 k_1^{-1} l_1(x) + h_2 k_2^{-1} l_2(x) \subset h_1(\operatorname{Ker} k_1)$$
$$= h_2(\operatorname{Ker} k_2) = j_0(\operatorname{Im} i_1 \cap \operatorname{Im} i_2).$$

Proof. For each $y_2 \in h_1 k_1^{-1} l_1(x)$ and $y_1 \in h_2 k_2^{-1} l_2(x)$, there exist $y_2' \in k_1^{-1} l_1(x)$ and $y_1' \in k_2^{-1} l_2(x)$ such that $h_1(y_2') = y_2$, $h_2(y_1') = y_1$. Since $k_1(y_2') = l_1(x)$, i.e. $j_1 i_2(y_2') = j_1 i_0(x)$, we have $i_2(y_2') - i_0(x) \in \operatorname{Ker} j_1 = \operatorname{Im} i_1$. Therefore there exists $y_1'' \in G_1$ such that $i_2(y_2') - i_0(x) = i_1(y_1'')$. Applying j_2, we have $-l_2(x) = k_2(y_2'')$. Since $k_2(y_1') = l_2(x)$, $k_2(y_1' + y_1'') = 0$, i.e. $y_1' + y_1'' \in \operatorname{Ker} k_2$. Then

$$y_1 + y_2 = h_2(y_1') + h_1(y_2') = h_2(y_1') + j_0 i_2(y_2') = h_2(y_1') + j_0(i_0(x) + i_1(y_1''))$$
$$= h_2(y_1') + j_0 i_1(y_1'') = h_2(y_1' + y_1'') \in h_2(\operatorname{Ker} k_2).$$

Moreover, $\operatorname{Im} i_1 \cap \operatorname{Im} i_2 = \operatorname{Im} i_1 \cap \operatorname{Ker} j_2 = i_1(\operatorname{Ker} k_2)$.

Applying j_0, $h_2(\operatorname{Ker} k_2) = j_0(\operatorname{Im} i_1 \cap \operatorname{Im} i_2)$.

Similarly $h_1(\operatorname{Ker} k_1) = j_0(\operatorname{Im} i_1 \cap \operatorname{Im} i_2)$.

(1.4) LEMMA. In the diagram given in (1.3), we have

$$x_1 - x_2 \in h_1(\operatorname{Ker} k_1),$$

where x_1 and x_2 are arbitrary elements of $h_1 k_1^{-1}(x)$ for each $x \in G_1'$.

Proof. Since $x_\alpha \in h_1 k_1^{-1}(x)$ $(\alpha=1, 2)$, there exists $y_\alpha \in k_1^{-1}(x)$ such that $h_1(y_\alpha) = x_\alpha$. Then $x_1 - x_2 = h_1(y_1 - y_2)$. Since $k_1(y_1 - y_2) = x - x = 0$, $x_1 - x_2 \in h_1(\operatorname{Ker} k_1)$.

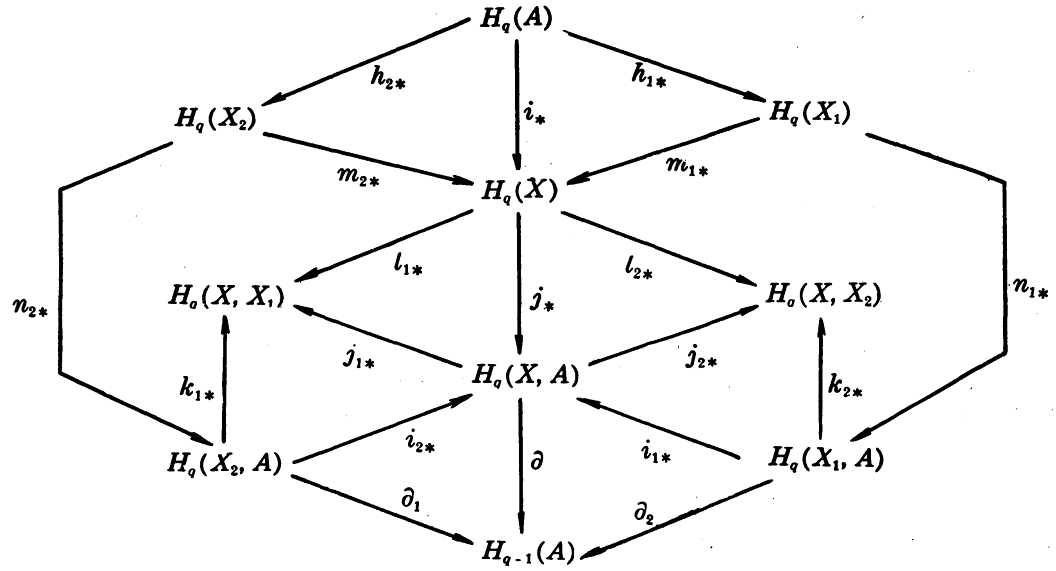
(1.5) LEMMA. If $\mathcal{C} = \mathcal{C}(\mathrm{I})$, then the conditions for a triad $(X; X_1, X_2)$ to be $\mathcal{C}$-excisive are equivalent to the following conditions: $X = X_1 \cup X_2$, $X, X_1, X_2, X_1 \cap X_2$ are arcwise connected and $H_q(X; X_1, X_2) \in \mathcal{C}$ for all q.

This is trivial.

(1.6) LEMMA. The conditions given in (1.5) are equivalent to the following conditions: $X = X_1 \cup X_2$, $X, X_1, X_2, X_1 \cap X_2$ are arcwise connected and the inclusion map $k_1: (X_2, X_1 \cap X_2) \to (X, X_1)$ induces the $\mathcal{C}$-isomorphism $k_{1*}: H_q(X_2, X_1 \cap X_2) \to H_q(X, X_1)$ for all q.

This lemma is also trivial.

