On Postnikov's complexes and spaces of loops

By

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M. M. Postnikov [1] defined the natural system of an arcwise-connected space and its complex which is a generalization of Eilenberg-MacLane's complex [3], and he obtained its geometrical realization [2].

It is the purpose of this paper to consider the properties of these systems and complexes of a space and of the space of loops on it, and to extend the theorems, concerned with the properties of Eilenberg-MacLane's complexes and their geometrically realized spaces (see [4], [5], for examples), to the case of Postnikov's complexes. In §§ 2 and 3, we shall construct the natural systems and their Postnikov's complexes of a topological space and of the space of loops on it, connecting them each other by some special relations (theorem 3.12).

In § 4, we have occasion to discuss the realization problem, and the following theorems are given:

THEOREM 4.2. If the natural systems of two spaces are isomorphic, then the natural systems of the spaces of loops on them are so also.

THEOREM 4.3. For a given system (G_i, k_i) satisfying some conditions, there exists a space of loops whose natural system and the given system are isomorphic if and only if G_1 operates trivially on G_i $(i\geqq 2)$.

In § 5, two problems will be considered, which are generalizations of Serre's [4] and of Cartan-Serre's fibering [6], i. e.,

THEOREM 5.1. For two systems $\mathbf{H}$ and $\mathbf{F}$ satisfying some conditions, there exists a fibering (E, X, F, p), in the sense of Serre [4], such that the natural systems of X and F are isomorphic to $\mathbf{H}$ and $\mathbf{F}$ respectively.

THEOREM 5.4. For two systems (G_i, k_i), (H_i, l_i) and groups F_i $(i=1, 2, \cdots)$ with some conditions, assume that the following sequence

$$\longrightarrow F_i \longrightarrow G_i \longrightarrow H_i \longrightarrow F_{i-1} \longrightarrow \cdots \longrightarrow H_2 \longrightarrow F_1 \longrightarrow G_1 \longrightarrow H_1 \longrightarrow 0$$

is exact. Then there exists a fibering (E, X, F, p), in the sense of Serre [4], such that the natural systems of E and X are isomorphic to (G_i, k_i) and (H_i, l_i) respectively and the homotopy exact sequence of this fiber space E is isomorphic to the given exact sequence.

In this paper we quote the notations, definitions and Postnikov's theorem from a report by P. J. Hilton [7], which we rewrite in § 1 of this paper without essential modifications.

§ 1. Preliminaries

1. The system

A non-decreasing sequence of $(p+1)$ terms of non-negative integers $\leqq r$ is called an $(r, p)^*$-sequence. If the terms are distinct it is called an (r, p)-sequence. If $\mathbf{a}$ is an (r, p)-sequence, we denote by $\mathbf{a}^{(i)}$ the $(r, p-1)$-sequence obtained by omitting the i-th term $(i=0, 1, \cdots, p)$, and by $\mathbf{a}^{-1}$ the $(r, r-p-1)$-sequence complementary to $\mathbf{a}$. We identify an $(r, 0)$-sequence with its single element. A function defined on $(r, p)^*$-sequences taking values in an additive abelian group G and taking the value 0 on non-(r, p)-sequence is called an (r, p)-function over G.

Given a multiplicative group G_1, let $K(G_1)$ be its cell-complex in the sense of Eilenberg-MacLane [3]. The face $A^{\mathbf{a}}$ of an r-cell A is an $(r-p-1)$-cell, obtained from A by deleting from its matrix representation the rows and columns whose numbers belong to $\mathbf{a}$. If θ_1 is an isomorphism $G_1 \approx H_1$, the mapping $\tilde{\theta}_1 : K(G_1) \to K(H_1)$ is given by $\tilde{\theta}_1 A = \| \theta_1(d_{ij}) \|$, where A is the matrix $\| d_{ij} \|$, $i, j=0,1,\cdots, r$.

A cell complex K is called a (G_1, σ)-complex if

1) to every r-cell A and every (r, p)-sequence $\mathbf{a}$ corresponds an $(r-p-1)$-cell $A^{\mathbf{a}}$,

2) $\sigma : K \to K(G_1)$ is dimension preserving and $\sigma(A^{\mathbf{a}}) = (\sigma A)^{\mathbf{a}}$,

3) the boundary of A is given by $\sum_{i=0}^{r} (-1)^i A^{(i)}$.

Let G_1 act as a group of left operators on G. Let C^r be an r-cochain of the (G_1, σ)-complex K over G. Define a coboundary ∇_σ by

$$\nabla_\sigma C^r(A) = \sigma_{01}(A) C^r(A^{(0)}) + \sum_{i=1}^{r+1} (-1)^i C^r(A^{(i)}),$$

for every $(r+1)$-cell A, $\sigma_{01}(A)$ being the element of the matrix $\sigma(A)$ with indices 0, 1.

We now construct the p-augmented complex of K over G with factor k, where k is a $(p+1)$-∇_σ-cocycle of K, and call the new complex K'. An r-cell of K' is to be a pair (A, φ) where A is an r-cell of K and φ is an (r, p)-function over G satisfying

$$\sigma_{a_0 a_1}(A)\varphi(\mathbf{a}^{(0)}) + \sum_{i=1}^{p+1} (-1)^i \varphi(\mathbf{a}^{(i)}) + k(A^{\mathbf{a}^{-1}}) = 0,$$

for every $(r, p+1)$-sequence $\mathbf{a} = (a_0, a_1, \cdots, a_{p+1})$

Given an (r, q)-sequence $\mathbf{a}$, and an $(r-q-1, p)^*$-sequence $\mathbf{b}$, we define the $(r, p)^*$-sequence $\mathbf{c} = \mathbf{a} \circ \mathbf{b}$ as follows: Take the sequence $0, 1, \cdots, r$. Remove the $(q+1)$ elements in $\mathbf{a}$ and renumber the remainder $0, 1, \cdots, r-q-1$. The sequence $\mathbf{b}$ picks out, perhaps with repetitions, $(p+1)$ elements in this sequence. These elements, with their original numbers, are the elements of **c**. Now define $(r\text{-}q\text{-}1,\ p)$-function $\varphi^{\mathbf{a}}$ by $\varphi^{\mathbf{a}}(\mathbf{b})=\varphi(\mathbf{a}\circ\mathbf{b})$ and define $(A,\varphi)^{\mathbf{a}}$ as $(A^{\mathbf{a}},\ \varphi^{\mathbf{a}})$. Finally, we define $\sigma(A,\ \varphi)=\sigma(A)$, so that K' is a $(G_1,\ \sigma)$-complex. If we identify the cell $(A,\ 0)$ with A, where dim $A\leqq p$, we have $K^{p-1}=K'^{p-1}$, $K^p\subset K'^p$.

Let us define a system $(G_1,\ G_2,\cdots,\ G_n,\cdots;\ k_1,\ k_2,\cdots,\ k_n,\cdots)$, which we call **G**. G_1 is a multiplicative group of left operators on the additive abelian groups G_i, $i\geqq 2$. Denote $K(G_1)$ by K_1 and let K_{i+1} be the $(i+1)$-augmentation of K_i over G_{i+1} with factor k_i where k_i is an $(i+2)$-∇_σ-cocycle of K_i over G_{i+1}, $i=1,\ 2,\cdots$. Then the sequence $(G_1,\ k_1,\ G_2,\ k_2,\cdots)$ is called a system, written $\mathbf{G}=(G_i\ ,\ k_i)$, the complex K_i is called the cell-complex of **G** and the sequence of complexes $K_1,\ K_2,\cdots$ is called the complex $K(\mathbf{G})$.

A mapping μ of the $(G_1,\ \sigma)$-complex K on the $(H_1,\ \sigma')$-complex L is called a θ_1-isomorphism if

1) θ_1 is an isomorphism $G_1\approx H_1$,
2) μ preserves dimension and is (1-1),
3) for every cell $A\in K$ and every sequence **a**, $\mu(A^{\mathbf{a}})=(\mu A)^{\mathbf{a}}$,
4) for every cell A, $\tilde{\theta}_1\sigma(A)=\sigma'(\mu A)$.

An isomorphic mapping, η, of the group G on the group H (on which H_1 acts as a group of left operators) is called a θ_1-isomorphism if $\eta(\alpha g)=(\theta_1\alpha)(\eta g)$, $\alpha\in G_1$, $g\in G$. If $\mu: K\to L$ and $\eta: G\to H$ are θ_1-isomorphisms, and if C^r is an r-cochain of L over H, we define μ^*C^r, an r-cochain of K over G, by

$$\mu^*C^r(A)=\eta^{-1}(C^r(\mu A)).$$

Now suppose L' is the p-augmentation of L over H with some factor l, and suppose further that there exists a p-cochain, d, of K over G, such that $k-\mu^*l=\nabla_\sigma d$. Define the θ_1-isomorphism V of K' on L' by $V(A,\ \varphi)=(\mu A,\ \psi)$, where the $(r,\ p)$-function ψ over H is given by

$$\psi(\mathbf{a})=\eta\{\varphi(\mathbf{a})+d(A^{\mathbf{a}^{-1}})\}.$$

It is called the η-prolongation of μ with cochain d.

We now say that two systems $\mathbf{G}=(G_i\ ,\ k_i)$ and $\mathbf{H}=(H_i\ ,\ l_i)$ are isomorphic if there is given for each i an isomorphism $\theta_i: G_i\approx H_i$ such that θ_i is a θ_1-isomorphism if $i>1$, and such that there exists for each i a θ_1-isomophism $\tilde{\theta}_i$ of K_i on L_i, $\tilde{\theta}_i$ being a θ_i-prolongation of $\tilde{\theta}_{i-1}$, where K_i and L_i are the cell-complexes of the systems **G** and **H** respectively.

