# On analytic tensors in certain Hermitian manifolds

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## §1. Introduction

Let $X_{2n}$ be a complex analytic manifold of $n$ complex dimension (topological dim. 2n) endowed with a Hermitian metric

$$ds^2=g_{jk}dz^j dz^k \quad (j,\ k=1,\ 2,\ \cdots,\ n,\ \bar{1},\ \bar{2},\ \cdots,\ \bar{n}) \tag{1.1}$$

where $g_{jk}(z, \bar{z})$ is a positive definite symmetric tensor satisfying

$$g_{\alpha\beta}=g_{\bar{\alpha}\bar{\beta}}=0,\ g_{\alpha\bar{\beta}}=\overline{g_{\bar{\alpha}\beta}} \quad (\alpha,\ \beta=1,\ 2,\ \cdots,\ n). \tag{1.2}$$

Hence, by virtue of (1.2), the metric form (1.1) can be written in the following

$$ds^2=2g_{\alpha\bar{\beta}}dz^\alpha dz^{\bar{\beta}} \tag{1.3}$$ [1].

Throughout this paper we shall assume that the Latin indices take the values 1, 2, $\cdots$, $n$, $\bar{1}$, $\bar{2}$, $\cdots$, $\bar{n}$ and the Greek indices run over the range 1, 2, $\cdots$, $n$.

The metric connection will be denoted by $E_{jk}{}^i$ and covariant differentiation with respect to this connection by $\nabla$, so that

$$\nabla_l g_{jk}=\partial_l g_{jk}-g_{sk}E_{lj}{}^s-g_{js}E_{lk}{}^s=0. \tag{1.4}$$

It is assumed that this connection $E_{jk}{}^i$ is so called unitary connection, that is, those components of $E_{jk}{}^i$ of different parity vanish and then the torsion

$$S_{jk}{}^i=\frac{1}{2}(E_{jk}{}^i-E_{kj}{}^i)$$

has only the following non-vanishing components:

$$S_{\beta\gamma}{}^\alpha=\frac{1}{2}(E_{\beta\gamma}{}^\alpha-E_{\gamma\beta}{}^\alpha),\ S_{\bar{\beta}\bar{\gamma}}{}^{\bar{\alpha}}=\frac{1}{2}(E_{\bar{\beta}\bar{\gamma}}{}^{\bar{\alpha}}-E_{\bar{\gamma}\bar{\beta}}{}^{\bar{\alpha}}).$$

From (1.4), we have

$$E_{\beta\gamma}{}^\alpha=g^{\alpha\bar{\delta}}\frac{\partial g_{\bar{\delta}\beta}}{\partial z^\gamma},\quad E_{\bar{\beta}\bar{\gamma}}{}^{\bar{\alpha}}=g^{\bar{\alpha}\delta}\frac{\partial g_{\delta\bar{\beta}}}{\partial z^{\bar{\gamma}}}, \tag{1.5}$$

so that

$$S_{\beta\gamma}{}^\alpha=\frac{1}{2}g^{\alpha\bar{\delta}}\left(\frac{\partial_{\bar{\delta}\beta}}{\partial z^\gamma}-\frac{\partial g_{\bar{\delta}\gamma}}{\partial z^\beta}\right),\quad S_{\bar{\beta}\bar{\gamma}}{}^{\bar{\alpha}}=\frac{1}{2}g^{\bar{\alpha}\delta}\left(\frac{\partial g_{\delta\bar{\beta}}}{\partial z^{\bar{\gamma}}}-\frac{\partial g_{\delta\bar{\gamma}}}{\partial z^{\bar{\beta}}}\right). \tag{1.6}$$

Hereafter, $X_{2n}$ always will mean a manifold endowed with such a connection $E_{ji}{}^h$ with torsion tensor $S_{ji}{}^h$. In this manifold $X_{2n}$, we consider a pure tensor of the following form:

(1.7) $$T_{i_1\cdots i_p}{}^{j_1\cdots j_q}=(T_{\alpha_1\cdots\alpha_p}{}^{\beta_1\cdots\beta_q},\ 0,\ \cdots,\ 0,\ T_{\bar\alpha_1\cdots\bar\alpha_p}{}^{\bar\beta_1\cdots\bar\beta_q})$$

and $T_{i_1\cdots i_p}{}^{j_1\cdots j_q}$ is called analytic, if it satisfies

(1.8) $$\partial_{\bar\gamma}T_{\alpha_1\cdots\alpha_p}{}^{\beta_1\cdots\beta_q}=0,\quad \partial_{\gamma}T_{\bar\alpha_1\cdots\bar\alpha_p}{}^{\bar\beta_1\cdots\bar\beta_q}=0.$$

But since $E_{ji}{}^k$ is unitary, it is easily seen that (1.8) is equivalent to

(1.9) $$\nabla_{\bar\gamma}T_{\alpha_1\cdots\alpha_p}{}^{\beta_1\cdots\beta_q}=0,\quad \nabla_{\gamma}T_{\bar\alpha_1\cdots\bar\alpha_p}{}^{\bar\beta_1\cdots\bar\beta_q}=0.$$

If the torsion tensor vanishes, then, by (1.6), $X_{2n}$ coincides with a Kählerian manifold.

The main purpose of this paper is to extend some properties of analytic tensors or vectors in the Kählerian manifold to the case of this Hermitian manifold with torsion.

Now, since our manifold $X_{2n}$ is a complex manifold, there exists a mixed tensor $F_j{}^i$ which has the numerical components [5]

(1.10) $$F_\alpha{}^\beta=i\delta_\alpha^\beta,\quad F_\alpha{}^{\bar\beta}=F_{\bar\alpha}{}^\beta=0,\quad F_{\bar\alpha}{}^{\bar\beta}=-i\delta_{\bar\alpha}^{\bar\beta}\quad (i=\sqrt{-1})$$

in all complex coordinate systems and which satisfies

(1.11) $$F_\gamma{}^\beta F_\beta{}^\alpha=-A_\gamma^\alpha,\quad F_{\bar\gamma}{}^{\bar\beta}F_{\bar\beta}{}^{\bar\alpha}=-A_{\bar\gamma}^{\bar\alpha},\quad i.e.\ F_i{}^jF_j{}^h=-A_i^h.$$

In this place, if we put $F_{ji}=F_j{}^rg_{ri}$, then $F_{ji}$ is hybrid in $j$, $i$ and $F_{ji}=-F_{ij}$ and $F_{ji}$ has the components

(1.12) $$F_{\alpha\beta}=F_{\bar\alpha\bar\beta}=0,\quad F_{\alpha\bar\beta}=ig_{\alpha\bar\beta},\quad F_{\bar\alpha\beta}=-ig_{\bar\alpha\beta}.$$

Moreover, we find

(1.13) $$\nabla_jF_i{}^h=0,\quad \nabla_jF_{ih}=0.$$

In fact,

$$\begin{aligned}\nabla_\beta F_\alpha{}^\gamma&=\partial_\beta F_\alpha{}^\gamma+E_{\beta\sigma}{}^\gamma F_\alpha{}^\sigma-E_{\beta\alpha}{}^\sigma F_\sigma{}^\gamma\\&=i\partial_\beta\delta_\alpha^\gamma+iE_{\beta\sigma}{}^\gamma\delta_\alpha^\sigma-iE_{\beta\alpha}{}^\sigma\delta_\sigma^\gamma=0\end{aligned}$$

and since $E_{ji}{}^h$ is unitary,

$$\begin{aligned}\nabla_\beta F_\alpha{}^{\bar\gamma}&=E_{\beta\sigma}{}^{\bar\gamma}F_\alpha{}^\sigma-E_{\beta\alpha}{}^{\bar\sigma}F_{\bar\sigma}{}^{\bar\gamma}=0,\\ \nabla_{\bar\beta}F_\alpha{}^{\bar\gamma}&=E_{\bar\beta\sigma}{}^{\bar\gamma}F_\alpha{}^\sigma-E_{\bar\beta\alpha}{}^{\bar\sigma}F_{\bar\sigma}{}^{\bar\gamma}=0.\end{aligned}$$

Next, we define the operators

(1.14) $$O_{il}^{mh}=\frac{1}{2}\left(A_i^m A_l^h - F_i{}^m F_l{}^h\right)$$

$$*O_{il}^{mh}=\frac{1}{2}\left(A_i^m A_l^h + F_i{}^m F_l{}^h\right)$$

and if a tensor is pure (hybrid) in two indices, then it is annihilated by transvection of $*O(O)$ on these indices and vice versa [5].