In order to define the generalized Mayer-Vietoris sequence of a $\mathcal{C}$-excisive triad (X, X_1, X_2), observe the following diagram, in which $A = X_1 \cap X_2$ and all homomorphisms other than $\partial, \partial_1, \partial_2$ are induced by inclusion maps. Commutativity holds in each triangle, and the lower hexagon satisfies the hypotheses of (1.3). Furthermore $i_{1*} n_{1*} = j_* m_{1*}$, $i_{2*} n_{2*} = j_* m_{2*}$, and k_{2*} is a $\mathcal{C}$-isomorphism, and by (1.6), k_{1*} is also a $\mathcal{C}$-isomorphism.



(1.7) DEFINITION. The generalized Mayer-Vietoris sequence of a $\mathcal{C}$-excisive triad $(X; X_1, X_2)$ with $A=X_1\cap X_2$ is the following sequence:

$$\cdots\rightarrow H_q(A)/L_q\xrightarrow{\psi}H_q(X_1)+H_q(X_2)\xrightarrow{\varphi}H_q(X)\xrightarrow{\varDelta}H_{q-1}(A)/L_{q-1}\rightarrow\cdots,$$

where $L_{q-1}=\partial_1(\operatorname{Ker} k_{1*})$ which is equal to $\partial_2(\operatorname{Ker} k_{2*})$ and to $\partial(i_{1*}H_q(X_1, A)\cap i_{2*}H_q(X_2, A))$ (by (1.3)),

$\psi(\{u\})=(h_{1*}(u), -h_{2*}(u))$ for $\{u\}\in H_q(A)/L_q$, the quotient class represented by $u\in H_q(A)$,

$$\varphi(v_1, v_2)=m_{1*}(v_1)+m_{2*}(v_2)\quad\text{for } v_\alpha\in H_q(X_\alpha)\ (\alpha=1, 2),$$

$$\varDelta(w)=-\partial_1 k_{1*}^{-1}l_{1*}(w)/L_{q-1}\quad\text{for } w\in H_q(X).$$

Since $h_{\alpha*}(L_q)=0$ $(\alpha=1, 2)$, ψ is a well-defined homomorphism. By (1.4), the difference of any two elements of $\partial_1 k_{1*}^{-1}l_{1*}(w)$ is contained in L_{q-1}, hence $\varDelta$ is well-defind.

(1.8) THEOREM. The generalized Mayer-Vietoris sequence of a $\mathcal{C}$-excisive triad $(X; X_1, X_2)$ is $\mathcal{C}$-exact, where $\mathcal{C}=\mathcal{C}(\mathrm{I})$. In detail, $\operatorname{Im}\psi\subset\operatorname{Ker}\varphi$, $\operatorname{Ker}\varphi/\operatorname{Im}\psi\in\mathcal{C}$, $\operatorname{Im}\varphi=\operatorname{Ker}\varDelta$, $\operatorname{Im}\varDelta=\operatorname{Ker}\psi$.

Proof. 1) By a manner similar to that given in [1] for the case of an ordinary excisive triad, it is possible to prove that $\operatorname{Im}\psi\subset\operatorname{Ker}\varphi$, $\operatorname{Im}\varphi\subset\operatorname{Ker}\varDelta$ and $\operatorname{Im}\varDelta=\operatorname{Ker}\psi$.

2) If $w\in H_q(X)$ and $\varDelta(w)=0$, then $\partial_1 k_{1*}^{-1}l_{1*}(w)\subset L_{q-1}=\partial_1(\operatorname{Ker} k_{1*})$. Therefore, for each $x\in k_{1*}^{-1}l_{1*}(w)$ there exists $y\in\operatorname{Ker} k_{1*}$ such that $\partial_1(x)=\partial_1(y)$, i.e., $x-y\in\operatorname{Ker}\partial_1=n_{2*}H_q(X_2)$. Thus there exists $v_2\in H_q(X_2)$ such that $x-y=n_{2*}(v_2)$. Applying k_{1*} we have that

$$l_{1*}(w)=k_{1*}n_{2*}(v_2)=l_{1*}m_{2*}(v_2),\ \text{i.e.,}\ w-m_{2*}(v_2)\in\operatorname{Ker} l_{1*}=m_{1*}H_q(X_1).$$

Consequently there exists $v_1\in H_q(X_1)$ such that $w-m_{2*}(v_2)=m_{1*}(v_1)$, i.e., $w=m_{1*}(v_1)+m_{2*}(v_2)=\varphi(v_1, v_2)$.

3) Setting $M=m_{1*}H_q(X_1)\cap m_{2*}H_q(X_2)$, we have

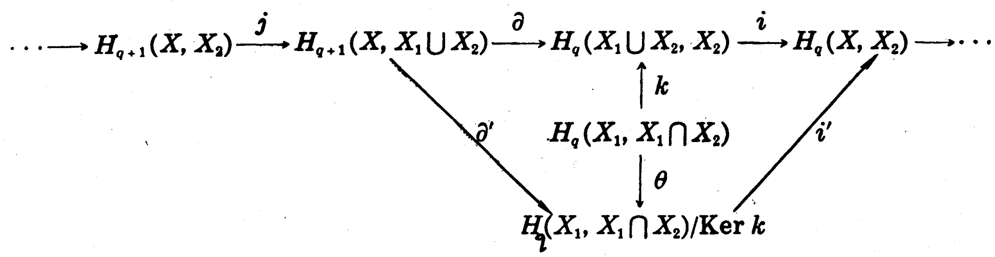
$$\operatorname{Ker}\varphi/\operatorname{Im}\psi \subset m_{1*}^{-1}(M)/h_{1*}H_q(A) + m_{2*}^{-1}(M)/h_{2*}H_q(A)\,,$$

and

$$m_{1*}^{-1}(M)/h_{1*}H_q(A) = \operatorname{Ext}\left(n_{1*}m_{1*}^{-1}(M),\ \frac{\operatorname{Ker} n_{1*}}{h_{1*}H_q(A)}\right).$$

Now $k_{2*}n_{1*}m_{1*}^{-1}(M) = l_{2*}m_{1*}m_{1*}^{-1}(M) = l_{2*}(M) \subset l_{2*}m_{2*}H_q(X_2) = 0$ and hence $n_{1*}m_{1*}^{-1}(M) \subset \operatorname{Ker} k_{2*} \in \mathcal{C}$. Since $\operatorname{Ker} n_{1*} = h_{1*}H_q(A)$, $m_{1*}^{-1}(M)/h_{1*}H_q(A) \in \mathcal{C}$. Similarly $m_{2*}^{-1}(M)/h_{2*}H_q(A) \in \mathcal{C}$. Consequently $\operatorname{Ker}\varphi/\operatorname{Im}\psi \in \mathcal{C}$. By 1) and 3) we have that $\operatorname{Ker}\varphi$ is $\mathcal{C}$-equal to $\operatorname{Im}\psi$. The proof of (1.8) is complete.