2. The natural system of a space

We wish to associate a system $\mathbf{G}=(G_i\ ,\ k_i)$ with an arcwise-connected topological space X. The groups G_i will be the homotopy groups of the space. Put $K_1=$

$K(G_1)$. A 0-dimensional singular simplex in x_0 is normal. A singular simplex of arbitrary dimension is called 0-normal if all its 0-faces are normal. As shown in [3], there is a natural mapping, w_1, of the 0-normal singular simplexes of X into K_1, say $w_1: S_1(X) \to K_1$, such that $w_1(T^r)$ is an r-cell A_1^r of K_1. Moreover, the 1-cells and 2-cells of K_1 are covered by w_1. We make the inductive hypothesis that K_i is constructed, and that a definition of $(i-1)$-normal singular simplexes has been given, moreover, that, if $S_i(X)$ is the complex consisting of $(i-1)$-normal singular simplexes, there is a mapping w_i: $S_i(X) \to K_i$, such that $w_i(T^r)$ is an r-cell of K_i, and that the i-cells and $(i+1)$-cells of K_i are covered by w_i. With each i-cell A_i^i of K_i we associate an $(i-1)$-normal T_N^i such that $w_i(T_N^i) = A_i^i$ and call it the normal i-dimensional singular simplex of X corresponding to A_i. Then T^r is i-normal if it is $(i-1)$-normal and all its i-faces are normal. For each $(i+1)$-cell, A_i^{i+1}, of K_i, choose an i-normal T_S^{i+1} with $w_i(T_S^i) = A_i^{i+1}$, and call it the standard $(i+1)$-singular simplex of X corresponding to A_i^{i+1}. Let the boundary of an $(i+2)$-dimensional Euclidean ordered simplex, $\dot{\triangle}^{i+2}$, be mapped into X so that the map of the r-th face defines T_{rS}^{i+1} an $(i+1)$-dimensional standard singular simplex of X corresponding to the r-th face $A_i^{i+2(r)}$ of an $(i+2)$-cell A_i^{i+2} of K_i. Taking the base point in $\dot{\triangle}^{i+2}$ as the zero vertex, we can choose the map so that it represents an element of $\pi_{i+1}(X)$. Associating the cell A_i^{i+2} with this element defines the factor k_i, which turns out to be an $(i+2)$-ν_σ-cocycle.

Let T^r be i-normal. Every $(r, i+1)$-sequence $\mathbf{a}$ determines an $(i+1)$-face $T^r(\mathbf{a})$ of T^r, spanned by the vertices whose numbers belong to $\mathbf{a}$. Let $T^r(\mathbf{a})_S$ be a standard $(i+1)$-simplex with $w_i(T^r(\mathbf{a})_S) = w_i(T^r(\mathbf{a}))$. The simplexes $T^r(\mathbf{a})$, $T^r(\mathbf{a})_S$ are distinguished from each other by an element $\varphi_{i+1}^r(\mathbf{a})$ of $\pi_{i+1}(X)$. It turns out that the pair $(w_i(T^r), \varphi_{i+1}^r)$ is an r-cell of K_{i+1}. We put

$$w_{i+1}(T^r) = (w_i(T^r), \ \varphi_{i+1}^r).$$

Continuing the construction, we obtain the sequence of factors of the system $\mathbf{G}$ and we also define, in each dimension, the concept of a normal singular simplex. $\mathbf{G}$ is called the natural system of the space X. Its construction involves a certain arbitrariness, but all natural systems of a space are isomorphic.

A system $\mathbf{G}^n$ is called n-segment of the system $\mathbf{G}$ if

$$H_i = \begin{cases} G_i, & (i \leqq n), \\ 0, & (i > n), \end{cases} \qquad l_i = \begin{cases} k_i, & (i < n), \\ 0, & (i \geqq n), \end{cases}$$

where $\mathbf{G}^n = (H_i, l_i)$, $\mathbf{G} = (G_i, k_i)$. Two systems are called n-isomorphic $(1 \leqq n \leqq \infty)$ if their $(n-1)$-segments are isomorphic. The complexes $K_{n-1}(\mathbf{G})$, $K_{n-1}(\mathbf{H})$ must be isomorphic when $\mathbf{G}$ and $\mathbf{H}$ are n-isomorphic.

THEOREM (Postnikov). Every system is n-isomorphic $(1 \leqq n \leqq \infty)$ to the natural system of some n-dimensional CW-complex.

§ 2. The natural system of a space of loops I

In §§ 2-4, X will denote an arcwise-connected simply-connected topological space. We shall denote the i-th homotopy group $\pi_i(X, x_0)$ and the natural system by π_i and (π_i, k_i) respectively. Let K_i and e^r be the cell-complex of (π_i, k_i) and the unique r-cell of $K_1=K(\pi_1)$ respectively. For the (π_1, σ)-complex K_1, $\sigma : K_1 \to K(\pi_1)$ is the identity map, and $\sigma_{\dot{a}_0 a_1}(A_i^r)$ is the unit element of π_1 for each cell A_i^r. Let us define the normal 1-dimensional singular simplex of X corresponding to e^1 and the standard 2-dimensional singular simplex of X corresponding to e^2 by the collapsed simplexes. Consequently we have $k_1=0$.

Let $\hat{X}$ be the space of loops on X with the base point x_0. Hereafter each notation covered by $\wedge$ denotes the notation concerned with the space of loops. In particular, $\hat{e}^r$ is the r-dimensional matrix $\| d_{ij} \|$ where d_{ij} is the unit element of $\hat{\pi}_1$ for each i and j.

THEOREM 2.1 *$\hat{\pi}_1$ operates trivially on $\hat{\pi}_n$ $(n \geqq 2)$.*

PROOF. We denote by E^n and I the n-element and the unit interval. Let $[\hat{f}]$ be an element of $\hat{\pi}_1$, i.e.,

$$\hat{f}=\hat{f}(y) : E^1, \dot{E}^1 \to \hat{X}, \hat{x}_0,$$
$$\hat{f}(y)(s)=f(y, s) : E^1 \times I, (E^1 \times I)^{\cdot} \to X, x_0.$$

where s being the parameter of loop.

Let $\hat{\beta}$ be an element of $\hat{\pi}_n$ and $\hat{g}$ be its representation:

$$\hat{g}=\hat{g}(y_1, y_2, \cdots, y_n) : E^n, \dot{E}^n \to \hat{X}, \hat{x}_0,$$
$$\hat{g}(y_1, y_2, \cdots, y_n)(s)=g(y_1, y_2, \cdots, y_n, s) : E^n \times I, (E^n \times I)^{\cdot} \to X, x_0.$$

We denote by $f^*(\hat{\beta})$ the image of $\hat{\beta}$ by $[\hat{f}]$ and let $\hat{h}_0$ be a representation of $f^*(\hat{\beta})$ defined as follows:

$$\hat{h}_0=\hat{h}_0(y_1, y_2, \cdots, y_n) : E^n, \dot{E}^n \to \hat{X}, \hat{x}_0,$$
$$\hat{h}_0(y_1, y_2, \cdots, y_n)(s)=h_0(y_1, y_2, \cdots, y_n, s) : E^n \times I, (E^n \times I)^{\cdot} \to X, x_0$$
$$=\begin{cases} f(2z-1, 2s), & \left(1 \geqq z \geqq \frac{1}{2}, \quad 0 \leqq s \leqq \frac{1}{2}\right), \\ x_0, & \left(1 \geqq z \geqq \frac{1}{2}, \quad \frac{1}{2} \leqq s \leqq 1\right), \\ x_0, & \left(\frac{1}{2} \geqq z \geqq 0, \quad 0 \leqq s \leqq \frac{1}{2}\right), \\ g(2y_1, \cdots, 2y_n, 2s-1) & \left(\frac{1}{2} \geqq z \geqq 0, \quad \frac{1}{2} \leqq s \leqq 1\right), \end{cases}$$

where $z=\sqrt{y_1^2+y_2^2+\cdots+y_n^2}$.

Let $\hat{h}_1$ be another representation of $\hat{\beta}$ defined as follows:

$$\hat{h}_1=\hat{h}_1(y_1,\ y_2,\cdots,y_n):E^n,\ \dot{E}^n\to\hat{X},\ \hat{x}_0,$$

$$\hat{h}_1(y_1,\ y_2,\cdots,y_n)(s)=h_1(y_1,\ y_2,\ ,y_n,\ s):E^n\times I,(E^n\times I)^{\cdot}\to X,\ x_0$$

$$=\begin{cases} x_0, & \left(1\geqq z\geqq\frac{1}{2},\quad 0\leqq s\leqq 1\right),\\ \begin{cases} x_0, & \left(\frac{1}{2}\geqq z\geqq 0,\quad 0\leqq s\leqq\frac{1}{2}\right),\\ g(2y_1,\cdots 2y_n,\ 2s-1), & \left(\frac{1}{2}\geqq z\geqq 0,\quad \frac{1}{2}\leqq s\leqq 1\right).\end{cases}\end{cases}$$

Then we have the homotopy between $\hat{h}_0$ and $\hat{h}_1$ defined by the following expression:

$$\hat{h}_t=\hat{h}_t(y_1,\ y_2,\cdots,y_n):E^n,\ \dot{E}^n\to\hat{X},\ \hat{x}_0,$$

$$\hat{h}_t(y_1,\ y_2,\cdots,y_n)(s)=h_t(y_1,\ y_2,\cdots,y_n,\ s):E^n\times I,(E^n\times I)^{\cdot}\to X,\ x_0$$

$$=\begin{cases}\begin{cases} f(1-2(1-t)(1-z),\ 2s), & \left(1\geqq z\geqq\frac{1}{2},\quad 0\leqq s\leqq\frac{1}{2}\right),\\ x_0, & \left(1\geqq z\geqq\frac{1}{2},\quad \frac{1}{2}\leqq s\leqq 1\right),\end{cases}\\ \begin{cases} f(t,\ 2s), & \left(\frac{1}{2}\geqq z\geqq 0,\quad 0\leqq s\leqq\frac{1}{2}\right),\\ g(2y_1,\cdots,2y_n,\ 2s-1), & \left(\frac{1}{2}\geqq z\geqq 0,\quad \frac{1}{2}\leqq s\leqq 1\right).\end{cases}\end{cases}$$

Thus the proof of theorem 2.1 is complete.