For instance, $*O_{il}^{mk}F_m{}^l=0$ and $O_{ji}^{ml}g_{ml}=0$.

Consequently, by virtue of (1.13), we see that (1.8) or (1.9) is equivalent to

(1.15) $$*O_{ht}^{sj_1}\nabla_s T_{i_1\cdots i_q}{}^{tj_2\cdots j_q}=0$$

or

(1.16) $$F_h{}^s\nabla_s T_{i_1\cdots i_p}{}^{j_1\cdots j_q}-F_s{}^{j_1}\nabla_h T_{i_1\cdots i_p}{}^{sj_2\cdots j_q}=0.$$

## §2. Curvature tensor

From the usual definition of the curvature tensor:

(2.1) $$E_{kji}{}^h=\partial_k E_{ji}{}^h-\partial_j E_{ki}{}^h+E_{kl}{}^h E_{ji}{}^l-E_{jl}{}^h E_{ki}{}^l,$$

we obtain

(2.2) $$E_{\bar\kappa\beta\alpha}{}^\mu=-E_{\beta\bar\kappa\alpha}{}^\mu=\partial_{\bar\kappa}E_{\beta\alpha}{}^\mu \qquad \text{(conj.)},$$

(2.3) $$E_{\bar\kappa\beta\alpha}{}^\mu-E_{\bar\kappa\alpha\beta}{}^\mu=2\partial_{\bar\kappa}S_{\beta\alpha}{}^\mu=2\nabla_{\bar\kappa}S_{\beta\alpha}{}^\mu \qquad \text{(conj.)}.$$

Applying the Ricci's identity to $g_{ji}$, we find

$$0=\nabla_l\nabla_k g_{ij}-\nabla_k\nabla_l g_{ij}=-E_{lki}{}^s g_{sj}-E_{lkj}{}^s g_{is}-2S_{lk}{}^s\nabla_s g_{ij}$$

and on putting $E_{ijkl}=g_{ls}E_{ijk}{}^s$, we obtaien

(2.4) $$E_{lkij}=-E_{lkji}.$$

Thus, we have non-vanishing components

$$E_{\alpha\bar\beta\gamma\bar\delta},\ E_{\alpha\bar\beta\bar\gamma\delta},\ E_{\bar\alpha\beta\gamma\bar\delta},\ E_{\bar\alpha\beta\bar\gamma\delta}$$

which satisfy

(2.5) $$E_{\alpha\bar\beta\gamma\bar\delta}=-E_{\bar\beta\alpha\gamma\bar\delta},\ E_{\alpha\bar\beta\gamma\bar\delta}=-E_{\alpha\bar\beta\bar\delta\gamma}.$$

Next, from (1.4) and (2.3), we have

(2.6) $$E_{\bar\kappa\beta\alpha\bar\mu}-E_{\alpha\bar\mu\bar\kappa\beta}=E_{\bar\kappa\beta\alpha\bar\mu}-E_{\bar\kappa\alpha\beta\bar\mu}-(E_{\alpha\bar\mu\bar\kappa\beta}-E_{\alpha\bar\kappa\bar\mu\beta})=2\nabla_{\bar\kappa}S_{\beta\alpha\bar\mu}-2\nabla_\alpha S_{\bar\mu\bar\kappa\beta}$$

where $S_{\beta\alpha\bar{\mu}}=g_{\delta\bar{\mu}}S_{\beta\alpha}{}^{\delta}$.

There are three kinds of Ricci's tensor

$$(2.7)\qquad E_{ji}\equiv g^{lm}E_{jlmi},\ S_{\alpha\bar{\beta}}\equiv g^{\gamma\bar{\delta}}E_{\gamma\bar{\delta}\alpha\bar{\beta}},\ S_{\bar{\beta}\alpha}\equiv g^{\bar{\delta}\gamma}E_{\bar{\delta}\gamma\bar{\beta}\alpha},$$
$$T_{\alpha\bar{\beta}}\equiv g^{\gamma\bar{\delta}}E_{\alpha\bar{\beta}\gamma\bar{\delta}},\ T_{\bar{\beta}\alpha}\equiv g^{\bar{\gamma}\delta}E_{\bar{\beta}\alpha\bar{\gamma}\delta}.$$

From this definition, we have immediately

$$(2.8)\qquad S_{\alpha\bar{\beta}}=S_{\bar{\beta}\alpha},\ T_{\alpha\bar{\beta}}=T_{\bar{\beta}\alpha}$$

and by virtue of (2.3), we have

$$(2.9)\qquad \begin{aligned} E_{\bar{\beta}\alpha}-E_{\alpha\bar{\beta}}&=g^{\gamma\bar{\mu}}E_{\bar{\beta}\gamma\bar{\mu}\alpha}-g^{\gamma\bar{\mu}}E_{\alpha\bar{\mu}\gamma\bar{\beta}}\\ &=g^{\gamma\bar{\mu}}(E_{\bar{\mu}\alpha\gamma\bar{\beta}}-E_{\bar{\mu}\gamma\alpha\bar{\beta}}+E_{\gamma\bar{\mu}\bar{\beta}\alpha}-E_{\gamma\bar{\beta}\bar{\mu}\alpha})\\ &=g^{\gamma\bar{\mu}}(2\nabla_{\bar{\mu}}S_{\alpha\gamma\bar{\beta}}+2\nabla_{\gamma}S_{\bar{\mu}\bar{\beta}\alpha})\\ &=2(\nabla^{\bar{\mu}}S_{\bar{\mu}\bar{\beta}\alpha}-\nabla^{\gamma}S_{\gamma\alpha\bar{\beta}}),\end{aligned}$$

$$(2.10)\qquad \begin{aligned} S_{\alpha\bar{\beta}}-E_{\alpha\bar{\beta}}&=g^{\gamma\bar{\delta}}(E_{\gamma\bar{\delta}\alpha\bar{\beta}}-E_{\alpha\bar{\delta}\gamma\bar{\beta}})\\ &=g^{\gamma\bar{\delta}}(E_{\bar{\delta}\alpha\gamma\bar{\beta}}-E_{\bar{\delta}\gamma\alpha\bar{\beta}})\\ &=2\nabla^{\gamma}S_{\alpha\gamma\bar{\beta}},\end{aligned}$$

$$(2.11)\qquad \begin{aligned} T_{\alpha\bar{\beta}}-E_{\alpha\bar{\beta}}&=g^{\gamma\bar{\delta}}(E_{\alpha\bar{\delta}\bar{\beta}\gamma}-E_{\alpha\bar{\beta}\bar{\delta}\gamma})\\ &=2\nabla_{\alpha}S_{\bar{\delta}\bar{\beta}}{}^{\bar{\delta}}.\end{aligned}$$

Consequently, if $S_{ji}{}^{h}$ is analytic, then these Ricci's tensors coincide with each other [2] and therefore when $S_{ji}{}^{h}$ is analytic, we shall write briefly $E_{ji}$ for these Ricci's tensors.

Moreover, in this case, from (2.6), we have

$$(2.12)\qquad E_{\bar{\kappa}\beta\alpha\bar{\mu}}=E_{\alpha\bar{\mu}\bar{\kappa}\beta}$$

and the Bianchi's identity:

$$(2.13)\qquad E_{jkl}{}^{i}+E_{klj}{}^{i}+E_{ljk}{}^{i}-2(\nabla_{j}S_{kl}{}^{i}+\nabla_{k}S_{lj}{}^{i}+\nabla_{l}S_{jk}{}^{i})$$
$$+4(S_{jk}{}^{t}S_{tl}{}^{i}+S_{kl}{}^{t}S_{tj}{}^{i}+S_{lj}{}^{t}S_{tk}{}^{i})=0$$

becomes the ordinary from

$$(2.14)\qquad E_{jkl}{}^{i}+E_{klj}{}^{i}+E_{ljk}{}^{i}=0.$$

Hence $\qquad E_{\bar{\beta}\gamma\delta\bar{\alpha}}+E_{\gamma\delta\bar{\beta}\bar{\alpha}}+E_{\delta\bar{\beta}\gamma\bar{\alpha}}=0 \qquad$ or

$$(2.15)\qquad E_{\bar{\beta}\gamma\bar{\alpha}\delta}=E_{\bar{\beta}\delta\bar{\alpha}\gamma}\ \ or\ \ E_{\bar{\beta}\gamma\delta\bar{\alpha}}=E_{\bar{\beta}\delta\gamma\bar{\alpha}}$$

(this is obtained also from (2.3)).

