Homotopy theory of c. s. s. pairs and triads

By

Kiyoshi AOKI, Eiitirô HONMA and Tetuo KANEKO

(Received July 9, 1959)

Our main purpose is to give the definitions of the homotopy groups of a c. s. s. pair (or triad) and of homotopy between maps of one of c. s. s. pair (or triad) into another after the combinatorial manner as [5] [6]. And we study the fundamental properties of these notions, for example, those properties mentioned by S. T. Hu [2] as the axioms of homotopy theory of topological space, or the exactness of the lower and upper homotopy sequences of c. s. s. triad. All of these properties may be verified combinatorially.

§ 1. The homotopy gruops of c. s. s. pairs.

In this note, K_n means the collection of all n-simplices of c. s. s. complex K, $\sigma\varepsilon^i$ and $\sigma\eta^i$ mean the *i-th face* and the *i-th degeneracy* of simplex σ.

DEFINITION 1.1. For $\sigma_i \in K_n$, symbol

$$\begin{matrix} (0) & \cdots & (l-1) & (l) & (l+1) & \cdots & (n+1) \\ [\sigma_0, & \cdots, & \sigma_{l-1}, & \square, & \sigma_{l+1}, & \cdots, & \sigma_{n+1}] \end{matrix}$$

is called an *equation in K*, and means that $\sigma_i\varepsilon^{j-1}=\sigma_j\varepsilon^i$, $0\leqq i<j\leqq n+1$, $i, j\neq l$ (*match condition*). If there exists $\sigma\in K_{n+1}$ such that $\sigma\varepsilon^i=\sigma_i$, $0\leqq i\leqq n+1$, $i\neq l$, σ and $\sigma\varepsilon^l$ are called *solvent* and *solution* of this equation respectively.

If each equation in K has at least one solvent, K is called a *Kan complex.* (This is a complex which satisfies the *extension condition* [3] [4]).

D. M. Kan [5] gave the following definition:

DEFINITION 1.2. Two simplices σ and τ of K_n ($n\geqq 0$) is called *homotopic* (notation $\sigma\sim\tau$ or $\rho:\sigma\sim\tau$) if

(a) their faces coincide, i.e. $\sigma\varepsilon^i=\tau\varepsilon^i$ for all i

(b) there exists $\rho\in K_{n+1}$ such that $\rho\varepsilon^n=\sigma$, $\rho\varepsilon^{n+1}=\tau$ and $\rho\varepsilon^i=\sigma\varepsilon^i\eta^{n-1}=\tau\varepsilon^i\eta^{n-1}$, $0\leqq i\leqq n-1$.

We have

PROPOSITION 1.3. *Let K be a Kan complex. Two n-simplices σ and τ of K are homotopic if and only if*

(a) $\sigma\varepsilon^i=\tau\varepsilon^i$ *for all i*

(b′) *there exists* $\rho'\in K_{n+1}$ *such that* $\rho'\varepsilon^{l-1}=\sigma$, $\rho'\varepsilon^{l}=\tau$ *and* $\rho'\varepsilon^{i}=\sigma\eta^{l-1}\varepsilon^{i}=\tau\eta^{l-1}\varepsilon^{i}$, $0\leqq i\leqq n+1$, $i\neq l-1$, l, *where* $1\leqq l\leqq n$.

Proof. Assume that $\rho:\sigma\sim\tau$. Put $\sigma_i=\sigma\eta^n\eta^{l-1}\varepsilon^i$, $0\leqq i\leqq n+1$, $i\neq l$. Then there exists the following equation in K:

$$\begin{matrix} (0) & \cdots & (l-1) & (l) & (l+1) & \cdots & (n+1) & (n+2) \\ [\sigma_0, & \cdots, & \sigma_{l-1}, & \rho, & \sigma_{l+1}, & \cdots, & \sigma_{n+1}, & \square] \end{matrix}$$

and let $\rho'\in K_{n+1}$ be a solution of this equation. We can see easily that $\rho'\varepsilon^{l-1}=\sigma$, $\rho'\varepsilon^{l}=\tau$ and $\rho'\varepsilon^{i}=\sigma\eta^{l-1}\varepsilon^{i}=\tau\eta^{l-1}\varepsilon^{i}$ for $i\neq l-1$, l. Thus the condition (b′) is satisfied.

Conversely assume that (a) and (b′) are satisfied, then a solution of the following equation in K:

$$\begin{matrix} (0) & \cdots & (l-1) & (l) & (l+1) & \cdots & (n+1) & (n+2) \\ [\sigma_0, & \cdots, & \sigma_{l-1}, & \square, & \sigma_{l+1}, & \cdots, & \sigma_{n+1}, & \rho'] \end{matrix}$$

is the homotopy of σ and τ. The proof is complete.