Define $\rho_{r+1}:\triangle^{r+1}\to\triangle^r\times I$ by

$$\rho_{r+1}(y_1,\ y_2,\cdots,y_{r+1})=\begin{cases}(ly_1,\ ly_2,\cdots,ly_{r+1}) & (y_1+y_2+\cdots+y_r\geqq y_{r+1}),\\ (my_1,\ my_2,\cdots,my_{r+1}), & (y_1+y_2+\cdots+y_r\leqq y_{r+1}),\end{cases}$$

where $l=\dfrac{y_1+y_2+\cdots+y_{r+1}}{y_1+y_2+\cdots+y_r}$ and $m=\dfrac{y_1+y_2+\cdots+y_{r+1}}{y_{r+1}}$, and

$$\triangle^{r+1}=\left\{(y_1,\ y_2,\cdots,y_{r+1}):\begin{matrix}0\leqq y_i\leqq 1,\quad i=1,2,\cdots,\ r+1\\ 0\leqq y_1+y_2+\cdots+y_{r+1}\leqq 1\end{matrix}\right\}$$

is an $(r+1)$-dimensional Euclidean simplex and $\triangle^r$ is the r-face $\triangle^{r+1(r+1)}$ of $\triangle^{r+1}$ contained in the hyperplane $y_{r+1}=0$.

Let $\hat{T}^r:\triangle^r\to\hat{X}$ be an r-dimensional singular simplex of $\hat{X}$ and define ξ_{r+1} by $\xi_{r+1}(P,\ s)=\hat{T}^r(P)(s)$ where $P\in\triangle^r$. Define $\tau:\hat{T}^r\to T^{r+1}$ by

$$\tau\hat{T}^r=\xi_{r+1}\circ\rho_{r+1}:\triangle^{r+1}\to X.$$

We use the same notation τ for the induced map: $[\hat{T}^r]\to[T^{r+1}]$ subject to the condition that $[\hat{T}^r]$ is an element of $\hat{\pi}_r$. It is easily seen that

1) *$\tau:[\hat{T}^r]\to[T^{r+1}]$ is an isomorphism of $\hat{\pi}_r$ onto π_{r+1},*

2) $T^{r+1(i)}=\tau(\hat{T}^{r(i)})$, $i=0, 1,\cdots, r$,

3) $T^{r+1(r+1)}$ *is the collapsed simplex* (see notations of [8]).

Define φ^{r+1}_{i+1*}, $(r+1.\ i+1)$-function over π_{i+1}, by

$$\varphi^{r+1}_{i+1*}(a_0, a_1,\cdots, a_{i+1})=\begin{cases}\tau\psi^r_i(a_0, a_1,\cdots, a_i), & (a_{i+1}=r+1),\\ 0, & (a_{i+1}<r+1),\end{cases}$$

where ψ^r_i is an (r, i)-function over $\hat{\pi}_i$ and $(a_0, a_1,\cdots, a_{i+1})$ is an $(r+1, i+1)$-sequence. And denote by α this transformation from ψ^r_i to φ^{r+1}_{i+1*}. Let $\hat{A}^r_i=\|\psi^r_i(i, j)\|$ be a matrix representation of an r-cell $\hat{K}_1$, and define α on $\hat{K}_1$ as follows:

$$\alpha\hat{A}^r_1=(e^{r+1}\ \ \alpha\psi^r_1).$$

If α was defined on $\hat{K}_i$, we define α on $\hat{K}_{i+1}$ as follows:

$$\alpha\hat{A}^r_{i+1}=(\alpha\hat{A}^r_i, \alpha\psi^r_{i+1}),$$

where $\hat{A}^r_{i+1}=(\hat{A}^r_i, \psi^r_{i+1})$ is an r-cell of $\hat{K}_{i+1}$. Then we have the following lemma.

LEMMA 2.2. *α is an isomorphism (into).*

This is trivial.

LEMMA 2.3. *If $\mathbf{a}=(a_0, a_1,\cdots, a_{p-1}, a_p)$ is an (r, p)-sequence, $\mathbf{c}=\mathbf{a}^{(p)}$ and ψ^{r-1}_{i-1} is an $(r-1, i-1)$-function over $\hat{\pi}_{i-1}$, then we have*

$$(\alpha\psi^{r-1}_{i-1})^{\mathbf{a}^{-1}}=\begin{cases}\alpha(\psi^{r-1}_{i-1}{}^{\mathbf{c}^{-1}}), & (a_p=r),\\ 0, & (a_p<r).\end{cases}$$

PROOF. 1°. $a_p=r$: Let $\mathbf{b}=(b_0, b_1,\cdots, b_i)$ be a $(p, i)^*$-sequence.

If $b_i=p$, $(\alpha\psi^{r-1}_{i-1})^{\mathbf{a}^{-1}}(b_0, b_1,\cdots, b_{i-1}, p)=\tau(\psi^{r-1}_{i-1}{}^{\mathbf{c}^{-1}}(b_0, b_1,\cdots, b_{i-1}))$,

if $b_i<p$, since the last element of $\mathbf{a}^{-1}\circ\mathbf{b}$ is less than r, we have

$$(\alpha\psi^{r-1}_{i-1})^{\mathbf{a}^{-1}}(b_0, b_1,\cdots, b_i)=(\alpha\psi^{r-1}_{i-1})(\mathbf{a}^{-1}\circ\mathbf{b})=0.$$

Thus we have $(\alpha\psi^{r-1}_{i-1})^{\mathbf{a}^{-1}}=\alpha(\psi^{r-1}_{i-1}{}^{\mathbf{c}^{-1}})$,

2.° $a_p<r$: Since the last element of $\mathbf{a}^{-1}\circ\mathbf{b}$ is less than r for each sequence $\mathbf{b}$, we have

$$(\alpha\psi^{r-1}_{i-1})^{\mathbf{a}^{-1}}(\mathbf{b})=(\alpha\psi^{r-1}_{i-1})(\mathbf{a}^{-1}\circ\mathbf{b})=0.$$

LEMMA 2.4. *If* $\mathbf{a}$, $\mathbf{c}$ *are the same sequences given in lemma* 2.3, *then*

$$(\alpha\hat{A}^{r-1}_{i-2})^{\mathbf{a}^{-1}}=\begin{cases}\alpha(\hat{A}^{r-1}_{i-2}{}^{\mathbf{c}^{-1}}), & (a_p=r),\\ (\cdots((e^p, 0), 0)\cdots, 0), & (a_p<r).\end{cases}$$

PROOF. 1°. $a_p=r$: In the case $i=3$, for an $(r-1)$-cell $\hat{A}^{r-1}_1=\|\psi^{r-1}_1(i, j)\|$, we have

$$(\alpha\hat{A}^{r-1}_1)^{\mathbf{a}^{-1}}=(e^r, \alpha\psi^{r-1}_1)^{\mathbf{a}^{-1}}=(e^{r\,\mathbf{a}^{-1}}, (\alpha\psi^{r-1}_1)^{\mathbf{a}^{-1}})$$

$$= (e^p, \alpha(\psi_1^{r-1\,\mathbf{c}^{-1}})) = \alpha(\hat{A}_1^{r-1\,\mathbf{c}^{-1}}).$$

Making the inductive hypothesis as follows: $(\alpha \hat{A}_{i-3}^{r-1})^{\mathbf{a}^{-1}} = \alpha(\hat{A}_{i-3}^{r-1\,\mathbf{c}^{-1}})$, let us consider an $(r-1)$-cell $\hat{A}_{i-2}^{r-1} = (\hat{A}_{i-3}^{r-1}, \psi_{i-2}^{r-1})$. Then we have

$$(\alpha \hat{A}_{i-2}^{r-1})^{\mathbf{a}^{-1}}) = ((\alpha \hat{A}_{i-3}^{r-1})^{\mathbf{a}^{-1}}, (\alpha \psi_{i-2}^{r-1})^{\mathbf{a}^{-1}}) = (\alpha(\hat{A}_{i-3}^{r-1\,\mathbf{c}^{-1}}), \alpha(\psi_{i-2}^{r-1\,\mathbf{c}^{-1}}))$$

$$= \alpha(\hat{A}_{i-3}^{r-1\,\mathbf{c}^{-1}}, \psi_{i-2}^{r-1\,\mathbf{c}^{-1}}) = \alpha(\hat{A}_{i-2}^{r-1\,\mathbf{c}^{-1}}).$$

2°. $a_p < r$: $(\alpha \hat{A}_{i-2}^{r-1})^{\mathbf{a}^{-1}} = ((\alpha \hat{A}_{i-3}^{r-1})^{\mathbf{a}^{-1}}, 0) = \cdots = (\cdots((e^p, 0), 0)\cdots, 0)$.

THEOREM 2.5. *If* $\hat{A}_1^{r-1} = \|\psi_1^{r-1}(i, j)\|$ *is an* $(r\text{-}1)$*-cell of* $\hat{K}_1$, $A_{2*}^r = \alpha \hat{A}_1^{r-1} = (e^r, \alpha \psi_1^{r-1})$ *is an* r*-cell of* K_2.

PROOF. In the first place we have to remark that ψ_1^{r-1} has the following relations:

$$\psi_1^{r-1}(i, j) + \psi_1^{r-1}(j, l) = \psi_1^{r-1}(i, l), \quad (i, j, l = 0, 1, \ \ , r\text{-}1).$$

When $r=2$, since $\alpha \psi_1^{r-1}$ is a (2, 2)-function over π_2, $(e^2, \alpha \psi_1^1)$ is a 2-cell of K_2. Consider the case $r \geqq 3$. Let $\mathbf{a} = (a_0, a_1, a_2, a_3)$ be an $(r, 3)$-sequence.

1°. $a_3 < r$: It is trivial.

2°. $a_3 = r$: $\alpha\psi_1^{r-1}(\mathbf{a}^{(0)}) + \sum_{j=1}^{3}(-1)^{(j)} \alpha \psi_1^{r-1}(\mathbf{a}^{(j)})$

$$= \tau(\psi_1^{r-1}(a_1, a_2) - \psi_1^{r-1}(a_0, a_2) + \psi_1^{r-1}(a_0, a_1)) = \tau(0) = 0.$$

THEOREM 2.6. $w_2(\tau \hat{T}^r) = \alpha(\hat{w}_1 \hat{T}^r)$.