Summarising these results, if the torsion tensor is analytic, then the curvature tensor has symmetric properties as in the Kählerian manifold [1].

Now, since, in our manifold $X_{2n}$, $E_{kji}{}^h$ is pure in ${}^h_i$, we have

(2.16) $$E_{kjl}{}^h F_i{}^l = E_{kji}{}^l F_l{}^h$$

and contracting with $g^{ji}$, we get

(2.17) $$E_{kml}{}^h F^{ml} = E_k{}^l F_l{}^h \qquad \text{or}$$

(2.18) $$\frac{1}{2}(E_{kml}{}^h - E_{klm}{}^h) F^{ml} = E_k{}^l F_l{}^h$$

where $$E_k{}^l = E_{ks} g^{sl} \text{ and } F^{ml} = F_s{}^l g^{sm}.$$

If the torsion tensor is analytic, then by virtue of (2.14), from (2.18), we have

(2.19) $$E_k{}^l F_l{}^h = -\frac{1}{2} E_{mlk}{}^h F^{ml} \qquad \text{or}$$

(2.20) $$E_k{}^s = \frac{1}{2} F^{ml} E_{mlk}{}^h F_h{}^s = \frac{1}{2} F^{ml} E_{mlh}{}^s F_k{}^h.$$

Here, if we consider a pure tensor $T_{i_1\cdots i_p}{}^{j_1\cdots j_q}$ and apply the Ricci's identity to $F_s{}^{j_1} F^{hk} \nabla_h \nabla_k T_{i_1\cdots i_p}{}^{sj_2\cdots j_q}$, then we have

(2.21) $$F_s{}^{j_1} F^{hk} \nabla_h \nabla_k T_{i_1\cdots i_p}{}^{sj_2\cdots j_q}$$

$$= \frac{1}{2} F_s{}^{j_1} F^{hk} (\nabla_h \nabla_k T_{i_1\cdots i_p}{}^{sj_2\cdots j_q} - \nabla_k \nabla_h T_{i_1\cdots i_p}{}^{sj_2\cdots j_q})$$

$$= \frac{1}{2} F_s{}^{j_1} F^{hk} (E_{hkt}{}^s T_{i_1\cdots i_p}{}^{tj_2\cdots j_q} + \sum_{r=2}^{q} E_{hkt}{}^{j_r} T_{i_1\cdots i_p}{}^{sj_2\cdots t\cdots j_q}$$

$$- \sum_{r=1}^{p} E_{hki_r}{}^t T_{i_1\cdots t\cdots i_p}{}^{sj_2\cdots j_q} - 2 S_{hk}{}^t \nabla_t T_{i_1\cdots i_p}{}^{sj_2\cdots j_q})$$

$$= \frac{1}{2} F_s{}^{j_1} F^{hk} E_{hkt}{}^s T_{i_1\cdots i_p}{}^{tj_2\cdots j_q} + \frac{1}{2} \sum_{r=1}^{q} F_s{}^t F^{hk} E_{hkt}{}^{j_r} T_{i_1\cdots i_p}{}^{j_1\cdots s\cdots j_q}$$

$$- \frac{1}{2} \sum_{r=1}^{p} F_t{}^s F^{hk} E_{hki_r}{}^t T_{i_1\cdots s\cdots i_p}{}^{j_1\cdots j_q} - 2 S_{hk}{}^t F_s{}^{j_1} F^{hk} \nabla_t T_{i_1\cdots i_p}{}^{sj_2\cdots j_q}.$$

But, since $E_{hka}{}^b$ is pure in ${}^b_a$, we have $F_s{}^{j_1} E_{hkt}{}^s = F_t{}^s E_{hks}{}^{j_1}$ and $F_t{}^s E_{hki_r}{}^t = F_{i_r}{}^t E_{hkt}{}^s$ and since $F^{hk}$ is hybrid in $h$, $k$ and $S_{hk}{}^t$ is pure in $h$, $k$, we have $F^{hk} S_{hkt} = 0$.

Consequently, (2.21) can be written as

(2.22) $$F_s{}^{j_1} F^{hk} \nabla_h \nabla_k T_{i_1\cdots i_p}{}^{sj_2\cdots j_q}$$

$$= \frac{1}{2} \sum_{r=1}^{q} F_s{}^t F^{hk} E_{hkt}{}^{j_r} T_{i_1\cdots i_p}{}^{j_1\cdots s\cdots j_q} - \frac{1}{2} \sum_{r=1}^{p} F_{i_r}{}^t F^{hk} E_{hkt}{}^s T_{i_1\cdots s\cdots i_p}{}^{j_1\cdots j_q}.$$

Thus, if the torsion tensor is analytic, by (2.20), we have

$$F_s{}^{j_1}F^{hk}\nabla_h\nabla_k T_{i_1\cdots i_p}{}^{sj_2\cdots j_q} \tag{2.23}$$

$$=\sum_{r=1}^{q} E_t{}^{j_r}T_{i_1\cdots i_p}{}^{j_1\cdots t\cdots j_q}-\sum_{r=1}^{p} E_{i_r}{}^{t}T_{i_1\cdots t\cdots i}{}^{j_1\cdots j_q}.$$

## §3. Lie derivatives

We consider an analytic pure tensor $T_{i_1\cdots i_p}{}^{j_1\cdots j_q}$ in $X_{2n}$ and the following Lie derivative of $T_{i_1\cdots i_p}{}^{j_1\cdots j_q}$ with respect to a contravariant analytic vector $v^i$:

$$\underset{v}{\pounds}T_{i_1\cdots i_p}{}^{j_1\cdots j_q}=v^a\nabla_a T_{i_1\cdots i_p}{}^{j_1\cdots j_q}+\sum_{r=1}^{p}T_{i_1\cdots t\cdots i_p}{}^{j_1\cdots j_q}v_{i_r}^{\cdot t} \tag{3.1}$$

$$-\sum_{r=1}^{q}T_{i_1\cdots i_p}{}^{j\cdots t\cdots j_q}v_t^{\cdot j_r}$$

where $v_k^{\cdot t}=\nabla_k v^t+2S_{sk}{}^{t}v^s=\partial_k v^t+E_{sk}{}^{t}v^s$ [5].

But since, by $\nabla_j F_{ih}=0$, $\nabla_h T_{i_1\cdots i_p}{}^{j_1\cdots j_q}$ is pure in all indices $i_1, \cdots, i_p, j_1, \cdots, j_q$ except $h$, $\underset{v}{\pounds}T_{i_1\cdots i_p}{}^{j_1\cdots j_q}$ is also a pure tensor.

Here we have the following

THEOREM 3.1. *For an analytic pure tensor $T_{i_1\cdots i_p}{}^{j_1\cdots j_q}$ and a contravariant analytic vector $v^i$ in $X_{2n}$, if the torsion tensor $S_{ji}{}^{h}$ is analytic, then the Lie derivative $\underset{v}{\pounds}T_{i_1\cdots i_p}{}^{j\cdots j_q}$ is also analytic.*

*Proof.* In order to prove that $\pounds T_{i_1\cdots i_p}{}^{j_1\cdots j_q}$ is analytic, since $\underset{v}{\pounds}T_{i_1\cdots i_p}{}^{j_1\cdots j_q}$ is a pure tensor, it is sufficient to show

$$F_d{}^s\nabla_s(\underset{v}{\pounds}T_{i_1\cdots i_p}{}^{j_1\cdots j_q})-F_s{}^{j_1}\nabla_d(\underset{v}{\pounds}T_{i_1\cdots i_p}{}^{sj_2\cdots j_q})=0. \tag{3.2}$$

First, when the left hand side of (3.2) is pure in $\overset{j_1}{d}$, that is, pure in all indices $d, i_1, \cdots, i_p, j_1, \cdots, j_q$, since $\nabla_d(\underset{v}{\pounds}T_{i_1\cdots i_p}{}^{sj_2\cdots j_q})$ is pure in $\overset{s}{d}$, (3.2) is evident.

Secondly, when the left hand side of (3.2) is hybrid in $\overset{j_1}{d}$, that is, hybrid with respect to $d$ and every one of $i_1, \cdots, i_p, j_1, \cdots, j_q$, we shall show that (3.2) is true.