The homology sequence of a $\mathcal{C}$-proper triad $(X; X_1, X_2)$ may be defined as follows: In the following diagram



the upper horizontal sequence is the homology exact sequence of the triple $(X, X_1 \cup X_2, X_2)$, k is a $\mathcal{C}$-isomorphism induced by inclusion map, θ is the canonical map. We define ∂' and i' as follows:

$$\partial'(x) = \theta k^{-1}\partial(x) \quad \text{for } x \in H_{q+1}(X, X_1 \cup X_2)\,,$$

$i'(\{y\}) = ik(y)$ for $\{y\} \in H_q(X_1, X_1 \cap X_2)/\operatorname{Ker} k$, the quotient class represented by $y \in H_q(X_1, X_1 \cap X_2)$.

(1.9) Definition. The homology sequence of the $\mathcal{C}$-proper triad $(X; X_1, X_2)$ is the following:

$$\cdots \to H_{q+1}(X, X_2) \xrightarrow{j} H_{q+1}(X, X_1 \cup X_2) \xrightarrow{\partial'} H_q(X_1, X_1 \cap X_2)/\operatorname{Ker} k \xrightarrow{i'} H_q(X, X_2) \to \cdots\,.$$

(1.10) Theorem. The homology sequence of the $\mathcal{C}$-proper triad $(X; X_1, X_2)$ is $\mathcal{C}$-exact, where $\mathcal{C} = \mathcal{C}(\mathrm{I})$. In detail, $\operatorname{Im} j = \operatorname{Ker}\partial'$, $\operatorname{Im}\partial' = \operatorname{Ker} i'$, $\operatorname{Im} i' \subset \operatorname{Ker} j$ and $\operatorname{Ker} j/\operatorname{Im} i' \in \mathcal{C}$.

Proof. It is clear that $\operatorname{Im} j = \operatorname{Ker}\partial'$, $\operatorname{Im}\partial' \subset \operatorname{Ker} i'$ and $\operatorname{Im} i' \subset \operatorname{Ker} j$.
Now

$$\operatorname{Ker} i'/\operatorname{Im}\partial' = \frac{k^{-1}(\operatorname{Im} k \cap \operatorname{Ker} i)/\operatorname{Ker} k}{k^{-1}\partial H_{q+1}(X, X_1 \cup X_2)/\operatorname{Ker} k} = \frac{k^{-1}(\operatorname{Im} k \cap \operatorname{Im}\partial)}{k^{-1}(\operatorname{Im}\partial)} = 0\,.$$

Thus $\operatorname{Im}\partial' = \operatorname{Ker} i'$.
Secondly

$$\operatorname{Ker} j/\operatorname{Im} i' = \operatorname{Im} i/\operatorname{Im}(i \circ k) = \frac{iH_q(X_1 \cup X_2, X_2)}{ikH_q(X_1, X_1 \cap X_2)} = \text{image of Coker } k \in \mathcal{C}.$$

§2. The Blakers and Massey theorem for a $\mathcal{C}$-excisive triad

(2.1) LEMMA. Let A and B be Abelian groups, and A' and B' be subgroups of A and B, respectively. If $h: A \to B$ is a $\mathcal{C}$-isomorphism and $h \mid A'$ is a $\mathcal{C}$-epimorphism from A' to B', then $h^*: A/A' \to B/B'$, the canonical homomorphism induced by h, is $\mathcal{C}$-isomorphism, where $\mathcal{C} = \mathcal{C}(\mathrm{I})$.

Proof. h^* is the composition of two homomorphisms

$$A/A' \xrightarrow{\bar{h}} B/h(A') \xrightarrow{\theta} B/B',$$

where $\bar{h}$ is induced by h and θ is the canonical epimorphism. Since $\bar{h}$ is $\mathcal{C}$-isomorphic [3, Proposition 3], h^* is $\mathcal{C}$-epimorphic.

Furthermore $\quad \mathrm{Ker}\, h^* = \bar{h}^{-1}(B'/h(A')) = \mathrm{Ext}\,(hh^{-1}(B')/h(A'),\ \mathrm{Ker}\, \bar{h}).$

Since $hh^{-1}(B')/h(A') \subset B'/h(A') \in \mathcal{C}$, $\mathrm{Ker}\, \bar{h} \in \mathcal{C}$, we have $\mathrm{Ker}\, h^* \in \mathcal{C}$.

(2.2) LEMMA. In the following diagram of Abelian groups and homomorphisms, assume that h_1 is a $\mathcal{C}$-epimorphism, h_2 is a $\mathcal{C}$-isomorphism and h_3 is a $\mathcal{C}$-monomorphism, where $\mathcal{C} = \mathcal{C}(\mathrm{I})$. If the commutativity holds in each square, then $h_2' = h_2 \mid \mathrm{Ker}\, f_2$ is a $\mathcal{C}$-isomorphism of $\mathrm{Ker}\, f_2$ with $\mathrm{Ker}\, g_2$ and $h_2'' = h_2 \mid \mathrm{Im}\, f_1$ is a $\mathcal{C}$-isomorphism of $\mathrm{Im}\, f_1$ with $\mathrm{Im}\, g_1$.

$$\begin{array}{ccccc} A_1 & \xrightarrow{f_1} & A_2 & \xrightarrow{f_2} & A_3 \\ \downarrow{\scriptstyle h_1} & & \downarrow{\scriptstyle h_2} & & \downarrow{\scriptstyle h_3} \\ B_1 & \xrightarrow{g_1} & B_2 & \xrightarrow{g_2} & B_3 \end{array}$$