PROOF. By the definition of w_2, we have $w_2(\tau \hat{T}^r) = (e^{r+1}, \varphi_2^{r+1})$ where $\varphi_2^{r+1}(\mathbf{a}) = [(\tau \hat{T}^r)(\mathbf{a})]$ for each $(r+1, 2)$-sequence $\mathbf{a} = (a_0, a_1, a_2)$.

On the other hand, $\hat{w}_1 \hat{T}^r = \|\psi_1^r(i, j)\|$ where $\psi_1^r(\mathbf{b}) = [\hat{T}^r(\mathbf{b})]$ for each $(r, 1)$-sequence $\mathbf{b}$.

1°. $a_2 = r+1$: Put $\mathbf{b} = \mathbf{a}^{(2)}$. Since $\tau[\hat{T}^r(\mathbf{b})] = [(\tau \hat{T}^r)(\mathbf{a})]$, we have $\varphi_2^{r+1}(\mathbf{a}) = \tau \psi_1^r(\mathbf{b})$.

2°. $a_2 < r+1$: $\varphi_2^{r+1}(\mathbf{a}) = 0$.

Thus we have $\varphi_2^{r+1} = \alpha \psi_1^r$ and then $w_2(\tau \hat{T}^r) = (e^{r+1}, \alpha \psi_1^r) = \alpha \|\psi_1^r(i, j)\| = \alpha(\hat{w}_1 \hat{T}^r)$.

DEFINITION 2.7. We define $\hat{T}_N^1$ corresponding to $\hat{e}^1$ by the collapsed simplex and $\hat{T}_N^1$ corresponding to other 1-cells by the method given in § 1.

DEFINITION 2.8. When $\hat{T}_N^1$ is the normal 1-dimdnsional singular simplex of $\hat{X}$ corresponding to $\hat{A}_1^1$, then we define T_N^2 corresponding to $A_{2*}^2 = \alpha \hat{A}_1^1$ by $\tau \hat{T}_N^1$.

REMARK. All 2-cells of K_2 are covered by α and therefore the definition of the normal 2-dimensional singular simplex of X corresponding to each 2-cell of K_2 was given by the above definition.

LEMMA 2.9. T_N^2 *corresponding to* $A_2^2 = (e^2, 0)$ *is the collapsed simplex.*

This is easily seen by definitions 2.7 and 2.8.

DEFINITION 2.10. We define $\hat{T}^2_S$ corresponding to $\hat{e}^2$ by the collapsed simplex and $\hat{T}^2_S$ corresponding to other 2-cells by the method given in § 1.

DEFINITION 2.11. When $\hat{T}^2_S$ is the standard 2-dimensional singular simplex of $\hat{X}$ corresponding to $\hat{A}^2_1$ we define T^3_S corresponding to $A^3_{2*}=\alpha \hat{A}^2_1$ by $\tau \hat{T}^2_S$ and T^3_S corresponding to other 3-cells by the method given in § 1.

LEMMA 2.12. *T^3_S corresponding to $A^3_{2*}=(e^3, 0)$ is the collapsed simplex.*
This is easily seen by definitions 2.10 and 2.11.

THEOREM 2.13. $k_2(e^4, 0)=0$

PROOF. Since $(e^4, 0)^{(j)}=(e^3, 0)$, $j=0, 1, 2, 3, 4$, and T^3_S corresponding to $(e^3, 0)$ is the collapsed simplex, we have $k_2(e^4, 0)=0$.

THEOREM 2.14. $\hat{k}_1=\tau^{-1}\circ k_2\circ\alpha$.

PROOF. Let $\hat{A}^3_1$, $\hat{T}^2_{jS}$ and $\hat{f}$ be a 3-cell of $\hat{K}_1$, the standard 2-dimensional singular simplex of $\hat{X}$ corresponding to the j-th face $\hat{A}^{3(j)}_1$ of $\hat{A}^3_1$ and a representation of an element of $\hat{\pi}_2$ defined by $\hat{f}\,|\,\triangle^{3(j)}=\hat{T}^2_{jS}$, $j=0, 1\ 2, 3$ respectively. Then, by the definition, we have

$$\hat{k}_1 : \hat{A}^3_1 \to [\hat{f}] \in \hat{\pi}_2.$$

By lemma 2.4, we have

$$(\alpha \hat{A}^3_1)^{(j)}=\alpha(\hat{A}^{3(j)}_1), \qquad (j=0, 1, 2, 3),$$
$$(\alpha \hat{A}^3_1)^{(4)}=(e^3, 0).$$

Consequently T^3_S corresponding to $(\alpha \hat{A}^3_1)^{(j)}$ is $\tau \hat{T}^2_{jS}$ $(j=0, 1, 2, 3)$ or the collapsed simplex $(j=4)$. Therefore we have $k_2(\alpha \hat{A}^3_1)=[f]$ where $f: \dot{\triangle}^4 \to X$ is defied as follows:

$$f\,|\,\triangle^{4(j)}=\tau \hat{T}^2_{jS}, \qquad (j=0, 1, 2, 3),$$

and $\quad f\,|\,\triangle^{4(4)}$ is the constant map.

Thus we have

$$\hat{k}_1=\tau^{-1}\circ k_2\circ\alpha.$$

§ 3. The natural system of a space of loops II

In this section, in the process of definition of the natural systems of X and $\hat{X}$, we assume that

$$K_{n-1},\quad S_{n-1}(X),\quad w_{n-1},\quad T^{n-1}_N,\quad T^n_S,\quad k_{n-1},$$
$$\hat{K}_{n-2},\quad \hat{S}_{n-2}(\hat{X}),\quad \hat{w}_{n-2},\quad \hat{T}^{n-2}_N,\quad \hat{T}^{n-1}_S,\quad \hat{k}_{n-2},\quad (n=3, 4,\cdots, i)$$

are defined by the method of § 1 and satisfying the following five relations:

1) When $\hat{A}^{r-1}_{n-2}$ is an $(r-1)$-cell of $\hat{K}_{n-2}$, $A^r_{n-1*}=\alpha\hat{A}^{r-1}_{n-2}$ is an r-cell of K_{n-1}.

2) $w_{n-1}(\tau\hat{T}^r)=\alpha(\hat{w}_{n-2}\hat{T}^r)$.

3) $\hat{T}^{n-2}_N$ corresponding to $(\cdots((\hat{e}^{n-2}, 0), 0),\cdots 0)$ is the collapsed simplex. When $\hat{T}^{n-2}_N$ is the normal $(n-2)$-dimensional singular simplex of $\hat{X}$ corresponding to $\hat{A}^{n-2}_{n-2}$, T^{n-1}_N corresponding to $A^{n-1}_{n-1*}=\alpha\hat{A}^{n-2}_{n-2}$ is $\tau\hat{T}^{n-2}_N$.

4) $\hat{T}^{n-1}_S$ corresponding to $(\cdots((\hat{e}^{n-1}, 0), 0)\cdots, 0)$ is the collapsed simplex. When $\hat{T}^{n-1}_S$ is the standard $(n-1)$-dimensional singular simplex of $\hat{X}$ corresponding to $\hat{A}^{n-1}_{n-2}$, T^n_S corresponding to $A^n_{n-1*}=\alpha\hat{A}^{n-1}_{n-2}$ is $\tau\hat{T}^{n-1}_S$.

5) $\hat{k}_{n-2}=\tau^{-1}\circ k_{n-1}\circ\alpha$.

REMARK 1. T^{n-1}_N corresponding to $(\cdots((e^{n-1}, 0), 0)\cdots, 0)$ is the collapsed simplex.
T^n_S corresponding to $(\cdots((e^n, 0), 0)\cdots, 0)$ is the collapsed simplex.

REMARK 2. If $\hat{T}^r$ belongs to $S_{n-1}(\hat{X})$, $\tau\hat{T}^r$ belongs to $S_n(X)$.

REMARK 3. $\hat{k}_{n-2}(\cdots((\hat{e}^n, 0), 0)\cdots, 0)=0$,
$k_{n-1}(\cdots((e^{n+1}, 0), 0)\cdots, 0)=0$.

THEOREM 3.1. *If* $\hat{A}^{r-1}_{i-1}$ *is an* $(r-1)$-cell of $\hat{K}_{i-1}$, then $A^r_{i*}=\alpha\hat{A}^{r-1}_{i-1}$ *is an r-cell of* K_i.

PROOF. The case $i=2$ was proved in theorem 2.5. We assume that $i>2$. Put

$$\hat{A}^{r-1}_{i-1}=(\hat{A}^{r-1}_{i-2}, \psi^{r-1}_{i-1}), \qquad \varphi^r_{i*}=\alpha\,\psi^{r-1}_{i-1},$$

where $A^r_{i-1*}=\alpha\hat{A}^{r-1}_{i-2}$ is an r-cell of K_{i-1} by the inductive hypothesis. Let $\mathbf{a}=(a_0, a_1,\cdots, a_{i+1})$ be an $(r, i+1)$-sequence.

