On the $b_p^{k,j}$, Coefficient of a Certain Symmetric Function.

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§ 1. Introduction.

A. Borel and J.-P. Serre have studied the cohomology mod a prime p of Lie group in the paper [1], using the cyclic reduced power defined by N. E. Steenrod.

This investigation has required the value of $b_p^{k,j}$ which appears in this paper. The definition of $b_p^{k,j}$ is as follows: Let $\sum x_1^p \cdots x_k^p x_{k+1} \cdots x_{j-k(p-1)}$ be a homogeneous symmetric polynomial in variables $x_1, x_2, \cdots, x_n$ of degree j where p, k and j are positive integers.

This polynomial is expressed by a polynomial $B_p^{k,j}$ in $\sigma_1, \cdots, \sigma_j$ where $\sigma_i (i=1, 2, \cdots, j)$ is fundamental symmetric expression in $x_1, \cdots, x_n$ of degree i, and we write the coefficient of σ_j $b_p^{k,j}$.

Regarding $b_p^{k,j}$, it is not necessary to know its value but it is sufficient to calculate the value with respect to mod. p.

For instance it is easily seen that $b_3^{1,j} \equiv j$ (*mod.* 3), but the general formula of $b_p^{k,j}$ has not been given. The particular case in which $p=2$ has been treated by Wu Wen Tsün and the result yields

$$b_2^{k,j} \equiv \binom{j-k-1}{k} \quad (mod.\ 2). \tag{1}$$

In addition to this formula, he has proved that

$$B_2^{k,j} \equiv \binom{j-k-1}{k}\sigma_j + \binom{j-k-2}{k-1}\sigma_1\sigma_{j-1} + \cdots + \binom{j-2k}{1}\sigma_{k-1}\sigma_{j-k+1} + \sigma_k\sigma_{j-k} \quad (mod.\ 2)$$

and also in the case $k=1$, $b_p^{1,j} \equiv j$ (*mod.* p) has been seen.

Now it is the purpose of this paper to show the following result about $b_p^{k,j}$:

$$b_p^{k,j} \equiv \binom{j-k(p-1)-1}{k} \quad (mod.\ p). \tag{2}$$

(1) is included in (2) in the particular case $p=2$.

§ 2. Reduction formula of $b_p^{k,j}$.

First of all we must show that there exists a following reduction formula

$$(3) \qquad \binom{n}{k} = \sum_{l=0}^{k} \binom{n-j+l(p-1)}{k-l} b_p^{l,j}$$

Proof. Let $P(x_1, \cdots, x_k)$ be a polynomial in variables $x_1, x_2, \cdots, x_n$ such as

$$P(x_1, \cdots, x_k) = (1-x_1^p) \cdots (1-x_k^p)(1-x_{k+1}) \cdots (1-x_n).$$

We write this polynomial in the form

$$\begin{aligned} P(x_1, \cdots, x_k) &= (1-x_1) \cdots (1-x_k)(1-x_{k+1}) \cdots (1-x_n) \\ &\quad \times (1+x_1+ \cdots +x_1^{p-1}) \times \cdots \times (1+x_k+ \cdots +x_k^{p-1}) \\ &= (1-\sigma_1+\sigma_2- \cdots +(-1)^j\sigma_j+ \cdots +(-1)^n\sigma_n) \\ &\quad \times (1+x_1+ \cdots +x_1^{p-1}) \times \cdots \times (1+x_k+ \cdots +x_k^{p-1}) \end{aligned}$$

where $\sigma_i (i=1, 2, \cdots, n)$ is fundamental symmetric expression in $x_1, x_2, \cdots, x_n$ of degree i.

Consider $\sum P(x_1, \cdots, x_k)$ where this summation runs through all combinations $(x_1, \cdots, x_k)$ taken from $x_1, \cdots, x_n$.

This polynomial is not only expressed by $\sigma_1, \cdots, \sigma_n$ but written in the form

$$(-1)^j \binom{n}{k} \sigma_j + (\textit{polynomial in } \sigma_1, \cdots, \sigma_n).$$

On the other hand, trying to develope $P(x_1, \cdots, x_k)$ into another form, we obtain another homogeneous symmetric polynomial of degree j, *i. e.*

$$\begin{aligned} P(x_1, \cdots, x_k) = \{1-(x_1^p+ \cdots +x_k^p)+(x_1^px_2^p+ \cdots +x_{k-1}^px_k^p) - \cdots \\ +(-1)^k x_1^p x_2^p \cdots x_k^p\} \times (1-x_{k+1}) \cdots (1-x_n). \end{aligned}$$

From this form on the right we get terms of degree j such as $x_1^px_2^p \cdots x_k^p x_{k+1} \cdots x_{j-k(p-1)}$, $x_1^px_2^p \cdots x_{k-1}^p x_{k+1} x_{k+2} \cdots x_{j-(k-1)(p-1)+1}, \cdots,$ $x_1^px_2^p \cdots x_{k-l}^p x_{k+1} x_{k+2} \cdots x_{j-(k-l)(p-1)+l}, \cdots,$ then we must calculate its sign and coefficient.