Proof. By the commutativity in each square, we have

$$h_2(\mathrm{Ker}\, f_2) \subset \mathrm{Ker}\, g_2, \quad h_2(\mathrm{Im}\, f_1) \subset \mathrm{Im}\, g_1.$$

Furthermore

$$\mathrm{Ker}\, h_2' = \mathrm{Ker}\, f_2 \cap \mathrm{Ker}\, h_2 \subset \mathrm{Ker}\, h_2 \in \mathcal{C},$$

$$\mathrm{Coker}\, h_2' = \mathrm{Ker}\, g_2/h_2(\mathrm{Ker}\, f_2) = \mathrm{Ext}\,(\mathrm{Ker}\, g_2/h_2 f_2^{-1}(\mathrm{Ker}\, h_3),\ h_2 f_2^{-1}(\mathrm{Ker}\, h_3)/h_2(\mathrm{Ker}\, f_2)).$$

Now the following relations hold:

$$h_2(A_2) \cap \mathrm{Ker}\, g_2 \subset h_2 f_2^{-1}(\mathrm{Ker}\, h_3) \subset \mathrm{Ker}\, g_2.$$

To prove the first, let x be an arbitrary element of $h_2(A_2) \cap \mathrm{Ker}\, g_2$. There exists $y \in A_2$ such that $h_2(y) = x$. Then $g_2 h_2(y) = g_2(x) = 0.$, i.e., $h_3 f_2(y) = 0$, hence $y \in f_2^{-1}(\mathrm{Ker}\, h_3)$. Thus $x = h_2(y) \in h_2 f_2^{-1}(\mathrm{Ker}\, h_3)$. To prove the second, let x be an element of $h_2 f_2^{-1}(\mathrm{Ker}\, h_3)$. There exists $y \in f_2^{-1}(\mathrm{Ker}\, h_3)$ such that $h_2(y) = x$. Then $g_2(x) = g_2 h_2(y) = h_3 f_2(y) \in h_3 f_2(f_2^{-1}(\mathrm{Ker}\, h_3)) \subset h_3(\mathrm{Ker}\, h_3) = 0$, i.e., $x \in \mathrm{Ker}\, g_2$.

By these relations we have the following canonical epimorphism:

$$\mathrm{Ker}\, g_2/(h_2(A_2) \cap \mathrm{Ker}\, g_2) \to \mathrm{Ker}\, g_2/h_2 f_2^{-1}(\mathrm{Ker}\, h_3).$$

Since

$$\mathrm{Ker}\, g_2/(h_2(A_2) \cap \mathrm{Ker}\, g_2) \approx (\mathrm{Ker}\, g_2 + h_2(A_2))/h_2(A_2) \subset B_2/h_2(A_2) \in \mathcal{C},$$

we obtain $\mathrm{Ker}\, g_2/h_2 f_2^{-1}(\mathrm{Ker}\, h_3) \in \mathcal{C}$.

On the other hand, $h_2 f_2^{-1}(\mathrm{Ker}\, h_3)/h_2(\mathrm{Ker}\, f_2)$ is the image of $f_2^{-1}(\mathrm{Ker}\, h_3)/\mathrm{Ker}\, f_2$ by the homomorphism induced by h_3, and

$$f_2^{-1}(\mathrm{Ker}\, h_3)/\mathrm{Ker}\, f_2 \approx f_2 f_2^{-1}(\mathrm{Ker}\, h_3) \subset \mathrm{Ker}\, h_3 \in \mathcal{C}\,.$$

Therefore $h_2 f_2^{-1}(\mathrm{Ker}\, h_3)/h_2(\mathrm{Ker}\, f_2) \in \mathcal{C}$. Thus $\mathrm{Coker}\, h_2' \in \mathcal{C}$ and h_2' is $\mathcal{C}$-isomorphic. The proof that h_2'' is a $\mathcal{C}$-isomorphism proceeds as follows:

$$\mathrm{Ker}\, h_2'' = \mathrm{Im}\, f_1 \cap \mathrm{Ker}\, h_2 \subset \mathrm{Ker}\, h_2 \in \mathcal{C}\,,$$

$$\mathrm{Coker}\, h_2'' = g_1(B_1)/(h_2 f_1(A_1)) = \mathrm{Ext}\,(g_1(B_1)/(h_2(A_2)\cap g_1(B_1)),\ (h_2(A_2)\cap g_1(B_1))/h_2 f_1(A_1))\,,$$

where

$$g_1(B_1)/(h_2(A_2)+g_1(B_1)) \approx (g_1(B_1)+h_2(A_2))/h_2(A_2) \subset B_2/h_2(A_2) \in \mathcal{C}\,,$$

$$(h_2(A_2)\cap g_1(B_1))/h_2 f_1(A_1) = (h_2(A_2)\cap g_1(B_1))/g_1 h_1(A_1) \subset g_1(B_1)/g_1 h_1(A_1)\,.$$

Since $g_1(B_1)/g_1 h_1(A_1)$ is the image of $B_1/h_1(A_1) \in \mathcal{C}$ by the homomorphism induced by g_1, we have $g_1(B_1)/g_1 h_1(A_1) \in \mathcal{C}$ and hence

$$(h_2(A_2)\cap g_1(B_1))/h_2 f_1(A_1) \in \mathcal{C}\,.$$

Consequently $\mathrm{Coker}\, h_2'' \in \mathcal{C}$ and h_2'' is a $\mathcal{C}$-isomorphism.

(2.3) LEMMA. In the diagram given in (2.2), if commutativity holds in each square and if further $f_2 \circ f_1$ and $g_2 \circ g_1$ are trivial, then h_2 induces a $\mathcal{C}$-isomorphism $h_2^* : \mathrm{Ker}\, f_2/\mathrm{Im}\, f_1 \to \mathrm{Ker}\, g_2/\mathrm{Im}\, g_1$.