Noticing that $F_s{}^{j_1}\nabla_d(\underset{v}{\pounds}T_{i_1\cdots i_p}{}^{sj_2\cdots j_q})=-F_d{}^s\nabla_s(\underset{v}{\pounds}T_{i_1\cdots i_p}{}^{j_1\cdots j_q})$,

we have

$$\frac{1}{2}[F_d{}^s\nabla_s(\underset{v}{\pounds}T_{i_1\cdots i_p}{}^{j_1\cdots j_q})-F_s{}^{j_1}\nabla_d(\underset{v}{\pounds}T_{i_1\cdots i_p}{}^{sj_2\cdots j_q})] \tag{3.3}$$

$$=F_d{}^s\nabla_s(\underset{v}{\pounds}T_{i_1\cdots i_p}{}^{j_1\cdots j_q})$$

$$=F_d{}^s\nabla_s(v^a\nabla^a T_{i_1\cdots i_p}{}^{j_1\cdots j_q}+\Sigma T_{i_1\cdots t\cdots i_p}{}^{j_1\cdots j_q}v_{i_r}^{\cdot t}-\Sigma T_{i_1\cdots i_p}{}^{j_1\cdots t\cdots j_q}v_t^{\cdot j_r})$$

$$=F_d{}^s(\nabla_s v^a)\nabla_a T_{i_1\cdots i_p}{}^{j_1\cdots j_q}+F_d{}^s v^a\nabla_s\nabla_a T_{i_1\cdots i_p}{}^{j_1\cdots j_q}+F_d{}^s\Sigma(\nabla_s T_{i_1\cdots t\cdots i_p}{}^{j_1\cdots j_q})v_{i_r}^{\cdot t}$$

$$+F_d{}^s\Sigma T_{i_1\cdots t\cdots i_p}{}^{j_1\cdots j_q}(\nabla_s v_{i_r}^{\cdot t})-F_d{}^s\Sigma(\nabla_s T_{i_1\cdots i_p}{}^{j_1\cdots t\cdots j_q})v_t^{\cdot j_r}-F_d{}^s\Sigma T_{i_1\cdots i_p}{}^{j_1\cdots t\cdots j_q}(\nabla_s v_t^{\cdot j_r}).$$

On the other hand, since $v^i$ and $T_{i_1\cdots i_p}{}^{j_1\cdots j_q}$ are analytic, we have $F_d{}^s(\nabla_s v^a)\times\nabla_a T_{i_1\cdots i_p}{}^{j_1\cdots j_q}=0$, because $F_d{}^s$ is pure in $\substack{s\\d}$, $\nabla_s v^a$ is pure in $\substack{a\\s}$ and $\nabla_a T_{i_1\cdots i_p}{}^{j_1\cdots j_q}$ is pure in $\substack{j_1\\a}$ and therefore $F_d{}^s(\nabla_s v^a)\nabla_a T_{i_1\cdots i_p}{}^{j_1\cdots j_q}$ must be pure in $\substack{j_1\\d}$ but, from the assumption, $F_d{}^s(\nabla_s v^a)\nabla_a T_{i_1\cdots i_p}{}^{j_1\cdots j_q}$ is hybrid in $\substack{j_1\\d}$.

Similarly, we have $F_d{}^s\Sigma\nabla_s T_{i_1\cdots t\cdots i_p}{}^{j_1\cdots j_q}=0$, $F_d{}^s\Sigma\nabla_s T_{i_1\cdots i_p}{}^{j_1\cdots t\cdots j_q}=0$ and hence

$$F_d{}^s\nabla_s(\underset{v}{\pounds}T_{i_1\cdots i_p}{}^{j_1\cdots j_q}) \tag{3.4}$$

$$=F_d{}^s v^a\nabla_s\nabla_a T_{i_1\cdots i_p}{}^{j_1\cdots j_q}+F_d{}^s\Sigma T_{i_1\cdots t\cdots i_p}{}^{j_1\cdots j_q}[\nabla_s\nabla_{i_r}v^t+2\nabla_s(S_{a i_r}{}^t v^a)]$$

$$-F_d{}^s\Sigma T_{i_1\cdots i_p}{}^{j_1\cdots t\cdots j_q}[\nabla_s\nabla_t v^{j_r}+2\nabla_s(S_{at}{}^{j_r}v^a)].$$

Moreover, since, from the same reason used in the preceding paragraph,

$$F_d{}^s\nabla_s(S_{a i_r}{}^t v^a)=F_d{}^s(\nabla_s S_{a i_r}{}^t)v^a+F_d{}^s S_{a i_r}{}^t\nabla_s v^a=0$$

and $F_d{}^s\nabla_s(S_{at}{}^{j_r}v^a)=0$, (3.3) can be written in the form

$$F_d{}^s\nabla_s(\underset{v}{\pounds}T_{i_1\cdots i_p}{}^{j_1\cdots j_q}) \tag{3.5}$$

$$=F_d{}^s v^a\nabla_s\nabla_a T_{i_1\cdots i_p}{}^{j_1\cdots j_q}+F_d{}^s\Sigma T_{i_1\cdots t\cdots i_p}{}^{j_1\cdots j_q}\nabla_s\nabla_{i_r}v^t$$

$$-F_d{}^s\Sigma T_{i_1\cdots i_p}{}^{j_1\cdots t\cdots j_q}\nabla_s\nabla_t v^{j_r}.$$

Next, applying the Ricci's identity to the three terms of the right hand side of (3.5) respectively, we have

$$(3.6)\qquad F_d{}^s v^a \nabla_s \nabla_a T_{i_1\cdots i_p}{}^{j_1\cdots j_q}$$

$$=F_d{}^s v^a \nabla_a \nabla_s T_{i_1\cdots i_p}{}^{j_1\cdots j_q}+F_d{}^s v^a \Sigma E_{sac}{}^{j_r} T_{i_1\cdots i_p}{}^{j_1\cdots c\cdots j_q}-F_d{}^s v^a \Sigma E_{sai_r}{}^c T_{i_1\cdots c\cdots i_p}{}^{j_1\cdots j_q}$$

$$-2F_d{}^s v^a S_{sa}{}^c \nabla_c T_{i_1\cdots i_p}{}^{j_1\cdots j_q}.$$

But, since the four terms of the right hand side of (3.6) are hybrid in $\overset{j_1}{d}$, we find that $F_d{}^s v^a \nabla_a \nabla_s T_{i_1\cdots i_p}{}^{j_1\cdots j_q}=0$, $F_d{}^s v^a S_{sa}{}^c \nabla_c T_{i_1\cdots i_p}{}^{j_1\cdots j_q}=0$, $E_{sac}{}^{j_r}$ is pure in $a$, $c$ and $E_{sai_r}{}^c$ is pure in $a$, $i_r$. Therefore, by (2.15), (3.6) can also be written as

$$(3.7)\qquad F_d{}^s v^a \nabla_s \nabla_a T_{i_1\cdots i_p}{}^{j_1\cdots j_q}=F_d{}^s v^a \Sigma E_{sca}{}^{j_r} T_{i_1\cdots i_p}{}^{j_1\cdots c\cdots j_q}$$

$$-F_d{}^s v^a \Sigma E_{si_r a}{}^c T_{i_1\cdots c\cdots i_p}{}^{j_1\cdots j_q}.$$

Similarly, we obtain

$$(3.8)\qquad F_d{}^s \Sigma T_{i_1\cdots t\cdots i_p}{}^{j_1\cdots j_q} \nabla_s \nabla_{i_r} v^t=F_d{}^s \Sigma E_{si_r a}{}^t v^a T_{i_1\cdots t\cdots i_p}{}^{j_1\cdots j_q}$$

$$-2F_d{}^s \Sigma S_{si_r}{}^a(\nabla_a v^t) T_{i_1\cdots t\cdots i_p}{}^{j_1\cdots j_q}=F_d{}^s v^a \Sigma E_{si_r a}{}^t T_{i_1\cdots t\cdots i_p}{}^{j_1\cdots j_q}$$

and

$$(3.9)\qquad F_d{}^s \Sigma T_{i_1\cdots i_p}{}^{j_1\cdots t\cdots j_q} \nabla_s \nabla_t v^{j_r}=F_d{}^s \Sigma E_{sta}{}^{j_r} v^a T_{i_1\cdots i_p}{}^{j_1\cdots t\cdots j_q}$$

$$-2F_d{}^s \Sigma S_{st}{}^a(\nabla_a v^{j_r}) T_{i_1\cdots i_p}{}^{j_1\cdots t\cdots j_q}=F_d{}^s v^a \Sigma E_{sta}{}^{j_r} T_{i_1\cdots i_p}{}^{j_1\cdots t\cdots j_q}.$$

Thus, by virtue of (3.7), (3.8) and (3.9), the right hand side of (3.5) vanishes. This is a generalization of the result obtained for a vector in the Kählerian manifold [3].