1°. $a_{i+1}<r$: Since $\varphi^r_{i*}(\mathbf{a}^{(j)})=0$ and $A^{r\;\mathbf{a}^{-1}}_{i-1*}=(\cdots((e^{i+1}, 0), 0)\cdots, 0)$, we have

$$\sum_{j=0}^{i+1}(-1)^j\varphi^r_{i*}(\mathbf{a}^{(j)})+k_{i-1}(A^{r\;\mathbf{a}^{-1}}_{i-1*})=0.$$

2°. $a_{i+1}=r$: Put $\mathbf{c}=\mathbf{a}^{(i+1)}$. Then we have

$$\sum_{j=0}^{i+1}(-1)^j\varphi^r_{i*}(\mathbf{a}^{(j)})+k_{i-1}(A^{r\;\mathbf{a}^{-1}}_{i-1*})=\sum_{j=0}^{i}(-1)^j\tau\,\psi^{r-1}_{i-1}(\mathbf{c}^{(j)})+k_{i-1}\alpha(\hat{A}^{r-1\;\mathbf{c}^{-1}}_{i-2})$$

$$=\tau(\sum_{j=0}^{i}(-1)^j\psi^{r-1}_{i-1}(\mathbf{c}^{(j)})+\hat{k}_{i-2}(\hat{A}^{r-1\;\mathbf{c}^{-1}}_{i-2}))=\tau(0)=0.$$

LEMMA 3.2. *If* T^r *is the constant map*: $\triangle^r\to x_0$,

$$w_i T^r=(\cdots((e^r, 0), 0)\cdots, 0).$$

PROOF. It is trivial in the cases $i=1, 2$. Assume that this lemma holds in the cases $i=1, 2,\cdots, j-1$. By the definition of w_j we have

$$w_j T^r=(w_{j-1}T^r, \varphi^r_j),$$

where $\varphi^r_j(\mathbf{a})=[T^r(\mathbf{a})-T^r(\mathbf{a})_S]$ for each (r, j)-sequence $\mathbf{a}$.

Since $T^r(\mathbf{a})$ is the collapsed simplex, we have $w_{j-1}T^r(\mathbf{a})=(\cdots((e^r, 0), 0)\cdots, 0)$ and

$T^r(\mathbf{a})_S$ is the collapsed simplex. Consequently φ_j^r is the constant map. Therefore we have

$$w_j T^r = (w_{j-1}T^r, 0) = \cdots = (\cdots((e^r, 0), 0)\cdots, 0).$$

THEOREM 3.3. $w_i(\tau \hat{T}^r) = \alpha(\hat{w}_{i-1}\hat{T}^r)$.

PROOF. By definition, we have

$$w_i(\tau\hat{T}^r) = (w_{i-1}(\tau\hat{T}^r), \varphi_i^{r+1}) = (\alpha(\hat{w}_{i-2}\hat{T}^r), \varphi_i^{r+1}),$$

where

$$\varphi_i^{r+1}(\mathbf{a}) = [\tau \hat{T}^r(\mathbf{a}) - (\tau\hat{T}^r(\mathbf{a}))_S]$$

and $\mathbf{a} = (a_0, a_1, \cdots, a_i)$ is an $(r+1, i)$-sequence. On the other hand,

$$\hat{w}_{i-1}\hat{T}^r = (\hat{w}_{i-2}\hat{T}^r, \psi_{i-1}^r),$$

where

$$\psi_{i-1}^r(\mathbf{b}) = [\hat{T}^r(\mathbf{b}) - (\hat{T}^r(\mathbf{b}))_S]$$

and $\mathbf{b}$ is an $(r, i-1)$-sequence.

1°. $a_i = r+1$, $\mathbf{b} = \mathbf{a}^{(i)}$: In this case it is easy to see that

$$\varphi_i^{r+1}(\mathbf{a}) = \tau\, \psi_{i-1}^r(\mathbf{b}).$$

2°. $a_i < r+1$: $\tau \hat{T}^r(\mathbf{a})$ is the collapsed simplex and therefore $(\tau \hat{T}^r(\mathbf{a}))_S$ is also the collapsed simplex. Consequently $\varphi_i^{r+1} = 0$. Thus we have

$$\varphi_i^{r+1} = \alpha\, \psi_{i-1}^r.$$

DEFINITION 3.4. We define $\hat{T}_N^{i-1}$ corresponding to $(\cdots((\hat{e}^{i-1}, 0), 0)\cdots, 0)$ by the collapsed simplex.

DEFINITION 3.5. When $\hat{T}_N^{i-1}$ is the normal $(i-1)$-dimensional singular simplex of $\hat{X}$ corresponding to $\hat{A}_{i-1}^{i-1}$, then we define T_N^i corresponding to $A_{i*}^i = \alpha \hat{A}_{i-1}^{i-1}$ by $\tau \hat{T}_N^{i-1}$.

LEMMA 3.6. *T_N^i corresponding to $(\cdots((e^i, 0), 0)\cdots, 0)$ is the collapsed simplex.* It is easily seen by definitions 3.4 and 3.5

DEFINITION 3.7. $\hat{T}_S^i$ corresponding to $(\cdots((\hat{e}^i, 0), 0)\cdots, 0)$ is defined by the collapsed simplex.

DEFINITION 3.8. When $\hat{T}_S^i$ is the standard i-dimensional singular simplex of $\hat{X}$ corresponding to $\hat{A}_{i-1}^i$, we define T_S^{i+1} corresponding to $A_{i*}^{i+1} = \alpha \hat{A}_{i-1}^i$ by $\tau \hat{T}_S^i$.

LEMMA 3.9. *T_S^{i+1} corresponding to $(\cdots((e^{i+1}, 0), 0)\cdots, 0)$ is the collapsed simplex.* This is easily seen by definition 3.7 and 3.8.

THEOREM 3.10. $k_i(\cdots((e^{i+2}, 0), 0)\cdots, 0) = 0$.

It is a trivial result of lemma 3.9.

THEOREM 3.11. $\hat{k}_{i-1}=\tau^{-1}\circ k_i\circ\alpha$.

PROOF. Let $\hat{A}^{i+1}_{i-1}$ be an $(i+1)$-cell of $\hat{K}_{i-1}$, $\hat{T}^i_{jS}$ be the standard i-dimensional singular simplex of $\hat{X}$ corresponding to the j-th face $\hat{A}^{i+1(j)}_{i-1}$ of $\hat{A}^{i+1}_{i-1}$ and $\hat{f}$ be a representation of an element of $\hat{\pi}_i$ defined by

$$\hat{f}\,|\,\triangle^{i+1(j)}=\hat{T}^i_{jS},\qquad (j=0,\ 1,\cdots,\ i+1).$$

Then we have

$$\hat{k}_{i-1}:\ A^{i+1}_{i-1}\to[\hat{f}].$$

By lemma 2.4, we have

$$(\alpha\,\hat{A}^{i+1}_{i-1})^{(j)}=\alpha(\hat{A}^{i+1(j)}_{i-1}),\qquad (j=0,\ 1,\cdots,i+1),$$
$$(\alpha\,\hat{A}^{i+1}_{i-1})^{(i+2)}=(\cdots((e^{i+1},\ 0),\ 0)\cdots,\ 0).$$

Consequently T^{i+1}_S corresponding to $(\alpha\,\hat{A}^{i+1}_{i-1})^{(j)}$ is $\tau\,\hat{T}^i_{jS}$ $(j=0,\ 1,\cdots,\ i+1)$ or the collapsed simplex $(j=i+2)$.

Therefore we have

$$k_i(\alpha\hat{A}^{i+1}_{i-1})=[f],$$

where $f:\ \triangle^{i+2}\to X$ is defined as follows:

$$f\,|\,\triangle^{i+2(j)}=\tau\,\hat{T}^i_{jS},\qquad (j=0,\ 1,\cdots,\ i+1),$$
$$f\,|\,\triangle^{i+2(i+2)}=\textit{the constant map}.$$

Namely $\qquad \hat{k}_{i-1}=\tau^{-1}\circ k_i\circ\alpha.$

We are now in a position to conclude the studies of §§2 and 3:

THEOREM 3.12. *Let X and $\hat{X}$ be an arcwise-connected simply-connected topological space and the space of loops on X respectively. Then we can construct the natural systems of X and $\hat{X}$ which satisfy the following relations for each $i\geqq 3$:*

1) *If $\hat{A}^{r-1}_{i-2}$ is an $(r-1)$-cell of $\hat{K}_{i-2}$, $A^r_{i-1*}=\alpha\,\hat{A}^{r-1}_{i-2}$ is an r-cell of K_{i-1}.*

2) $w_{i-1}(\tau\,\hat{T}^r)=\alpha(\hat{w}_{i-2}\hat{T}^r)$,

3) *$\hat{T}^{i-2}_N$ corresponding to $(\cdots((\hat{e}^{i-2},\ 0),\ 0)\cdots,\ 0)$ is the collapsed simplex.*

T^{i-1}_N corresponding to $(\cdots((e^{i-1},\ 0),\ 0)\cdots,\ 0)$ is the collapsed simplex.

If $\hat{T}^{i-2}_N$ is the normal $(i$-$2)$-dimensional singular simplex of $\hat{X}$ corresponding to $\hat{A}^{i-2}_{i-2}$, $\tau\,\hat{T}^{i-2}_N$ is the normal $(i$-$1)$-dimensional singnlar simplex of X corresponding to $A^{i-1}_{i-1*}=\alpha\hat{A}^{i-2}_{i-2}$.

4) *$\hat{T}^{i-1}_S$ corresponding to $(\cdots((\hat{e}^{i-1},\ 0),\ 0)\cdots,\ 0)$ is the collapsed simplex.*

T^i_S corresponding to $(\cdots((e^i,\ 0),\ 0)\cdots,\ 0)$ is the collapsed simplex.

If $\hat{T}^{i-1}_S$ is the standard $(i$-$1)$-dimensional singular simplex of $\hat{X}$ corresponding to $\hat{A}^{i-1}_{i-2}$, $\tau\,\hat{T}^{i-1}_S$ is the standard i-dimensional singular simplex of X corresponding to $A^i_{i-1*}=\alpha\,\hat{A}^{i-1}_{i-2}$.

5) $\hat{k}_{i-2}(\cdots((\hat{e}^i, \ 0), 0)\cdots, 0)=0,$
$k_{i-1}(\cdots((e^{i+1}, \ 0), 0)\cdots, 0)=0,$
$\hat{k}_{i-2}=\tau^{-1}\circ k_{i-1}\circ\alpha.$

§ 4. Isomorphism of natural systems

Let X and Y be two arcwise-connected, simply-connected topological spaces, and $\hat{X}$ and $\hat{Y}$ the spaces of loops on X and Y respectively. We make the assumption that the natural systems $\mathbf{G}=(G_i, k_i)$, $\hat{\mathbf{G}}=(\hat{G}_i, \hat{k}_i)$, $\mathbf{H}=(H_i, l_i)$ and $\hat{\mathbf{H}}=(\hat{H}_i, \hat{l}_i)$ of X, $\hat{X}$, Y and $\hat{Y}$, respectively, have been defined such that they satisfy the relations given in theorem 3.12.