Proof. It follows from (2.2) that h_2 induces $\mathcal{C}$-isomorphisms h_2' and h_2''. By the trivialities of $f_2 \circ f_1$ and $g_2 \circ g_1$ we have that $\mathrm{Im}\, f_1 \subset \mathrm{Ker}\, f_2$ and $\mathrm{Im}\, g_1 \subset \mathrm{Ker}\, g_2$. Then the assertion follows from (2.1).

The following theorem concerned with the associated triad is a generalization of Theorem 3.3 of [2].

(2.4) PROPOSITION. Let $\mathcal{C} = \mathcal{C}(\mathrm{I}, \mathrm{II}_B)$. If a triad $(X; X_1, X_2)$ is $\mathcal{C}$-excisive, and X is 1-connected, then the associated triad (X^*, X_1^*, X_2^*) of $(X; X_1, X_2)$ is $\mathcal{C}$-excisive.

The truth of this theorem follows from the definition of $\mathcal{C}$-excisive triad once we have extended Theorem 2.2 of [2] in the following form:

(2.5) PROPOSITION. Let (E, p, B) and (E', p', B') be fibre spaces in the sense of Serre [4] with fibre F, A and A' be subspaces of B and B' respectively, $D = p^{-1}(A)$, $D' = p'^{-1}(A')$, B, A, B', A' and F be arcwise connected and $\pi_1(B)$, $\pi_1(B')$ operate trivially on $H_q(F)$ for all q, finally let $f : (E, D) \to (E', D')$ be a fibre preserving map, $f' : (B, A) \to (B', A')$ be induced by f. If $f'_* : H_q(B, A) \to H_q(B', A')$ is $\mathcal{C}$-isomorphic for all q, then $f_* : H_q(E, D) \to H_q(E', D')$ is $\mathcal{C}$-isomorphic for all q, where $\mathcal{C} = \mathcal{C}(\mathrm{I}, \mathrm{II}_B)$.

Application of (2.3), Corollary in [5, p. 263] and the five lemma in the case of $\mathcal{C}$-notion [5] enables us to prove (2.5) in a way similar to that of Theorem 2.2 of [2].

(2.6) REMARK. It should be noted that we have $\pi_q(X^*, X_\alpha^*) \approx \pi_q(X, X_\alpha)$ $(\alpha = 1, 2)$ [5] and $\pi_q(X^*; X_1^*, X_2^*) \approx \pi_q(X; X_1, X_2)$ [2], for all q.

The following theorem is due to J.-P. Serre [5]:

THEOREM S. Let (E, p, B) be a fibre space with fibre F, A be a subspace of B and $D=p^{-1}(A)$. Assume that B, A and F are arcwise connected and the local system formed by $H_q(F)$ on B is trivial for all q. If $H_q(B, A)\in\mathcal{C}$ for $0\leqq q<m$ and $H_q(F)\in\mathcal{C}$ for $0<q<r$, then the projection p induces

$$p_*: H_q(E, D)\to H_q(B, A)$$

such that p^* is $\mathcal{C}$-isomorphic for $q\leqq m+r-1$ and $\mathcal{C}$-epimorphic for $q\leqq m+r$, where $\mathcal{C}=\mathcal{C}(\mathrm{I}, \mathrm{II}_\mathrm{B})$.

As an immediate consequence of Theorem S, we have the following:

(2.7) PROPOSITION. Let $\mathcal{C}=\mathcal{C}(\mathrm{I}, \mathrm{II}_\mathrm{B})$. Let $(X; X_1, X_2)$ be a triad such that

X is 1-connected, X_1 and X_2 are arcwise connected,

$H_q(X)\in\mathcal{C}$ for $0<q\leqq r$, $H_q(X, X_\alpha)\in\mathcal{C}$ for $q<m_\alpha$ $(\alpha=1, 2)$.

Let $(X^*; X_1^*, X_2^*)$ be the associated triad of $(X; X_1, X_2)$ and $p:(X^*; X_1^*, X_2^*)\to(X; X_1, X_2)$ be the projection. Then p induces

$$p_{\alpha*}: H_q(X^*, X_\alpha^*)\to H_q(X, X_\alpha)$$

which is $\mathcal{C}$-isomorphic for $q\leqq m_\alpha+r-1$ and $\mathcal{C}$-epimorphic for $q\leqq m_\alpha+r$ $(\alpha=1, 2)$.

Proof. Since X^* is contractible, we have $H_q(X^*)\in\mathcal{C}$ for each $q>0$. The axiom (II_B) implies the axiom (II_A) of [5]. Therefore applying Proposition 3.A of [5, p. 269] it follows, from the assumptions: $H_q(X)\in\mathcal{C}$ for $0<q\leqq r$, that $H_q(F)\in\mathcal{C}$ for $0<q<r$, where F is the fibre of (X^*, p, X). Our theorem now follows from Theorem S.

Before we study the Blakers and Massey triad theorem, it is convenient to state the following lemma:

(2.8) LEMMA. If $(X; X_1, X_2)$ is a $\mathcal{C}$-excisive triad where $\mathcal{C}=\mathcal{C}(\mathrm{I})$, then the following sequence is $\mathcal{C}$-exact:

$$\cdots\to H_q(Z)/L_q'\xrightarrow{i'}H_q(X_1\times X_2)/i(L_q')\xrightarrow{j'}H_q(X_1\times X_2, Z)\xrightarrow{\partial'}H_{q-1}(Z)/L_{q-1}'\to\cdots$$

where $Z=(X_1\times X_2)\cap X_d$, X_d is the diagonal of $X\times X$, L_q' is the inverse image of L_q (for the definition of L_q, see (1.7)) by the isomorphism τ of $H_q(Z)$ with $H_q(X_1\cap X_2)$ induced by the homeomorphism $Z\approx X_1\cap X_2$, i' and j' are induced by the inclusion maps $i: H_q(Z)\to H_q(X_1\times X_2)$ and $j: H_q(X_1\times X_2)\to H_q(X_1\times X_2, Z)$, and ∂' is the composition of the two homomorphisms

$$H_q(X_1\times X_2, Z)\xrightarrow{\partial}H_{q-1}(Z)\xrightarrow{\theta}H_{q-1}(Z)/L_{q-1}',$$

where ∂ is the boundary homomorphism and θ is the canonical map.