Observing that $\underset{v}{\pounds} T_{i_1\cdots i_p}{}^{j_1\cdots j_q}$ is a pure tensor, from this theorem, we have the following

COROLLARY 3.1. *When $T_{i_1\cdots i_p}{}^{j_1\cdots j_q}$ is an analytic pure tensor and $\underset{1}{v^i}, \underset{2}{v^i}, \cdots, \underset{m}{v^i}$ are $m$ contravariant analytic vectors in $X_{2n}$, if the torsion tensor $S_{ji}{}^h$ is analytic, then the following tensors are all analytic:*

$$\underset{\underset{1}{v}}{\pounds} T_{i_1\cdots i_p}{}^{j_1\cdots j_q},\quad \underset{\underset{2}{v}}{\pounds}\underset{\underset{1}{v}}{\pounds} T_{i_1\cdots i_p}{}^{j_1\cdots j_q},\quad \cdots,\quad \underset{\underset{m}{v}}{\pounds}\ \underset{\underset{m-1}{v}}{\pounds} \cdots \underset{\underset{1}{v}}{\pounds} T_{i_1\cdots i_p}{}^{j_1\cdots j_q}$$

*where $\underset{\underset{2}{v}}{\pounds}\underset{\underset{1}{v}}{\pounds} T_{i_1\cdots i_p}{}^{j_1\cdots j_q}$ implies $\underset{\underset{2}{v}}{\pounds}(\underset{\underset{1}{v}}{\pounds} T_{i_1\cdots i_p}{}^{j_1\cdots j_q})$ and so on.*

## §4. A necessary and sufficient condition that a pure tensor be analytic

In this section, we assume that $X_{2n}$ is compact and $S_{ji}{}^{i}=0$.

Then, for any vector $v^i$, we have

$$(4.1)\qquad \begin{aligned}\nabla_i v^i &= \overset{\circ}{\nabla}_i v^i + 2S_{li}{}^{i} v^l \\ &= \overset{\circ}{\nabla}_i v^i = \frac{1}{\sqrt{g}}\,\frac{\partial\sqrt{g}\,v^i}{\partial z^i}\end{aligned}$$

where $\overset{\circ}{\nabla}$ denotes covariant differentiation with respect to the Christoffel symbol $\left\{ {i \atop jk} \right\}$ and $g$ is the determinant formed with $g_{jk}$. But, since $X_{2n}$ is orientable, by virtue of Green's theorem, for any vector field $v^i$, we have

$$(4.2)\qquad \int_{X_{2n}} \nabla_i v^i d\sigma = 0$$

where $d\sigma$ is the volume element.

Using (4.2), we can prove the following

THEOREM 4.1. *If, in a compact Hermitian manifold $X_{2n}$, $S_{ji}{}^{i}=0$, then a pure tensor $T_{i_1\cdots i_p}{}^{j_1\cdots j_q}$ is analytic if and only if*

$$\nabla^h\nabla_h T_{i_1\cdots i_p}{}^{j_1\cdots j_q} + \sum_{r=1}^{q} E^*{}_t{}^{j_r} T_{i_1\ldots i_p}{}^{j_1\cdots t\cdots j_q} - \sum_{r=1}^{p} E^*{}_{i_r}{}^{t} T_{i_1\cdots t\cdots i_p}{}^{j_1\cdots j_q} = 0$$

*where* $\quad E^*{}_j{}^i = \frac{1}{2} F^{ab} E_{abt}{}^{i} F_j{}^t.$

*If the torsion tensor $S_{ji}{}^{h}$ satisfies $S_{ji}{}^{i}=0$ and $S_{ji}{}^{h}$ is analytic, then a pure tensor $T_{i_1\cdots i_p}{}^{j_1\cdots j_q}$ is analytic if and only if*

$$\nabla^h\nabla_h T_{i_1\cdots i_p}{}^{j_1\cdots j_q} + \sum_{r=1}^{q} E_t{}^{j_r} T_{i_1\cdots i_p}{}^{j_1\cdots t\cdots j_q} - \sum_{r=1}^{q} E_{i_r}{}^{t} T_{i_1\cdots t\cdots i_p}{}^{j_1\cdots j_q} = 0.$$

*Proof.* If a pure tensor $T_{i_1\cdots i_p}{}^{j_1\cdots j_q}$ is analytic, then, from (1.16) we have

$$(4.3)\qquad \nabla_h T_{i_1\cdots i_p}{}^{j_1\cdots j_q} + F_s{}^{j_1} F_h{}^{l} \nabla_l T_{i_1\cdots i_p}{}^{s j_2\cdots j_q} = 0$$

and operating $\nabla^h$ to (4.3)

$$(4.4)\qquad \nabla^h\nabla_h T_{i_1\cdots i_p}{}^{j_1\cdots j_q} + F_s{}^{j_1} F_h{}^{l} \nabla^h \nabla_l T_{i_1\cdots i_p}{}^{s j_2\cdots j_q} = 0.$$

Using (2.22), we can write (4.4) in the form

(4.5) $$\nabla^h\nabla_h T_{i_1\cdots i_p}{}^{j_1\cdots j_q}+\sum_{r=1}^{q}F_s{}^t F^{hk}E_{hkt}{}^{j_r}T_{i_1\cdots i_p}{}^{j_1\cdots s\cdots j_q}$$
$$-\frac{1}{2}\sum_{r=1}^{p}F_{i_r}{}^t F^{hk}E_{hkt}{}^s T_{i_1\cdots s\cdots i_p}{}^{j_1\cdots j_q}=0$$

or

(4.6) $$\nabla^h\nabla_h T_{i_1\cdots i_p}{}^{j_1\cdots j_q}+\sum_{r=1}^{q}E^*{}_s{}^{j_r}T_{i_1\cdots i_p}{}^{j_1\cdots s\cdots j_q}-\sum E^*{}_{i_r}{}^s T_{i_1\cdots s\cdots i_p}{}^{j_1\cdots j_q}=0.$$

In this place, if $S_{ji}{}^h$ is analytic, from (2.20), we have $E^*{}_k{}^s=E_k{}^s$.

Next, in order to prove the converse, putting

(4.7) $$P_{hi_1\cdots i_p}{}^{j_1\cdots j_q}=-\nabla_h T_{i_1\cdots i_p}{}^{j_1\cdots j_q}-F_s{}^{j_1}F_h{}^l\nabla_l T_{i_1\cdots i_p}{}^{sj\cdots j_q}$$

and calculating the square of $P_{hi_1\cdots i_p}{}^{j_1\cdots j_q}$, we have

$$\frac{1}{2}P_{hi_1\cdots i_p}{}^{j_1\cdots j_q}P^{hi_1\cdots i_p}{}_{j_1\cdots j_q}=(\nabla_h T_{i_1\cdots i_p}{}^{j_1\cdots j_q})\nabla^h T^{i_1\cdots i_p}{}_{j_1\cdots j_q}$$
$$+F_s{}^{j_1}F_h{}^l(\nabla^h T^{i_1\cdots i_p}{}_{j_1\cdots j_q})\nabla_l T_{i_1\cdots i_q}{}^{sj_2\cdots j_q}.$$

Therefore, we find

(4.8) $$\frac{1}{2}P_{hi_1\cdots i_p}{}^{j_1\cdots j_p}P^{hi_1\cdots i_p}{}_{j_1\cdots j_q}+\nabla^h(T^{i_1\cdots i_p}{}_{j_1\cdots j_q}P_{hi_1\cdots i_p}{}^{j_1\cdots j_q})$$
$$=\frac{1}{2}P_{hi_1\cdots i_p}{}^{j_1\cdots j_q}P^{hi_1\cdots i_p}{}_{j_1\cdots j_q}+(\nabla^h T^{i_1\cdots i_p}{}_{j_1\cdots j_q})P_{hi_1\cdots i_q}{}^{j_1\cdots j_q}$$
$$+T^{i_1\cdots i_p}{}_{j_1\cdots j_p}\nabla^h P_{hi_1\cdots i_p}{}^{j_1\cdots j_q}=T^{i_1\cdots i_p}{}_{j_1\cdots j_p}\nabla^h P_{hi_1\cdots i_p}{}^{j_1\cdots j_q}$$

and then, if $S_{ji}{}^i=0$, then, from (4.2), we have

(4.9) $$0=\int_{X_{2n}}\nabla^h(T^{i_1\cdots i_p}{}_{j_1\cdots j_q}P_{hi_1\cdots i_p}{}^{j_1\cdots j_q})d\sigma$$
$$=\int_{X_{2n}}[T^{i_1\cdots i_p}{}_{j_1\cdots j_q}\nabla_h P_{hi_1\cdots i_p}{}^{j_1\cdots j_q}-\frac{1}{2}P_{hi_1\cdots i_p}{}^{j_1\cdots j_q}P^{hi_1\cdots i_p}{}_{j_1\cdots j_q}]d\sigma.$$