Let K_i, $\hat{K}_i$, L_i and $\hat{L}_i$ be the cell-complexes of the above systems $\mathbf{G}$, $\hat{\mathbf{G}}$, $\mathbf{H}$ and $\hat{\mathbf{H}}$ respectively. Put $e^i=\|d_{mn}\|$, $E^i=\|D_{mn}\|$, $d_{mn}=1\in G_1$, $D_{mn}=1\in H_1$, $m=0, 1,\cdots, i$, $n=0, 1,\cdots, i$.

Assume that $\mathbf{G}$ and $\mathbf{H}$ are isomorphic, i. e., there exists for each i an isomorphism $\theta_i: G_i\simeq H_i$ such that θ_i is a θ_1-isomorphism if $i>1$, and such that there exists for each i a θ_1-isomorphism $\tilde{\theta}_i$ of K_i on L_i, $\tilde{\theta}_i$ being a θ_i-prolongation of $\tilde{\theta}_{i-1}$ with i-cochain d_{i-1}.

LEMMA 4.1. $\tilde{\theta}_{i+1}\alpha\hat{K}_i=\alpha\hat{L}_i.$

PROOF. In the first place, we intend to prove the case $i=1$. Let $\|\psi_1^r(i, j)\|$ be an r-cell of $\hat{K}_1$. Then we have

$$\alpha\|\psi_1^r(i, j)\|=(e^{r+1}, \alpha\psi_1^r)\in K_2,$$
$$\tilde{\theta}_2\alpha\|\psi_1^r(i, j)\|=(E^{r+1}, \Phi_2^{r+1})\in L_2,$$

where

$$\Phi_2^{r+1}(\mathbf{a})=\theta_2(\alpha\psi_1^r(\mathbf{a})+d_1(e^{r+1\mathbf{a}^{-1}}))$$

for each $(r+1, 2)$-sequence $\mathbf{a}=(a_0, a_1, a_2)$. Since $k_1=0$ and $l_1=0$, we have $d_1=0$ and

$$\Phi_2^{r+1}(\mathbf{a})=\theta_2\alpha\psi_1^r(\mathbf{a})=\begin{cases}\theta_2\tau\psi_1^r(\mathbf{a}^{(2)}), & (a_2=r+1),\\ 0, & (a_2<r+1).\end{cases}$$

Define an $(r, 1)$-function Ψ_1^r as follows:

$$\Psi_1^r(a_0, a_1)=\tau^{-1}\Phi_2^{r+1}(a_0, a_1, r+1)$$

for each $(r, 1)$-sequence (a_0, a_1). Then we have that $\|\Psi_1^r(i, j)\|_{i, j=0, 1,\cdots, r}$ is an r-cell of $\hat{L}_1$, and

$$\alpha\|\Psi_1^r(i, j)\|=(E^{r+1}, \Phi_2^{r+1})\in L_2.$$

Conversely, let $\|\Psi_1^r(i, j)\|$ be an r-cell of $\hat{L}_1$. Then we have

$$\alpha\|\Psi_1^r(i, j)\|=(E^{r+1}, \alpha\Psi_1^r)\in L_2.$$

Put $\varphi_2^{r+1}=\theta_2^{-1}\circ\alpha\circ\Psi_1^r$. Then we have

$$\varphi_2^{r+1}(\mathbf{a}) = \theta_2^{-1} \alpha \Psi_1^r(\mathbf{a}) = 0$$

for each $(r+1, 2)$-sequence $\mathbf{a} = (a_0, a_1, a_2)$ with $a_2 < r+1$.
Put

$$\psi_1^r(a_0, a_1) = \tau^{-1} \varphi_2^{r+1}(a_0, a_1, r+1).$$

Then

$$\tilde{\theta}_2 \alpha \| \psi_1^r(i, j) \| = (E^{r+1}, \alpha \Psi_1^r).$$

Secondly we make the inductive assumption that the following relation

$$\tilde{\theta}_i \alpha \hat{K}_{i-1} = \alpha \hat{L}_{r-1}$$

has been proved. Let $\hat{A}_i^r = (\hat{A}_{i-1}^r, \psi_i^r)$ be an r-cell of $\hat{K}_i$. Then $\tilde{\theta}_{i+1} \alpha \hat{A}_i^r = (\tilde{\theta}_i \alpha \hat{A}_{i-1}^r, \Phi_{i+1}^{r+1})$ is an $(r+1)$-cell of L_{i+1},
where

$$\Phi_{i+1}^{r+1}(\mathbf{a}) = \theta_{i+1}(\alpha \psi_i^r(\mathbf{a}) + d_i((\alpha \hat{A}_{i-1}^r)^{\mathbf{a}^{-1}}))$$

for each $(r+1, i+1)$-sequence $\mathbf{a} = (a_0, a_1, \cdots, a_{i+1})$.
In the case $a_{i+1} < r+1$,

$$\alpha \psi_i^r(\mathbf{a}) = 0,$$

$$d_i((\alpha \hat{A}_{i-1}^{r-1})^{\mathbf{a}^{-1}} = d_i(\cdots((e^{i+1}, 0), 0)\cdots, 0) = 0,$$

i. e., $$\Phi_{i+1}^{r+1}(a_0, a_1, \cdots, a_{i+1}) = 0 \quad \text{for} \quad a_{i+1} < r+1.$$

Put

$$\Psi_i^r(a_0, a_1, \cdots, a_i) = \tau^{-1} \Phi_{i+1}^{r+1}(a_0, a_1, \cdots, a_i, r+1).$$

By the inductive hypothesis there exists an r-cell $\hat{B}_{i-1}^r$ of $\hat{L}_{i-1}$ such that

$$\alpha \hat{B}_{i-1}^r = \tilde{\theta}_i \alpha \hat{A}_{i-1}^r.$$

Since $(\tilde{\theta}_i \alpha \hat{A}_{i-1}^r, \Phi_{i+1}^{r+1})$ is an $(r+1)$-cell of L_{i+1}, for each $(r+1, i+2)$-sequence $\mathbf{a}$ we have

$$\sum_{j=0}^{i+2} (-1)^j \Phi_{i+1}^{r+1}(\mathbf{a}^{(j)}) + l_i((\tilde{\theta}_i \alpha \hat{A}_{i-1}^r)^{\mathbf{a}^{-1}} = 0.$$

Especially in the case $\mathbf{a} = (a_0, a_1, \cdots, a_{i+1}, r+1)$ and $\mathbf{b} = (a_0, a_1, \cdots, a_{i+1})$, we have

$$\sum_{j=0}^{i+1} (-1)^j \Psi_i^r(\mathbf{b}^{(j)}) + \hat{l}_{i-1}(\hat{B}_{i-1}^{r\ \mathbf{b}^{-1}}) = 0.$$

Namely, $\tilde{\theta}_i \alpha \hat{K}_i \subset \alpha \hat{L}_i$.
Conversely, let $\hat{B}_i^r = (\hat{B}_{i-1}^r, \Psi_i^r)$ be an r-cell of $\hat{L}_i$. Then we have

$$\alpha \hat{B}_i^r = (\alpha \hat{B}_{i-1}^r, \alpha \Psi_i^r),$$

and by the inductive hypothesis there exists an r-cell $\hat{A}_{i-1}^r$ of $\hat{K}_{i-1}$ such that

$$\tilde{\theta}_i \alpha \hat{A}^r_{i-1} = \alpha \hat{B}^r_{i-1}$$

Let φ^{r+1}_{i+1} be an $(r+1, i+1)$-function defined as follows:

$$\varphi^{r+1}_{i+1}(\mathbf{a}) = \theta^{-1}_{i+1}\alpha \Psi^r_i(\mathbf{a}) - d_i((\alpha \hat{A}^r_{i-1})^{\mathbf{a}^{-1}})$$

for each $(r+1, i+1)$-sequence $\mathbf{a}$. Then

$$\varphi^{r+1}_{i+1}(a_0, a_1, \cdots, a_{i+1}) = 0 \quad \text{for} \quad a_{i+1} < r+1.$$

Define ψ^r_i as follows:

$$\psi^r_i(a_0, a_1, \cdots, a_i) = \tau^{-1}\varphi^{r+1}_{i+1}(a_0, a_1, \cdots, a_i, r+1)$$

for each (r, i)-sequence $(a_0, a_1, \cdots, a_i)$.
Then we have

$$\alpha \Psi^r_i(\mathbf{a}) = \theta_{i+1}(\alpha \psi^r_i(\mathbf{a}) + d_i((\alpha \hat{A}^r_{i-1})^{\mathbf{a}^{-1}}),$$

and if $(\hat{A}^r_{i-1}\ \psi^r_i)$ is an r-cell of $\hat{K}_i$ we have

$$\tilde{\theta}_{i+1} \alpha(\hat{A}^r_{i-1}, \psi^r_i) = (\tilde{\theta}_i \alpha \hat{A}^r_{i-1}, \alpha \Psi^r_i) = \alpha \hat{B}^r_i.$$

Therefore we can complete this proof by showing that $(\hat{A}^r_{i-1}, \psi^r_i)$ is an r-cell of $\hat{K}_i$. Since $(\alpha \hat{B}^r_{i-1}, \alpha \Psi^r_i)$ is an $(r+1)$-cell of L_{i+1},

$$\sum_{j=0}^{i+2}(-1)^j \Psi^r_i(\mathbf{a}^{(j)}) + l_i((\alpha\hat{B}^r_{i-1})^{\mathbf{a}^{-1}} = 0$$

for each $(r+1, i+2)$-sequence $\mathbf{a} = (a_0, a_1, \cdots, a_{i+2})$. Consequently,

$$\sum_{j=0}^{i+2}(-1)^j \theta_{i+1}\alpha \psi^r_i(\mathbf{a}^{(j)}) + \sum_{j=0}^{i+2}(-1)^j\theta_{i+1} d_i((\alpha \hat{A}^r_{i-1})^{(\mathbf{a}^{(j)})^{-1}}) + l_i \tilde{\theta}_i(\alpha \hat{A}^r_{i-1})^{\mathbf{a}^{-1}} = 0.$$