Proof. It is clear that

$$\mathrm{Im}\, i'=\mathrm{Ker}\, j',\quad \mathrm{Im}\, j'=\mathrm{Ker}\,\partial\subset\mathrm{Ker}\,\partial',\quad \mathrm{Im}\,\partial'=\mathrm{Ker}\, i/L_{q-1}'\subset\mathrm{Ker}\, i'.$$

Furthermore $\mathrm{Ker}\,\partial'/\mathrm{Im}\, j'=\partial^{-1}(L_{q-1}')/\mathrm{Ker}\,\partial$ is isomorphic with a subgroup of L_{q-1}'.

Since $(X; X_1, X_2)$ is $\mathcal{C}$-excisive, it follows from (1.2) and (1.3) that $L_{q-1}\in\mathcal{C}$, hence $L'_{q-1}\in\mathcal{C}$. Thus $\operatorname{Ker}\partial'/\operatorname{Im} j'\in\mathcal{C}$, and consequently $\operatorname{Ker}\partial'$ is $\mathcal{C}$-equal to $\operatorname{Im} j'$.
Secondly

$$\operatorname{Ker} i'/\operatorname{Im}\partial' = \frac{i^{-1}i(L'_{q-1})/L'_{q-1}}{\operatorname{Ker} i/L'_{q-1}} \approx \frac{i^{-1}i(L'_{q-1})}{\operatorname{Ker} i} \approx i(L'_{q-1})\in\mathcal{C}\,.$$

Therefore $\operatorname{Ker} i'$ is $\mathcal{C}$-equal to $\operatorname{Im}\partial'$.
This completes the proof of $\mathcal{C}$-exactness.

Now Theorem 3.4 of [2] may be generalized as follows:

(2.9) THEOREM. (The generalized Blakers and Massey triad theorem) Let $\mathcal{C}=\mathcal{C}(\mathrm{I}, \mathrm{II}_B, \mathrm{III})$. If $(X; X_1, X_2)$ is a $\mathcal{C}$-excisive triad with $A=X_1\cap X_2$ such that

X is 1-connected, (X, X_1) and (X, X_2) are 2-connected, $\pi_3(X; X_1, X_2)=0$,

$H_q(X, X_1)\in\mathcal{C}$ for $q<m$, $H_q(X, X_2)\in\mathcal{C}$ for $q<n$,

then $\pi_q(X; X_1, X_2)\in\mathcal{C}$ for $q<m+n-1$,

$\pi_{m+n-1}(X; X_1, X_2)$ is $\mathcal{C}$-isomorphic with $H_m(X, X_1)\otimes H_n(X, X_2)$.

(2.10) REMARK. In (2.9), if $(X; X_1, X_2)$ is excisive in the ordinary sense, and if further A is 1-connected, then the assumption: $\pi_3(X; X_1, X_2)=0$, is an immediate consequence of the other assumptions, and it may be verified as follows:

Since $\pi_2(X, X_\alpha)=0$ and $\pi_1(X)=0$, $\pi_1(X_\alpha)=0$ $(\alpha=1, 2)$. Then by the Hurewicz isomorphism theorem, $H_q(X, X_2)=0$, $\pi_3(X, X_2)\approx H_3(X, X_2)$. From the excision property of $(X; X_1, X_2)$, $H_2(X_1, A)\approx H_2(X, X_2)=0$. Then, since X_1 and A are 1-connected, we have $\pi_2(X_1, A)=0$ and $\pi_3(X_1, A)\approx H_3(X_1, A)$. Consequently $\pi_3(X_1, A)\approx\pi_3(X, X_2)$ (by the isomorphism induced by the inclusion map). Then from the exactness of the homotopy sequence of triad $(X; X_1, X_2)$, it follows that $\pi_3(X; X_1, X_2)=0$.

Proof of (2.9). The proof proceeds after the manner of J. C. Moore [2]. By (2.4), (2.6) and (2.7) it may be assumed that X is contractible, hence the following relations 1° and 2° hold [2]:

1°. $\pi_q(X_1\times X_2, Z)\approx\pi_{q+1}(X; X_1, X_2)$ for all q, where Z is the same set as given in (2.8).

2°. In the following diagram, we have that $j_1\,|\operatorname{Ker}\mu$ is an isomorphism of $\operatorname{Ker}\mu$ with $H_q(X_1\times X_2, X_1\vee X_2)$, where j_1 is injection, μ is the natural homomorphism defined using the projections of $X_1\times X_2$ on its factors. Now consider the following diagram:

$$\begin{array}{ccc} H_q(X_1\times X_2) & \xrightarrow{j_1} & H_q(X_1\times X_2, X_1\vee X_2) \\ \downarrow\mu & & \\ H_q(X_1)+H_q(X_2) & & \end{array}$$

$$\begin{array}{ccccccc} \cdots\to H_q(Z)/L_q' & \xrightarrow{i'} & H_q(X_1\times X_2)/i(L_q') & \xrightarrow{j'} & H_q(X_1\times X_2, Z) & \xrightarrow{\partial'} & H_{q-1}(Z)/L'_{q-1}\to\cdots \\ \approx\downarrow\tau' & & \downarrow\mu' & & & & \\ H_q(A)/L_q & \xrightarrow{\psi} & H_q(X_1)+H_q(X_2) & & & & \end{array}$$

where the upper horizontal sequence is $\mathcal{C}$-exact one given in (2.8), τ' and μ' are induced by τ and μ (for the definition of τ, see (2.8)), ψ is the map given in (1.7), and commutativity holds in the square.