Here, (4.9) shows that if $\nabla^h P_{hi_1\cdots i_p}{}^{j_1\cdots j_q}=0$, then $P_{hi_1\cdots i_p}{}^{j_1\cdots j_q}=0$.
On the other hand, since $\nabla^h P_{hi_1\cdots i_p}{}^{j_1\cdots j_q}=0$ is (4.6) itself, the proof is complete. This theorem formally coincides with the case of the Kählerian manifold [4]. From the theorem 4.1, we have the following

THEOREM 4.2. *If, in a compact Hermitian manifold $X_{2n}$ with the torsion tensor satisfying $S_{ji}{}^i=0$, $\underset{t}{v}{}^i(t=1,\ 2,\ \cdots,\ p)$* and *$\underset{t}{u}{}_j(t=1,\ 2,\ \cdots,\ q)$ are contravariant analytic vectors and covariant analytic vectors respectively, then for analytic pure tensor $T_{i_1\cdots i_p}{}^{j_1\cdots j_q}$*

*we have*

$$T_{i_1\cdots i_p}{}^{j_1\cdots j_q} \underset{1}{v}^{i_1}\cdots \underset{p}{v}^{i_p}\cdot \underset{1}{u}_{j_1}\cdots \underset{q}{u}_{j_q}=constant.$$

*Proof.*
$$\Delta(T_{i_1\cdots i_p}{}^{j_1\cdots j_q} \underset{1}{v}^{i_1}\cdots \underset{p}{v}^{i_p}\cdot \underset{1}{u}_{j_1}\cdots \underset{q}{u}_{j_q})$$

$$=T_{i_1\cdots i_p}{}^{j_1\cdots j_q}\nabla^l\nabla_l(\underset{1}{v}^{i_1}\cdots \underset{p}{v}^{i_p}\underset{1}{u}_{j_1}\cdots \underset{q}{u}_{j_q})+2\nabla_l(T_{i_1\cdots i_p}{}^{j_1\cdots j_q})\nabla^l(\underset{1}{v}^{i_1}\cdots \underset{p}{v}^{i_p}\underset{1}{u}_{j_1}\cdots \underset{q}{u}_{j_q})$$

$$+(\nabla^l\nabla_l T_{i_1\cdots i_p}{}^{j_1\cdots j_q})\underset{1}{v}^{i_1}\cdots \underset{p}{v}^{i_p}\underset{1}{u}^{j_1}\cdots \underset{q}{u}^{j_q}$$

where $\Delta$ denotes the Laplacean *w. r. t.* $\nabla$.
Here

$$(4.10)\qquad (\nabla_l T_{i_1\cdots i_p}{}^{j_1\cdots j_q})\nabla^l(\underset{1}{v}^{i_1}\cdots \underset{p}{v}^{i_p}\underset{1}{u}_{j_1}\cdots \underset{q}{u}_{j_q})$$

$$=(\nabla_l T_{i_1\ldots i_p}{}^{j_1\cdots j_q})[(\nabla^l \underset{1}{v}^{i_1})\underset{2}{v}^{i_2}\cdots \underset{p}{v}^{i_p}\underset{1}{u}_{j_1}\cdots \underset{q}{u}_{j_q}+\cdots+\underset{1}{v}^{i_1}\cdots \underset{q-1}{u}_{j_{q-1}}\nabla^l \underset{q}{u}_{j_q}]$$

but since $\nabla_l T_{i_1\cdots i_p}{}^{j_1\cdots j_q}$ is pure in $l$, $i_1$ and $\nabla^l \underset{1}{v}^{i_1}$ is hybrid in $l$, $i_1$ and since $\nabla_l T_{i_1\ldots i_p}{}^{j_1\cdots j_q}$ is pure in ${}^{j_q}_{l}$ and $\nabla^l \underset{q}{u}_{j_q}$ is hybrid in ${}^{l}_{j_q}$, the right hand side of (4.10) vanishes.

Similarly, we have

$$(4.11)\qquad \nabla^l\nabla_l(\underset{1}{v}^{i_1}\cdots \underset{p}{v}^{i_p}\underset{1}{u}_{j_1}\cdots \underset{q}{u}_{j_q})$$

$$=\sum_{t=1}^{p}(\nabla^l\nabla_l \underset{t}{v}^{i_t})(\underset{1}{v}^{i_1}\cdots \underset{t-1}{v}^{i_{t-1}}\underset{t+1}{v}^{i_{t+1}}\cdots \underset{p}{v}^{i_p}\underset{1}{u}_{j_1}\cdots \underset{q}{u}_{j_q})+\sum_{r=1}^{q}(\nabla^l\nabla_l \underset{t}{u}_{j_t})\underset{1}{v}^{i_1}\cdots \underset{p}{v}^{i_p}\underset{1}{u}_{j_1}\cdots \underset{t-1}{u}_{j_{t-1}}\underset{t+1}{u}_{j_{t+1}}\cdots \underset{q}{u}_{j_q}.$$

On the other hand, from theorem 4.1., we have

$$\nabla^l\nabla_l \underset{t}{v}^{i_t}=-E^*{}_s{}^{i_t}\underset{t}{v}^s \text{ and } \nabla^l\nabla_l \underset{t}{u}_{j_t}=E^*{}_{j_t}{}^s \underset{t}{u}_s.$$

Consequently,

$$\Delta(T_{i_1\cdots i_p}{}^{j_1\cdots j_q}\underset{1}{v}^{i_1}\cdots \underset{p}{v}^{i_p}\underset{1}{u}_{j_1}\cdots \underset{q}{u}_{j_q})$$

$$=T_{i_1\cdots i_p}{}^{j_1\cdots j_q}[-\sum_{t=1}^{p}E^*_s{}^{i_t}\underset{t}{v}^s\underset{1}{v}^{i_1}\cdots \underset{t-1}{v}^{i_{t-1}}\underset{t+1}{v}^{i_{t+1}}\cdots \underset{p}{v}^{i_p}\underset{1}{u}_{j_1}\cdots \underset{q}{u}_{j_q}+\sum_{t=1}^{q}E^*{}_{j_t}{}^s\underset{t}{u}_s\underset{1}{v}^{i_1}\cdots \underset{p}{v}^{i_p}\underset{1}{u}_{j_1}\cdots \underset{t-1}{u}_{j_{t-1}}\underset{t+1}{u}_{j_{t+1}}\cdots \underset{q}{u}_{j_q}]$$

$$-\sum_{t=1}^{q}E^*{}_s{}^{j_t}T_{i_1\cdots i_p}{}^{j_1\cdots s\cdots j_q}\underset{1}{v}^{i_1}\cdots \underset{p}{v}^{i_p}\underset{1}{u}_{j_1}\cdots \underset{q}{u}_{j_q}+\sum_{t=1}^{p}E^*{}_{i_t}{}^s T_{i_1\cdots s\cdots i_p}{}^{j_1\cdots j_q}\underset{1}{v}^{i_1}\cdots \underset{p}{v}^{i_p}\underset{1}{u}_{j_1}\cdots \underset{q}{u}_{j_q}=0.$$

Similarly, we have the following

THEOREM 4.3. *If, in a compact Hermitian manifold $X_{2n}$ with the torsion tensor satisfying $S_{ji}{}^{i}=0$, $\overset{1}{T}_{i_1\cdots i_p}{}^{j_1\cdots j_q}$ and $\overset{2}{T}_{j_1\cdots j_q}{}^{i_1\cdots i_p}$ are analytic pure tensors, then we haue*

$$\overset{1}{T}_{i_1\cdots i_p}{}^{j_1\cdots j_q}\overset{2}{T}_{j_1\cdots j_q}{}^{i_1\cdots i_p}=constant.$$

Now, we shall call an anti-symmetric tensor $T_{i_1\cdots i_p}$ pseudo-harmonic if it satisfies the condition:

$$\nabla_{[r}T_{i_1\cdots i_p]}=0 \text{ and } g^{rs}\nabla_s T_{ri_2\cdots i_p}=0 \tag{4.12}$$

and in a compact orientable metric manifold whose torsion tensor $S_{jih}$ satisfies $S_{ji}{}^{i}=0$, an anti-symmetric tensor $T_{i_1\cdots i_p}$ is pseudo-harmonic if and only if

$$\nabla^l\nabla_l T_{i_1\cdots i_p}-\sum_{r=1}^{p}E_{i_r}{}^{t}T_{i_1\cdots t\cdots i_p}+\sum_{s<t}^{p}T_{i_1\cdots i_{s-1}ai_{s+1}\cdots i_{t-1}bi_{t+1}\cdots i_p}(E^{a}{}_{i_t i_s}{}^{b}-E^{a}{}_{i_s i_t}{}^{b})$$
$$+2\sum_{s=1}^{p}(\nabla_t T_{i_1\cdots i_{s-1}ai_{s+1}\cdots i_p})S_{i_s}{}^{at}=0 \tag{4.13}$$ [6].