In the first place, since $(-1)^k x_1^p x_2^p \cdots x_k^p$ combines with $j-kp$ terms taken from $-x_{k+1}, -x_{k+2}, \cdots, -x_n$, its sign is

$(-1)^k(-1)^{j-kp} = (-1)^{j-k(p-1)}$ and coefficient is 1.

$(-1)^{k-1} (x_1^p \cdots x_{k-1}^p + \cdots + x_2^p \cdots x_k^p)$ combines with $j-(k-1)p$ terms taken from $-x_{k+1}, -x_{k+2}, \cdots, -x_n$ and therefore its sign is

$(-1)^{k-1}(-1)^{j-(k-1)p}=(-1)^{j-(k-1)(p-1)}$ and coefficient is

$$\binom{\{n-(k-1)\}-\{j-(k-1)p\}}{1}=\binom{n-j+(k-1)(p-1)}{1}$$

and so on. $(-1)^{k-r}(x_1^p\cdots x_{k-r}^p+\cdots+x_{r+1}^p\cdots x_k^p)$ combining with $j-(k-r)p$ terms taken from $-x_{k+1}, -x_{k+2},\cdots, -x_n$, its sign is $(-1)^{k-r}(-1)^{j-(k-r)p}=(-1)^{j-(k-r)(p-1)}$ and coefficient is

$$\binom{\{n-(k-r)\}-\{j-(k-r)p\}}{r}=\binom{n-j+(k-r)(p-1)}{r}$$

where $r=0, 1,\cdots, k$.

As in the above paragraph considering $\sum \mathbf{P}(x_1,\cdots, x_k)$, we obtain a homogeneous symmetric polynomial of degree j

$$(-1)^{j-k(p-1)}\sum x_1^p\cdots x_k^p x_{k+1}\cdots x_{j-k(p-1)}+(-1)^{j-(k-1)(p-1)}\binom{n-j+(k-1)(p-1)}{1}$$
$$\times\sum x_1^p\cdots x_{k-1}^p x_k\cdots x_{j-(k-1)(p-1)}+\cdots+(-1)^{j-(k-r)(p-1)}\binom{n-j+(k-r)(p-1)}{r}$$
$$\times\sum x_1^p\cdots x_{k-r}^p x_{k-r+1}\cdots x_{j-(k-r)(p-1)}+\cdots+(-1)^j\binom{n-j}{k}\sum x_1\cdots x_j.$$

Of course this last polynomial is expressed by $\sigma_1,\cdots,\sigma_j$, and from this expression we obtain the following value as the coefficient of σ_j:

$$(-1)^{j-k(p-1)}b_p^{k,j}+(-1)^{j-(k-1)(p-1)}\binom{n-j+(k-1)(p-1)}{1}b_p^{k-1,j}+\cdots$$
$$+(-1)^{j-(k-r)(p-1)}\binom{n-j+(k-r)(p-1)}{r}b_p^{k-r,j}+\cdots+(-1)^j\binom{n-j}{k}b_p^{0,j}$$ where we put $b_p^{0,j}=1$.

Let $k-r=l$, this becomes $\sum_{l=0}^{k}(-1)^{j-l(p-1)}\binom{n-j+l(p-1)}{k-l}b_p^{l,j}$ and must coincide with $(-1)^j\binom{n}{k}$, the coefficient of σ_j derived in the preceding paragraph and therefore we obtain

$$(-1)^j\binom{n}{k}=\sum_{l=0}^{k}(-1)^{j-l(p-1)}\binom{n-j+l(p-1)}{k-l}b_p^{l,j}.$$

If p is any prime number such as $p\geqq 3$, this reduction formula becomes

$$\binom{n}{k}=\sum_{l=0}^{k}\binom{n-j+l(p-1)}{k-l}b_p^{l,j}$$

and also when $p=2$ this equality is valid with respect to mod. 2.

§ 3. proof of (2).

To this end we shall prove that the reduction formula (3) is independent of n, *i. e.*

(4) $$\binom{m}{k}=\sum_{l=0}^{k}\binom{m-j+l(p-1)}{k-l}b_p^{l,j}$$

is valid for every integer $m\geqq n$. (Observe that we do not change n which may be contained in each $b_p^{l,j}$ but only n expressed explicitly in (3)).