Since $(X; X_1, X_2)$ is $\mathcal{C}$-excisive, its generalized Mayer-Vietoris sequence:

$$\cdots \to H_q(A)/L_q \xrightarrow{\psi} H_q(X_1)+H_q(X_2) \xrightarrow{\phi} H_q(X) \xrightarrow{\Delta} H_{q-1}/L_{q-1} \to \cdots$$

is $\mathcal{C}$-exact (by (1.8)). Now X is contractible, hence $H_q(X)=0$. Then by (1.8), $\operatorname{Ker}\psi=\operatorname{Im}\Delta=0$, i.e., ψ is a monomorphism. Therefore we have $\operatorname{Im} i' \cap \operatorname{Ker}\mu'=0$, i.e., $\operatorname{Ker} j' \cap \operatorname{Ker}\mu'=0$, and moreover i' is monomorphic. Thus $j' \mid \operatorname{Ker}\mu'$ is a monomorphism of $\operatorname{Ker}\mu'$ into $H_q(X_1\times X_2, Z)$. Since $\operatorname{Im}\partial' \subset \operatorname{Ker} i'=0$, $H_q(X_1\times X_2, Z)=\operatorname{Ker}\partial'$, hence

$$H_q(X_1\times X_2, Z)/j'(\operatorname{Ker}\mu')=\operatorname{Ker}\partial'/j'(\operatorname{Ker}\mu')=\operatorname{Ext}(\operatorname{Ker}\partial'/\operatorname{Im} j', \operatorname{Im} j'/j'(\operatorname{Ker}\mu')),$$

where $\operatorname{Ker}\partial'/\operatorname{Im} j' \in \mathcal{C}$ (by (2.8)). And by (1.1) with $H_1=0$,

$$\frac{H_q(X_1\times X_2)/i(L_q')}{\operatorname{Im} i'+\operatorname{Ker}\mu'} \in \mathcal{C},$$

hence $\operatorname{Im} j'/j'(\operatorname{Ker}\mu') \in \mathcal{C}$.

Consequently $H_q(X_1\times X_2, Z)$ is the $\mathcal{C}$-isomorphic image of $\operatorname{Ker}\mu'$ by j'.

Furthermore, since $i(L_q')\in\mathcal{C}$, $\operatorname{Ker}\mu$ is $\mathcal{C}$-isomorphic with $\operatorname{Ker}\mu'=\operatorname{Ker}\mu/i(L_q')$.

Combining these results, we find that $H_q(X_1\times X_2, Z)$ is $\mathcal{C}$-isomorphic with $H_q(X_1\times X_2, X_1\vee X_2)$.

Then using the Künneth theorem, we see that $H_q(X_1\times X_2, Z)$ is $\mathcal{C}$-isomorphic with

$$\sum_{\substack{r+s=q\\ r,s>0}} H_r(X_1)\otimes H_s(X_2) + \sum_{r+s=q-1} H_r(X_1) * H_s(X_2)$$

for all q. Since the axiom ($\mathrm{II_B}$) is equivalent to the axiom ($\mathrm{II_B}'$) of [5],

$$H_r(X_1)\otimes H_s(X_2)\in\mathcal{C} \text{ for non-zero } r \text{ and } s \text{ such that } r+s<m+n-2,$$

$$H_r(X_1) * H_s(X_2)\in\mathcal{C} \text{ for all } r \text{ and } s \text{ such that } r+s<m+n-2.$$

Hence

$$H_q(X_1\times X_2, Z)\in\mathcal{C} \text{ for } q<m+n-2,$$

$$H_{m+n-2}(X_1\times X_2, Z) \text{ is } \mathcal{C}\text{-isomorphic with } H_{m-1}(X_1)\otimes H_{n-1}(X_2).$$

Since $\pi_1(X_\alpha)=\pi_2(X, X_\alpha)=0$, X_α is 1-connected ($\alpha=1, 2$), and hence $X_1\times X_2$ is also. Moreover $\pi_2(X_1\times X_2, Z)\approx\pi_3(X; X_1, X_2)=0$. Then by the relative Hurewicz isomorphism theorem [5],

$$\pi_q(X_1\times X_2, Z)\in\mathcal{C} \text{ for } q<m+n-2,$$

$$\pi_{m+n-2}(X_1\times X_2, Z) \text{ is } \mathcal{C}\text{-isomorphic with } H_{m-1}(X_1)\otimes H_{n-1}(X_2).$$

This completes the proof of (2.9).

(2.11) COROLLARY. Let $\mathcal{C}=\mathcal{C}(\mathrm{I}, \mathrm{II_B}, \mathrm{III})$. If $(X; X_1, X_2)$ is a $\mathcal{C}$-proper triad such that

$X_1 \cup X_2$ is 1-connected, $\pi_3(X_1 \cup X_2; X_1, X_2) = 0$,
$(X_1 \cup X_2, X_1)$ and $(X_1 \cup X_2, X_2)$ are 2-connected,
$H_q(X_1 \cup X_2, X_1) \in \mathcal{C}$ for $q < m$, $H_q(X_1 \cup X_2, X_2) \in \mathcal{C}$ for $q < n$,
$\pi_q(X, X_1 \cup X_2) \in \mathcal{C}$ for $q < r$,

then $\pi_q(X; X_1, X_2) \in \mathcal{C}$ for $q < \min(m+n-1, r)$.

Proof. By (2.9), we have that $\pi_q(X_1 \cup X_2; X_1, X_2) \in \mathcal{C}$ for $q < m+n-1$. Therefore by the exactness of the following homotopy sequence of tetrad $(X; X_1 \cup X_2, X_1, X_2)$:

$$\cdots \to \pi_q(X_1 \cup X_2; X_1, X_2) \to \pi_q(X; X_1, X_2) \to \pi_q(X; X_1 \cup X_2; X_1, X_2)$$
$$\to \pi_{q-1}(X_1 \cup X_2; X_1, X_2) \to \cdots,$$

we have that $\pi_q(X; X_1, X_2) \to \pi_q(X; X_1 \cup X_2, X_1, X_2)$ is $\mathcal{C}$-isomorphic for $q < m+n-1$. Since $\pi_q(X; X_1 \cup X_2, X_1, X_2) \approx \pi_q(X, X_1 \cup X_2)$ for all q, it follows, from the assumption: $\pi_q(X, X_1 \cup X_2) \in \mathcal{C}$ for $q < r$, that $\pi_q(X; X_1, X_2) \in \mathcal{C}$ for $q < \min(m+n-1, r)$.