Here, let $T_{i_1\cdots i_p}$ be an anti-symmetric pure tensor in $X_{2n}$. Since, in $X_{2n}$, $\nabla_l T_{i_1\cdots i_p}$ and $\nabla^l\nabla_l T_{i_1\cdots i_p}$ are also pure in $i_1\cdots i_p$, $(\nabla_t T_{i_1\cdots a\cdots i_p})S_{i_s}{}^{at}$ is pure in $t$, $a$ but $S_{i_s}{}^{at}$ is hybrid in $a$, $t$ and therefore $(\nabla_t T_{i_1\cdots a\cdots i_p})S_{i_s}{}^{at}$ vanishes. Moreover, since $E^{a}{}_{i_s i_t}{}^{b}$ is pure in $i_s$, $i_t$, if $S_{ji}{}^{h}$ is analytic, then, by (2.15), we have

$$E^{a}{}_{i_s i_t}{}^{b}=E^{a}{}_{i_t i_s}{}^{b}.$$

Hence, from (4.13), we find that in a compact Hermitian manifold $X_{2n}$ with the analytic torsion tensor $S_{ji}{}^{h}$ satisfying $S_{ji}{}^{i}=0$, a necessary and sufficient condition that an anti-symmetric pure tensor $T_{i_1\cdots i_p}$ be pseudo-harmonic is that $T_{i_1\cdots i_p}$ satisfy

$$\nabla^l\nabla_l T_{i_1\cdots i_p}-\sum_{r=1}^{p}E_{i_r}{}^{a}T_{i_1\cdots a\cdots i_p}=0 \tag{4.14}$$

Therefore, from the theorem 4.1, we have the following theorem (Cf. [1]).

THEOREM 4.4. *In a compact Hermitian manifold $X_{2n}$ with the analytic torsion tensor $S_{ji}{}^{h}$ satisfying $S_{ji}{}^{i}=0$, an anti-symmetric pure tensor $T_{i_1\cdots i_p}$ is analytic if and only if $T_{i_1\cdots i_p}$ is pseudo-harmonic.*

## §5. The equation $g^{ji}\nabla_j\nabla_i f=\lambda f$

Let $X_{2n}$ be the same manifold that we considered in the preceding paragraph and $v^i$ be an arbitrary vector in $X_{2n}$. Then, by virtue of the Ricci's identity, we have

(5.1)
$$\nabla_j(v^k\nabla_k v^j)=v^k\nabla_j\nabla_k v^j+(\nabla_j v_k)\nabla^k v^j$$
$$=v^k(\nabla_k\nabla_j v^j+E_{jks}{}^j v^s-2S_{jk}{}^a\nabla_a v^j)+(\nabla_j v_k)\nabla^k v^j$$
$$=v^k\nabla_k\nabla_j v^j+E_{ks}v^s v^k+(\nabla_j v_k)\nabla^k v^j-2v^k S_{jk}{}^a\nabla_a v^j$$

and

(5.2)
$$\nabla_k(v^k\nabla_j v^j)=v^k\nabla_k\nabla_j v^j+(\nabla_k v^k)\nabla_j v^j.$$

If $X_{2n}$ is compact and $S_{ji}{}^i=0$, then, integrating (5.1)-(5.2) on the whole space, by virtue of Green's theorem, we have

(5.3)
$$\int_{X_{2n}}[E_{ks}v^k v^s+(\nabla_j v_k)\nabla^k v^j-(\nabla_k v^k)\nabla_j v^j-2v_k(\nabla_a v_j)S^{jka}]d\sigma=0$$

and similarly from $\nabla_k(v_j\nabla^k v^j)$,

(5.4)
$$\int_{X_{2n}}[v_j\nabla_l\nabla^l v^j+(\nabla_k v_j)\nabla^k v^j]d\sigma=0.$$

In this place, forming (5.3)+(1+$\varepsilon$)×(5.4) where $\varepsilon$ is an arbitrary positive constant, we get

(5.5)
$$\int_{X_{2n}}[E_{ij}v^i v^j+(1+\varepsilon)v_j\nabla_l\nabla^l v^j+(1+\varepsilon)(\nabla_k v_j)\nabla^k v^j+(\nabla_j v_k)\nabla^k v^j$$
$$-(\nabla^k v_k)\nabla_j v^j-2v_k S^{jka}\nabla_a v_j]d\sigma=0$$

but since $\frac{1}{2}(\nabla^k v^j+\nabla^j v^k)(\nabla_k v_j+\nabla_j v_k)=(\nabla_k v_j)\nabla^k v^j+(\nabla_j v_k)\nabla^k v^j$, (5.5) becomes

(5.6)
$$\int_{X_{2n}}[E_{ij}v^i v^j+(1+\varepsilon)v_j\nabla_l\nabla^l v^j+\varepsilon(\nabla_k v_j)\nabla^k v^j+\frac{1}{2}(\nabla^k v^j+\nabla^j v^k)(\nabla_k v_j+\nabla_j v_k)$$
$$-(\nabla^k v_k)\nabla_j v^j-2v_k S^{jka}\nabla_a v_j]d\sigma=0.$$

Now, moreover assuming that $S_{ji}{}^h$ is analytic, we consider an equation of the form

(5.7)
$$g^{ji}\nabla_j\nabla_i f=\lambda f \qquad (\lambda=\text{constant}<0)$$ [5]

or

$$g^{jk}\frac{\partial^2 f}{\partial z^j\partial z^k}-g^{jk}E_{jk}{}^i\frac{\partial f}{\partial z^i}=\lambda f$$

here, since $g^{jk}$ is hybrid in $j$, $k$ and $E_{jk}{}^i$ is pure in $j$, $k$, $g^{jk}E_{jk}{}^i=0$ and therefore (5.7) can be also written as

(5.8)
$$g^{jk}\frac{\partial^2 f}{\partial z^j\partial z^k}=\lambda f.$$

And from (5.7), we have

(5.9)
$$\nabla_h\nabla_l\nabla^l f-\lambda\nabla_h f=0$$

but, by the Ricci's identity, we get

$$\begin{aligned}(5.10)\qquad \nabla_h\nabla_l\nabla^l f &= \nabla_l\nabla_h\nabla^l f - E_{lhs}{}^l\nabla^s f - 2S_{hl}{}^a\nabla_a\nabla^l f\\ &= \nabla^l(\nabla_l\nabla_h f - 2S_{hl}{}^a\nabla_a f) - E_{hs}\nabla^s f - 2S_{hl}{}^a\nabla_a\nabla^l f\\ &= \nabla^l\nabla_l\nabla_h f - 2(\nabla^l S_{hl}{}^a)\nabla_a f - 2S_{hl}{}^a\nabla^l\nabla_a f - E_{hs}\nabla^s f\\ &\quad -2S_{hl}{}^a\nabla_a\nabla^l f\end{aligned}$$

and, from the assumption that $S_{ji}{}^h$ is analytic, $\nabla_k S_{hl}{}^a$ is pure in $k$, $l$ and hence

$$(5.11)\qquad \nabla^l S_{hl}{}^a = g^{kl}\nabla_k S_{hl}{}^a = 0.$$

Thus, (5.9) can be written in the following form

$$(5.12)\qquad \nabla^l\nabla_l\nabla_h f - 2S_{hl}{}^a\nabla^l\nabla_a f - E_h{}^s\nabla_s f - 2S_{hl}{}^a\nabla_a\nabla^l f - \lambda\nabla_h f = 0.$$