Especially, in the case $a_{i+2} = r+1$, $\mathbf{b} = (a_0, a_1, \cdots, a_{i+1})$, we have

$$\sum_{j=0}^{i+1}(-1)^j\tau \psi^r_i(\mathbf{b}^{(j)}) + \sum_{j=0}^{i+2}(-1)^j d_i(((\alpha \hat{A}^r_{i-1})^{\mathbf{a}^{-1}})^{(j)}) + \theta^{-1}_{i+1} l_i \tilde{\theta}_i \alpha(\hat{A}^r_{i-1}{}^{\mathbf{b}^{-1}}) = 0,$$

$$\sum_{j=0}^{i+1}(-1)^j \psi^r_i(\mathbf{b}^{(j)}) + \tau^{-1}(pd_i + \theta^{-1}_{i+1} l_i \tilde{\theta}_i)\alpha(\hat{A}^r_{i-1}{}^{\mathbf{b}^{-1}}) = 0.$$

Substituting k_i for $pd_i + \theta^{-1}_{i+1} l_i \tilde{\theta}_i$ and $\hat{k}_{i-1}$ for $\tau^{-1} \circ k_i \circ \alpha$ we obtain

$$\sum_{j=0}^{i+1}(-1)^j \psi^r_i(\mathbf{b}^{(j)}) + \hat{k}_{i-1}(\hat{A}^r_{i-1}{}^{\mathbf{b}^{-1}}) = 0.$$

Thus $(\hat{A}^r_{i-1}, \psi^r_i)$ is an r-cell of $\hat{K}_i$.

THEOREM 4.2. *On the assumptions mentioned above,* $\hat{\mathbf{G}} \approx \hat{\mathbf{H}}$ *That is to say, there exists for each* i *an isomorphism* $\eta_i : \hat{G}_i \approx \hat{H}_i$ *such that* η_i *is an* η_1-*isomorphism if* $i > 1$, *and such that there exists for each* i *an* η_1-*isomorphism* $\tilde{\eta}_i$ *of* $\hat{K}_i$ *on* $\hat{L}_i$, $\tilde{\eta}_i$ *being an* η_i-*prolongation of* $\tilde{\eta}_{i-1}$ *with some* i-*cochain* $\hat{d}_{i-1}$.

PROOF. Put

$$\eta_i=\tau^{-1}\circ\theta_{i+1}\circ\tau,\qquad \tilde{\eta}_i=\alpha^{-1}\circ\tilde{\theta}_{i+1}\circ\alpha \quad\text{and}\quad \hat{d}_{i-1}=\tau^{-1}\circ d_i\circ\alpha.$$

By lemma 4.1, it is justified that $\tilde{\eta}_i$ is a mapping from $\hat{K}_i$ onto $\hat{L}_i$. It is easy to see that

η_i is an isomorphism from $\hat{G}_i$ onto $\hat{H}_i$,
$\tilde{\eta}_i$ preserves dimension and is $(1-1)$,
$\hat{d}_{i-1}$ is an i-cochain of $\hat{K}_{i-1}$ over $\hat{G}_i$.

Since $\hat{G}_1$ and $\hat{H}_1$ operate trivially on $\hat{G}_i$ and $\hat{H}_i$ $(i\geqq 2)$, respectively, η_i is an η_1-isomorphism.

For each cell $\|\psi_1^r(i, j)\|$ of $\hat{K}_1$, we have

$$\tilde{\eta}_1\|\psi_1^r(i, j)\|=\alpha^{-1}\tilde{\theta}_2\alpha\|\psi_1^r(i, j)\|=\|\tau^{-1}\theta_2\tau(\psi_1^r(i, j))\|=\|\eta_1\psi_1^r(i, j)\|.$$

Put $\hat{B}_i^r=\tilde{\eta}_i\hat{A}_i^r$, i. e., $\alpha\hat{B}_i^r=\tilde{\theta}_{i+1}\alpha\hat{A}_i^r$.

Let $\mathbf{a}=(a_0, a_1,\cdots, a_j)$ be an (r, j)-sequence and we denote it by $\mathbf{b}$ when we consider it as an $(r+1, j)$-sequence, then

$$\alpha(\hat{B}_i^{r\mathbf{a}})=(\alpha\hat{B}_i^r)^{\mathbf{b}}=(\tilde{\theta}_{i+1}\alpha\hat{A}_i^r)^{\mathbf{b}}=\tilde{\theta}_{i+1}\alpha(\hat{A}_i^{r\mathbf{a}}),$$

i. e., $$(\tilde{\eta}_i\hat{A}_i^r)^{\mathbf{a}}=\tilde{\eta}_i(\hat{A}_i^{r\mathbf{a}}).$$

Let $\hat{K}_1=K(\hat{G}_1)$ and $\hat{L}_1=K(\hat{H}_1)$ be $(\hat{G}_1, \sigma)$-complex and $(\hat{H}_1, \sigma')$-complex, respectively. Defining $\hat{A}_i^r$ by $(\cdots((\hat{A}_1^r, \psi_2^r), \psi_3^r)\cdots, \psi_i^r)$, we have

$$\sigma(\hat{A}_i^r)=\sigma(\hat{A}_1^r)=\hat{A}_1^r$$

and then $\tilde{\eta}_1\sigma(\hat{A}_i^r)=\tilde{\eta}_1\hat{A}_1^r$.

On the other hand by the definition of $\tilde{\eta}_i$,

$$\sigma'(\tilde{\eta}_i\hat{A}_i^r)=\sigma'(\alpha^{-1}\tilde{\theta}_{i+1}\alpha\hat{A}_i^r)=\sigma'(\alpha^{-1}\tilde{\theta}_2\alpha\hat{A}_1^r)=\alpha^{-1}\tilde{\theta}_2\alpha\hat{A}_1^r=\tilde{\eta}_1\hat{A}_1^r.$$

Thus, we have

$$\tilde{\eta}_1\sigma(\hat{A}_i^r)=\sigma'(\tilde{\eta}_i\hat{A}_i^r)$$

for each r-cell $\hat{A}_i^r$ of $\hat{K}_i$.

Let us cirsider the property of $\hat{d}_{i-1}$:

$$\begin{aligned}\nabla\hat{d}_{i-1}(\hat{A}_{i-1}^{i+1})&=\sum_{j=0}^{i+1}(-1)^j\hat{d}_{i-1}(\hat{A}_{i-1}^{i+1(j)})=\tau^{-1}(\nabla d_i(\alpha\hat{A}_{i-1}^{i+1}))\\&=\tau^{-1}(k_i\alpha\hat{A}_{i-1}^{i+1}-\theta_{i+1}^{-1}l_i\tilde{\theta}_i(\alpha\hat{A}_{i-1}^{i+1}))\\&=\hat{k}_{i-1}\hat{A}_{i-1}^{i+1}-\eta_{i+1}^{-1}\hat{l}_{i-1}\tilde{\eta}_i(\hat{A}_{i-1}^{i+1}).\end{aligned}$$

To finish the proof, we must prove that

$$\tilde{\eta}_i\hat{A}_i^r=(\tilde{\eta}_{i-1}\hat{A}_{i-1}^r, \Psi_i^r)$$

for each r-cell $\hat{A}_i^r=(\hat{A}_{i-1}^r, \psi_i^r)$ of $\hat{K}_i$, where Ψ_i^r is defined as follows:

$$\Psi_i^r(\mathbf{a}) = \eta_i(\psi_i^r(\mathbf{a}) + \hat{d}_{i-1}(\hat{A}_{i-1}^{r\ \mathbf{a}^{-1}}))$$

for each (r, i)-sequence $\mathbf{a}$.
Since $\tilde{\eta}_i = \alpha^{-1} \circ \tilde{\theta}_{i+1} \circ \alpha$,

$$\tilde{\eta}_i \hat{A}_i^r = \alpha^{-1}(\tilde{\theta}_i \alpha \hat{A}_{i-1}^r, \Phi_{i+1}^{r+1}) = (\alpha^{-1} \tilde{\theta}_i \alpha \hat{A}_{i-1}^r, \alpha^{-1} \Phi_{i+1}^{r+1})$$
$$= (\tilde{\eta}_{i-1} \hat{A}_{i-1}^r, \alpha^{-1} \Phi_{i+1}^{r+1}),$$

where $\Phi_{i+1}^{r+1}(\mathbf{a}) = \theta_{i+1}(\alpha \psi_i^r(\mathbf{a}) + d_i((\alpha \hat{A}_{i-1}^r)^{\mathbf{a}^{-1}}))$

for each $(r+1, i+1)$-sequence $\mathbf{a}$.
Let $\mathbf{b} = (b_0, b_1 \cdots, b_i)$ be an (r, i)-sequence, and let $\mathbf{a}$ be an $(r+1, i+1)$-sequence defined by $\mathbf{a} = (b_0, b_1, \cdots, b_i, r+1)$. Then we have

$$(\alpha^{-1} \Phi_{i+1}^{r+1})(\mathbf{b}) = \tau^{-1}(\Phi_{i+1}^{r+1}(\mathbf{a}))$$
$$= \tau^{-1} \theta_{i+1}(\alpha \psi_i^r(\mathbf{a}) + d_i \alpha (\hat{A}_{i-1}^{r\ \mathbf{b}^{-1}})) = \eta_i \psi_i^r(\mathbf{b}) + \eta_i \hat{d}_{i-1}(\hat{A}_{i-1}^{r\ \mathbf{b}^{-1}}).$$

Thus $\tilde{\eta}_i$ is an η_i-prolongation of $\tilde{\eta}_{i-1}$.