Considering $b_p^{k-1,j}$ for $\sum x_1^p \cdots x_{k-1}^p x_k \cdots x_{j-(k-1)(p-1)}$, we obtain from (3)

$$\binom{n}{k-1}=\sum_{l=0}^{k-1}\binom{n-j+l(p-1)}{k-1-l}b_p^{l,j}$$

and combining this with (3)

$$\binom{n+1}{k}=\sum_{l=0}^{k}\binom{n-j+l(p-1)}{k-l}b_p^{l,j}+\sum_{l=0}^{k-1}\binom{n-j+l(p-1)}{k-1-l}b_p^{l,j}$$

$$=\sum_{l=0}^{k-1}\left\{\binom{n-j+l(p-1)}{k-l}+\binom{n-j+l(p-1)}{k-1-l}\right\}b_p^{l,j}+b_p^{k,j}$$

$$=\sum_{l=0}^{k-1}\binom{n+1-j+l(p-1)}{k-l}b_p^{l,j}+b_p^{k,j}=\sum_{l=0}^{k}\binom{n+1-j+l(p-1)}{k-l}b_p^{l,j}.$$

By induction more generally

$$\binom{n+r}{k}=\sum_{l=0}^{k}\binom{n+r-j+l(p-1)}{k-l}b_p^{l,j}$$ for every positive integer r and then put $m=n+r$.

Let us now turn to the proof of (2).

Since the expression (4) is a polynomial in variable m of degree k and its equality is valid for every positive integer $m\geqq n$, setting the constant term equal to zero, we get

(5) $$\sum_{l=0}^{k}(-1)^l\binom{j-l(p-1)+k-l-1}{k-l}b_p^{l,j}=0.$$

As (2) is true for $k=0$, 1, if we assume

$b_p^{l,j}\equiv\binom{j-l(p-1)-1}{l}$ (*mod. p*) for $0\leqq l<k$, it is sufficient to prove this equality for $l=k$.

From this assumption and (5), we obtain

$$(-1)^k b_p^{k,j}+\sum_{l=0}^{k-1}(-1)^l\binom{j-l(p-1)+k-l-1}{k-l}\binom{j-l(p-1)-1}{l}\equiv 0 \quad (\textit{mod. } p)$$

and therefore the proof may be reduced to

$$(-1)^k\binom{j-k(p-1)-1}{k}+\sum_{l=0}^{k-1}(-1)^l\binom{j-l(p-1)+k-l-1}{k-l}\binom{j-l(p-1)-1}{l}\equiv 0$$

$$(mod.\ p),$$

i. e. $$\sum_{l=0}^{k}(-1)^l\binom{j-l(p-1)+k-l-1}{k-l}\binom{j-l(p-1)-1}{l}\equiv 0 \quad (mod.\ p).$$

However, since each term in the last expression may be reformed as follows:

$$\binom{j-l(p-1)+k-l-1}{k-l}\binom{j-l(p-1)-1}{l}=\frac{(j-l(p-1)+k-l-1)!}{(k-l)!(j-l(p-1)-1)!}$$

$$\times\frac{(j-l(p-1)-1)!}{l!(j-l(p-1)-l-1)!}=\frac{k!}{(k-l)!l!}\times\frac{(j-l(p-1)+k-l-1)!}{k!(j-l(p-1)-l-1)!}$$

$$=\binom{k}{l}\binom{j+k-1-lp}{k},$$

it is sufficient to prove that

$$\sum_{l=0}^{k}(-1)^l\binom{k}{l}\binom{j+k-1-lp}{k}\equiv 0 \quad (mod.\ p).$$

In order to prove this, letting $\binom{j+k-1-lp}{k}=\frac{1}{k!}\sum_{i=0}^{k}(-1)^i A_i p^i l^i$ where A_i is polynomial in j and k, we have

$$\sum_{l=0}^{k}(-1)^l\binom{k}{l}\binom{j+k-1-lp}{k}=\sum_{l=0}^{k}(-1)^l\binom{k}{l}\left\{\frac{1}{k!}\sum_{i=0}^{k}(-1)^i A_i p^i l^i\right\}$$

$$=\frac{1}{k!}\sum_{l=0}^{k}(-1)^l\binom{k}{l}\sum_{i=0}^{k}(-1)^i A_i p^i l^i=\frac{1}{k!}\sum_{i=0}^{k}(-1)^i A_i p^i\sum_{l=0}^{k}(-1)^l\binom{k}{l}l^i.$$

In this place thanks to Euler's formula [2], we have

$$\sum_{l=0}^{k}(-1)^l\binom{k}{l}l^i=0 \qquad \text{if} \quad 0\leqq i\leqq k-1,$$

$$=(-1)^k k! \qquad \text{if} \quad i=k$$

and consequently

$$\sum_{l=0}^{k}(-1)^l\binom{k}{l}\binom{j+k-1-lp}{k}=\frac{1}{k!}(A_k p^k(-1)^{2k}k!)=A_k p^k\equiv 0 \qquad (mod.\ p).$$

Thus the proof of (2) is complete.

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References

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[2] L. E. Dickson, History of the theory of numbers, Washington, Vol. I, 1919, p. 63.