Next, transvecting (5.12) with $F^h_{\cdot i}$ and noticing $\nabla_j F^h_{\cdot i}=0$ where $F^h_{\cdot i}=F_{ri}g^{rh}$, we find

$$(5.13)\qquad \nabla^l\nabla_l(F^h_{\cdot i}\nabla_h f) - 2F^h_{\cdot i}S_{hl}{}^a\nabla^l\nabla_a f - F^h_{\cdot i}E_h{}^s\nabla_s f - 2F^h_{\cdot i}S_{hl}{}^a\nabla_a\nabla^l f$$
$$-\lambda F^h_{\cdot i}\nabla_h f = 0.$$

Here, if we put $v_h = F^l_{\cdot h}\nabla_l f$, then we have

$$F^h_{\cdot i}S_{hl}{}^a\nabla^l\nabla_a f = F^a_{\cdot h}S_{il}{}^h\nabla^l\nabla_a f = S_{il}{}^h\nabla^l(F^a_{\cdot h}\nabla_a f) = S_{il}{}^h\nabla^l v_h,$$

$$F^h_{\cdot i}S_{hl}{}^a\nabla_a\nabla^l f = F^h_{\cdot l}S_{ih}{}^a\nabla_a\nabla^l f = S_{ih}{}^a\nabla_a(F^h_{\cdot l}\nabla^l f) = -S_{ih}{}^a\nabla_a v^h$$

and

$$F^h_{\cdot i}E_h{}^s\nabla_s f = F^s_{\cdot h}E_i{}^h\nabla_s f = E_i{}^h v_h.$$

Consequently, again (5.13) can be written as

$$(5.14)\qquad \nabla^l\nabla_l v_i - 2S_{il}{}^h\nabla^l v_h - E_i{}^h v_h + 2S_{ih}{}^a\nabla_a v^h - \lambda v_i = 0.$$

Substituting this equation into the integrand of (5.6), we have

$$\begin{aligned}(5.15)\qquad & E_{ij}v^i v^j + (1+\varepsilon)v_j\nabla_l\nabla^l v^j + \varepsilon(\nabla_k v_j)\nabla^k v^j + \frac{1}{2}(\nabla^k v^j + \nabla^j v^k)(\nabla_k v_j + \nabla_j v_k)\\ & -(\nabla^k v_k)\nabla_j v^j - 2v_k S^{jka}\nabla_a v_j\\ & = E_{ij}v^i v^j + (2+2\varepsilon)v_j S^{jlh}\nabla_l v_h + (1+\varepsilon)v_j v_h E^{jh} - (2+2\varepsilon)v_j S^{j\cdot a}_{\ h}\nabla_a v^h + \lambda(1+\varepsilon)v_j v^j\\ & +\varepsilon(\nabla_k v_j)\nabla^k v^j + \frac{1}{2}(\nabla^k v^j + \nabla^j v^k)(\nabla_k v_j + \nabla_j v_k) - (\nabla^k v_k)\nabla^j v_j - 2v_k S^{jka}\nabla_a v_j\\ & = (2+\varepsilon)E_{ij}v^i v^j + \lambda(1+\varepsilon)v_j v^j + \frac{1}{2}(\nabla^k v^j + \nabla^j v^k)(\nabla_k v_j + \nabla_j v_k)\\ & +2(1+\varepsilon)(\nabla_l v_h + \nabla_h v_l)S^{jlh}v_j - 2(1+2\varepsilon)S^{jlh}v_j\nabla_h\nabla_l + \varepsilon(\nabla_k v_j)\nabla^k v^j\\ & -(\nabla_k v^k)\nabla_j v^j\end{aligned}$$

$$=(2+\varepsilon)E_{ij}v^iv^j+(1+\varepsilon)\lambda v^iv_i+\frac{1}{2}[\nabla_k v_j+\nabla_j v_k-2(1+\varepsilon)v^iS_{jik}]$$
$$\times[\nabla^k v^j+\nabla^j v^k-2(1+\varepsilon)v_aS^{jak}]-2(1+\varepsilon)^2S^{jak}S_{jik}v_av^i$$
$$+\varepsilon(\nabla_k v_j+\frac{1+2\varepsilon}{\varepsilon}v^iS_{jik})(\nabla^k v^j+\frac{1+2\varepsilon}{\varepsilon}v_aS^{jak})-\frac{(1+2\varepsilon)^2}{\varepsilon}S^{jak}S_{j\cdot k}^{\ i}v_av_i-(\nabla_k v^k)\nabla_j v^j$$
$$=[(2+\varepsilon)E_{ij}+(1+\varepsilon)\lambda g_{ji}-\frac{2\varepsilon^3+8\varepsilon^2+6\varepsilon+1}{\varepsilon}S_{lim}S_j^{l\cdot m}]v^iv^j$$
$$+\frac{1}{2}[\nabla_k v_j+\nabla_j v_k-2(1+\varepsilon)v^iS_{jik}][\nabla^k v^j+\nabla^j v^k-2(1+\varepsilon)v_aS^{jak}]$$
$$+\varepsilon(\nabla_k v_j+\frac{1+2\varepsilon}{\varepsilon}v^iS_{jik})(\nabla^k v^j+\frac{1+2\varepsilon}{\varepsilon}v_aS^{jak})-(\nabla_k v^k)\nabla_j v^j$$

and

(5.16)
$$\nabla^k v_k=F^{lk}\nabla_k\nabla_l f$$
$$=\frac{1}{2}F^{lk}(\nabla_k\nabla_l f-\nabla_l\nabla_k f)$$
$$=F^{lk}S_{kl}{}^a\nabla_a f=0,$$

because $F^{lk}$ is hybrid in $l$, $k$ and $S_{kl}{}^a$ is pure in $k$, $l$ and therefore (5.6) becomes

(5.17)
$$\int_{X_{2n}}\Big[\{(2+\varepsilon)E_{ji}+(1+\varepsilon)\lambda g_{ji}-\frac{2\varepsilon^3+8\varepsilon^2+6\varepsilon+1}{\varepsilon}S_{lim}S^{l\cdot m}_{\ \ j}\}v^iv^j$$
$$+\frac{1}{2}\{\nabla_k v_j+\nabla_j v_k-2(1+\varepsilon)v^iS_{jik}\}\{\nabla^k v^j+\nabla^j v^k-2(1+\varepsilon)v_aS^{jak}\}$$
$$+\varepsilon(\nabla_k v_j+\frac{1+2\varepsilon}{\varepsilon}v^iS_{jik})(\nabla^k v^j+\frac{1+2\varepsilon}{\varepsilon}v_aS^{jak})\Big]d\sigma=0.$$

Thus, we have the following

THEOREM 5.1. *If, in a compact Hermitian manifold $X_{2n}$ with the analytic torsion tensor satisfying $S_{ji}{}^i=0$, then the form*

(5.18)
$$\Big[(2+\varepsilon)E_{ji}+(1+\varepsilon)\lambda g_{ji}-\frac{2\varepsilon^3+8\varepsilon^2+6\varepsilon+1}{\varepsilon}S_{lim}S^{l\cdot m}_{\ \ j}\Big]v^iv^j$$

*($\varepsilon$ = an arbitrary constant $>0$) is positive definite, then the quation*

$$g^{ji}\nabla_j\nabla_i f=\lambda f \qquad (\lambda=\text{constant}<0)$$

*has no solution other than zero.*

Here, since $\varepsilon$ is arbitrary, say, putting $\varepsilon=1$, (5.18) becomes

(5.19)
$$(3E_{ji}+2\lambda g_{ji}-17S_{lim}S^{l\cdot m}_{\ \ j})v^iv^j.$$

When the torsion tensor vanishes, that is, when $X_{2n}$ is a Kählerian manifold, (5.18) becomes

(5.20)
$$[(2+\varepsilon)E_{ji}+(1+\varepsilon)\lambda g_{ji}]v^iv^j.$$

In this case, since $\varepsilon$ is arbitrary, from (5.20), we have

$$(2E_{ji}+\lambda g_{ji})v^i v^j, \tag{5.21}$$

that is, in a Kählerian manifold, if (5.20) is positive definite, then (5.7) has no solution other than zero.

This is a well known result in a compact Kählerian manifold [5].

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