REMARK. We can extend theorem 4.2 to the following form and its proof is very similar to that of theorem 4.2:

Let (G_i, k_i), (G_i', k_i'), (H_i, l_i) and (H_i', l_i') be systems, not necessarily being the natural systems of spaces, and assume that

$G_1 = 0$ and $H_1 = 0$,

G_1' operates trivially on G_i' $(i \geqq 2)$, H_1' operates trivially on H_i' $(i \geqq 2)$, there exists an isomorphism τ such that $\tau: G'_{i-1} \approx G_i$ $(i \geqq 2)$ and

$$\tau: H'_{i-1} \approx H_i \ (i \geqq 2),$$

$k'_{i-1} = \tau^{-1} \circ k_i \circ \alpha$ and $l'_{i-1} = \tau^{-1} \circ l_i \circ \alpha$ where α is the isomorphism defined in § 2,

$k_i(\cdots((e^{i+2}, 0), 0) \cdots, 0) = 0$ and $l_i(\cdots((E^{i+2}, 0), 0) \cdots, 0) = 0$ where e^{i+2} and E^{i+2} are the matrices defined at the beginning of this section,

(G_i, k_i) and (H_i, l_i) are isomorphic.

Then we have that (G_i', k_i') and (H_i', l_i') are isomorphic.

THEOREM 4.3. *Let (G_i', k_i') be a system such that $k'_i(\cdots((e'^{i+2}, 0), 0) \cdots, 0) = 0$. Then there exists a space of loops whose natural system is isomorphic to (G_i', k_i') if and only if G_1' operates trivially on G_i' $(i \geqq 2)$.*

PROOF. The condition is evidently necessary. To prove the sufficiency, let G_1' be a multiplicative group of left operators which operate trivially on G_i' $(i \geqq 2)$. Define a system (G_i, k_i) as follows:

$G_1 = 0$,

there exists an isomorphism $\tau: G'_{i-1} \approx G_i$ $(i \geqq 2)$,

$k_i=\tau\circ k'_{i-1}\circ\alpha^{-1}$ where α is the isomorphism defined in § 2,
$k_i=0$ on the complementary of the image of α

Then, by Postnikov's theorem (see § 1), there exists a topological space X whose natural system is isomorphic to (G_i, k_i). Let $\hat{X}$ be the space of loops on X, then it is easy to see that the natural system of $\hat{X}$ and (G_i', k_i') are isomorphic.

§ 5. Fibering

1. By theorem 4.3, we have the following theorem:

THEOREM 5.1. *For two systems* $\mathbf{G}'=(G_i', k_i')$ *and* $\mathbf{G}=(G_i, k_i)$ *given in theorem* 4.3 *and in its proof, there exists a fibering* (E, X, F, p) *such that the natural systems of* X *and* F *are isomorphic to* $\mathbf{G}$ *and* $\mathbf{G}'$ *respectively.*

2. Let G_1 and H_1 be multiplicative groups of left operators on abelian groups G_i and H_i $(i\geqq 2)$, respectively, and assume that the following sequence is exact:

$$\longrightarrow F_i \xrightarrow{f_i} G_i \xrightarrow{g_i} H_i \xrightarrow{h_i} F_{i-1} \longrightarrow \cdots \longrightarrow H_2 \xrightarrow{h_2} F_1 \xrightarrow{f_1} G_1 \xrightarrow{g_1} H_1 \xrightarrow{h_1} 0.$$

We now consider two systems $\mathbf{G}=(G_i, k_i)$ and $\mathbf{H}=(H_i, l_i)$ and denote their cell-complexes by K_i and L_i. In this section we assume that the following relations hold:

1) g_i is an onto-homomorphism: $G_i\to H_i$,

2) $g_i(x_1x_i)=g_1(x_1)g_i(x_i)$ for all elements $x_1\in G_1$ and $x_i\in G_i$,

3) $g_{i+1}\circ k_i=l_i\circ \bar{g}_i$, defining $\bar{g}_1$: $K_1\to L_1$ by $\bar{g}_1\,\|d_{ij}\|=\|g_1 d_{ij}\|$ and $\bar{g}_i$ on K_i by $\bar{g}_i A_i^r=(\bar{g}_{i-1}A_{i-1}^r, g_i\circ\varphi_i^r)$ for each r-cell $A_i^r=(A_{i-1}^r, \varphi_i^r)$ of K_i, inductively.

LEMMA 5.2. *We have*

$$g_1(\sigma_{a_0a_1}(A_j^r))=\sigma_{a_0a_1}(\bar{g}_j A_j^r)$$

for each r-*cell* $A_j^r=(\cdots((A_1^r, \varphi_2^r), \varphi_3^r)\cdots, \varphi_j^r)$ *of* K_j *and for each* (r, i)-*sequence* $\mathbf{a}=(a_0, a_1,\cdots, a_i)$.

PROOF. By definitions

$$g_1(\sigma_{a_0a_1}(A_j^r))=g_1(\sigma_{a_0a_1}(A_1^r))=\sigma_{a_0a_1}(\bar{g}_1 A_1^r).$$

On the other hand

$$\sigma_{a_0a_1}(\bar{g}_j A_j^r)=\sigma_{a_0a_1}(\cdots((\bar{g}_1A_1^r, g_2\circ\varphi_2^r), g_3\circ\varphi_3^r)\cdots, g_j\circ\varphi_j^r)=\sigma_{a_0a_1}(\bar{g}_1 A_1^r).$$

LEMMA 5.3. *If* $\bar{g}_{i-1}(K_{i-1})\subset L_{i-1}$, $\bar{g}_i(K_i)\subset L_i$.

PROOF. Let $A_i^r=(A_{i-1}^r, \varphi_i^r)$ be an r-cell of K_i, then we have

$$\sigma_{a_0a_1}(A_{i-1}^r)\varphi_i^r(\mathbf{a}^{(0)})+\sum_{j=1}^{i+1}(-1)^j\varphi_i^r(\mathbf{a}^{(j)})+k_{i-1}(A_{i-1}^{r\,\mathbf{a}^{-1}})=0$$

for each $(r, i+1)$-sequence $\mathbf{a}=(a_0, a_1,\cdots, a_{i+1})$.

Transforming this expression by g_i we have

$$\sigma_{a_0 a_1}(\bar{g}_{i-1} A^r_{i-1}) \cdot g_i \varphi^r_i(\mathbf{a}^{(0)}) + \sum_{j=0}^{i+1} (-1)^j g_i \varphi^r_i(\mathbf{a}^{(j)}) + g_i k_{i-1}(A^{r\ \mathbf{a}^{-1}}_{i-1}) = 0.$$

But

$$g_i k_{i-1}(A^{r\ \mathbf{a}^{-1}}_{i-1}) = l_{i-1} \bar{g}_{i-1}(A^{r\ \mathbf{a}^{-1}}_{i-1}) = l_{i-1}((\bar{g}_{i-1} A^r_{i-1})^{\mathbf{a}^{-1}}).$$

Thus we have that $\bar{g}_i A^r_i$ is contained in L_i.

THEOREM 5.4. *There exists a fibering (E, X, F, p), in the sense of Serre* [4], *such that the natural systems of E and X are isomorphic to* **G** *and* **H** *respectively and its homotopy exact sequence is the above given exact sequence.*

PROOF. Define a mapping $\bar{g}: K(\mathbf{G}) \to K(\mathbf{H})$ by $\bar{g}(A^r_1, A^r_2, \cdots) = (\bar{g}_1 A^r_1, \bar{g}_2 A^r_2, \cdots)$. By Postnikov's theorem we have two spaces Y and X whose natural systems are isomorphic to **G** and **H** respectively. Let $g^*: Y \to X$ be the barycentric extension of $\bar{g}$. Construct a fibering (E, X, F, p) by the method of Cartan-Serre [6] as follows:

Let E be a space of pairs $(y, \omega(t))$ where $y \in Y$ and $\omega(t)$ is a path of X such that $\omega(0) = g^*(y)$. Y is a deformation retract of E and therefore all natural systems of E are isomorphic to **G**.

The map $p: E \to X$ such that $p(y, \omega(t)) = \omega(1)$ makes E a fiber space with base space X and fiber F which is a subspace of E consisting of pairs $(y, \omega(t))$ such that $\omega(1)$ is a fixed point of X.

In the following diagram which consists of the given exact sequence and the homotopy exact sequence of this fiber space E:

$$\begin{array}{ccccccccc}
\pi_{i+1}(E) & \xrightarrow{j'} & \pi_{i+1}(X) & \xrightarrow{\partial} & \pi_i(F) & \xrightarrow{j} & \pi_i(E) & \xrightarrow{j'} & \pi_i(X) \\
\downarrow \rho' & & \downarrow \rho'' & & & & \downarrow \rho' & & \downarrow \rho'' \\
G_{i+1} & \xrightarrow[g_{i+1}]{} & H_{i+1} & \xrightarrow[h_{i+1}]{} & F_i & \xrightarrow[f_i]{} & G_i & \xrightarrow[g_i]{} & H_i
\end{array}$$

it is easily verified that $\rho'' \circ j' = g_i \circ \rho'$. And then we have, for each $a \in \pi_i(F)$,

$$g_i \rho' j(a) = \rho'' j' j(a) = 0,$$

that is to say,

$$\rho' j(a) \in g_i^{-1}(0) = \textit{image of } f_i.$$

Since g_{i+1} maps G_{i+1} onto H_{i+1}, f_i is an isomorphism and therefore we can define

$$\rho: \pi_i(F) \to F_i \text{ by } \rho(a) = f_i^{-1} \rho' j(a).$$

And then we have $f_i \circ \rho = \rho' \circ j$.

On the other hand, since $\rho \circ \partial = f_i^{-1} \circ \rho' \circ j \circ \partial = 0$ and $h_{i+1} \circ \rho'' = 0$, we have

$$\rho \circ \partial = h_{i+1} \circ \rho''.$$

By the five lemma, we can conclude that $\rho: \pi_i(F) \to F_i$ is an isomorphism (onto). Thus the proof is complete.

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