§3. The Hurewicz isomorphism theorem for triad

The matter of this section has no intimate relations with that of preceding sections. The following theorem is a formal generalization of the relative Hurewicz isomorphism theorem due to J.-P. Serre [5].

(3.1) THEOREM. Let $(X; X_1, X_2)$ be a triad such that

X, X_1, X_2 and $X_1 \cap X_2$ are 1-connected,
the inclusion maps $\pi_2(X_2) \to \pi_2(X)$ and $\pi_2(X_1 \cap X_2) \to \pi_2(X_1)$ are epimorphic,
$\pi_q(X_1, X_1 \cap X_2) \in \mathcal{C}$ for $q \leqq n$,

where $\mathcal{C} = \mathcal{C}(\mathrm{I}, \mathrm{II_B}, \mathrm{III})$. If $\pi_q(X; X_1, X_2) \in \mathcal{C}$ for $q < n$, we have that $H_q(X; X_1, X_2) \in \mathcal{C}$ for $0 < q < n$ and the natural map $\omega_2 : \pi_q(X; X_1, X_2) \to H_q(X; X_1, X_2)$ is $\mathcal{C}$-isomorphic for $q = n$ and $\mathcal{C}$-epimorphic for $q = n+1$.

Proof. The Hurewicz isomorphism theorem implies that $H_q(X_1, X_1 \cap X_2) \in \mathcal{C}$ for $q \leqq n$. Therefore the exactness of the homology sequence of $(X; X_1, X_2)$ implies that the homomorphism

$$j_* : H_q(X, X_2) \to H_q(X; X_1, X_2),$$

induced by inclusion, is $\mathcal{C}$-isomorphic for $q \leqq n$ and is $\mathcal{C}$-epimorphic for $q = n+1$. Similarly by the assumptions: $\pi_q(X_1, X_1 \cap X_2) \in \mathcal{C}$ for $q \leqq n$, and by the exactness of the homotopy sequence of $(X; X_1, X_2)$, we have that

$$j_0 : \pi_q(X, X_2) \to \pi_q(X; X_1, X_2)$$

is $\mathcal{C}$-isomorphic for $q \leqq n$.

From the hypotheses: $\pi_q(X; X_1, X_2) \in \mathcal{C}$ for $q < n$, we have that $\pi_q(X, X_2) \in \mathcal{C}$ for $q < n$, and by the application of the Hurewicz isomorphism theorem we see that $H_q(X, X_2) \in \mathcal{C}$ for $q < n$ and the natural homomorphism

$$\omega_1 : \pi_q(X, X_2) \rightarrow H_q(X, X_2)$$

is $\mathcal{C}$-isomorphic for $q \leqq n$ and $\mathcal{C}$-epimorphic for $q=n+1$.
Consequently $H_q(X; X_1, X_2) \in \mathcal{C}$ for $q<n$, and from the following commutative diagrams, where j_0, j_* and ω_1 in the left diagram are $\mathcal{C}$-isomorphisms and j_* and ω_1 in the right diagram are $\mathcal{C}$-epimorphisms, it follows that the natural homomorphism $\omega_2 : \pi_q(X; X_1, X_2) \rightarrow H_q(X; X_1, X_2)$ is $\mathcal{C}$-isomorphic for $q=n$ and $\mathcal{C}$-epimorphic for $q=n+1$.

$$\begin{array}{ccc} \pi_n(X, X_2) & \xrightarrow{j_0} & \pi_n(X; X_1, X_2) \\ \omega_1 \downarrow & & \omega_2 \downarrow \\ H_n(X, X_2) & \xrightarrow{j_*} & H_n(X; X_1, X_2) \end{array} \qquad \begin{array}{ccc} \pi_{n+1}(X, X_2) & \xrightarrow{j_0} & \pi_{n+1}(X; X_1, X_2) \\ \omega_1 \downarrow & & \omega_2 \downarrow \\ H_{n+1}(X, X_2) & \xrightarrow{j_*} & H_{n+1}(X; X_1, X_2) \end{array}$$

Thus the proof is complete.

(3.2) COROLLARY. Let $(X; X_1, X_2)$ be the triad mentioned in (3.1). If $H_q(X; X_1, X_2) \in \mathcal{C} = \mathcal{C}(\mathrm{I}, \mathrm{II}_\mathrm{B}, \mathrm{III})$ for $0<q<n$, we have that

$$\pi_q(X; X_1, X_2) \in \mathcal{C} \text{ for } 2 \leqq q < n,$$

and the natural homomorphism $\omega_2 : \pi_q(X; X_1, X_2) \rightarrow H_q(X; X_1, X_2)$ is $\mathcal{C}$-isomorphic for $q=n$ and $\mathcal{C}$-epimorphic for $q=n+1$.

Proof. Since the inclusion map : $\pi_2(X_2) \rightarrow \pi_2(X)$ is epimorphic and $\pi_1(X_2)=0$, $\pi_2(X, X_2)=0$. Since $\pi_1(X_1)=0$, $\pi_1(X_1, X_1 \cap X_2)=0$. Then by the exactness of the homotopy sequence of $(X; X_1, X_2)$ we have that $\pi_2(X; X_1, X_2)=0 \in \mathcal{C}$.
Our corollary now follows from (3.1).

References

[1] S. Eilenberg and N. Steenrod: Foundations of algebraic topology, Princeton University Press, 1952.

[2] J. C. Moore: *Some applications of homology theory to homotopy problems*, Ann. of Math. 58 (1953), 325–350.

[3] M. Nakaoka: *On a theorem of Eilenberg-MacLane*, Journ. Inst. Polyt. Osaka City Univ. 5 (1954), 32–39.

[4] J.-P. Serre: *Homologie singulière des espaces fibrés*, Ann. of Math. 54 (1951), 425–505.

[5] J.-P. Serre: *Groupes d'homotopie et classes de groupes abéliens*, Ann. of Math. 58 (1953), 258–294.