If K and L are c. s. s. complexes such that L is a subcomplex of K, (K, L) is called a *c. s. s. pair*. A c. s. s. pair (K, L) such that K and L are Kan complexes is called a *Kan pair*.

Let (K, L) be a c. s. s. pair and put

$$\Lambda_n(K, L)=\{\sigma:\sigma\in K_n,\ \sigma\varepsilon^0\in L_{n-1}\}\ \text{for}\ n\geqq 1.$$

DEFINITION 1.4. Two simplices σ and τ of $\Lambda_n(K, L)$ are called *homotopic loosened in* L (notation $\sigma\sim\tau$ *lsd.* L or $\rho:\sigma\sim\tau$ *lsd.* L) if

(a) $\sigma\varepsilon^i=\tau\varepsilon^i$ for all $i\neq 0$

(b) there exists ρ in $\Lambda_{n+1}(K, L)$ such that $\rho\varepsilon^n=\sigma$, $\rho\varepsilon^{n+1}=\tau$ and $\rho\varepsilon^i=\sigma\varepsilon^i\eta^{n-1}=\tau\varepsilon^i\eta^{n-1}$ for $1\leqq i\leqq n-1$.

We call ρ the *homotopy loosened in* L between σ and τ.

PROPOSITION 1.5. *Let* (K, L) *be a Kan pair. Then the relation* $\sim$*lsd.* L *is an equivalence relation on the simplices of* $\Lambda_n(K, L)$ *for all* $n\geqq 1$.

Proof. It is clear that $\rho=\sigma\eta^n:\sigma\sim\sigma$ lsd. L. Now let us show that $\rho_n:\sigma\sim\tau$ lsd. L and $\rho_{n+1}:\sigma\sim\omega$ lsd. L imply $\tau\sim\omega$ lsd. L. Put $\sigma_i=\sigma\eta^n\eta^n\varepsilon^i$, then the following table

$$\begin{matrix} (0) & \cdots & (i) & \cdots & (n-2) & (n-1) & (n) & (n+1) \\ [\sigma_1\varepsilon^0, & \cdots, & \sigma_{i+1}\varepsilon^0, & \cdots, & \sigma_{n-1}\varepsilon^0, & \sigma_n\varepsilon^0, & \sigma_{n+1}\varepsilon^0, & \square] \end{matrix}$$

is an equation in L. Let $\gamma\in L_{n+1}$ be a solvent of this equation. Then

$$\begin{matrix} (0) & (1) & \cdots & (i) & \cdots & (n-1) & (n) & (n+1) & (n+2) \\ [\gamma, & \sigma_1, & \cdots, & \sigma_i, & \cdots, & \sigma_{n-1}, & \rho_n, & \rho_{n+1}, & \square] \end{matrix}$$

is an equation in K. Let $\beta\in K_{n+1}$ be a solution of this equation, then we can see that $\beta:\tau\sim\omega$ lsd. L.

PROPOSITION 1.6. *Let* (K, L) *be a Kan pair. Two simplices* σ *and* τ *of* $\Lambda_n(K, L)$

are homotopic loosened in L if and only if

(a) $\sigma\varepsilon^i=\tau\varepsilon^i$ *for all* $i\neq 0$

(b′) *there exists* $\rho'\in\Lambda_{n+1}(K, L)$ *such that* $\rho'\varepsilon^{l-1}=\sigma$, $\rho'\varepsilon^l=\tau$ *and* $\rho'\varepsilon^i=\sigma\eta^{l-1}\varepsilon^i=\tau\eta^{l-1}\varepsilon^i$ *for* $i\neq 0, l-1, l$, *where* $2\leqq l\leqq n$.

Proof. Assume that $\sigma\sim\tau$ lsd. L and let ρ be its homotopy. Put $\sigma_i=\sigma\eta^n\eta^{l-1}\varepsilon^i$. Then there is the following equation in L:

$$\overset{(0)}{[\sigma_1\varepsilon^0}, \overset{\cdots}{\cdots}, \overset{(l-2)}{\sigma_{l-1}\varepsilon^0}, \overset{(l-1)}{\rho\varepsilon^0}, \overset{(l)}{\sigma_{l+1}\varepsilon^0}, \overset{\cdots}{\cdots}, \overset{(n)}{\sigma_{n+1}\varepsilon^0}, \overset{(n+1)}{\square]}.$$

Let $\gamma\in L_{n+1}$ be a solvent of this equation. Then a solution $\rho'\in K_{n+1}$ of the equation

$$\overset{(0)}{[\gamma}, \overset{(1)}{\sigma_1}, \overset{(2)}{\sigma_2}, \overset{\cdots}{\cdots}, \overset{(l-1)}{\sigma_{l-1}}, \overset{(l)}{\rho}, \overset{(l+1)}{\sigma_{l+1}}, \overset{\cdots}{\cdots}, \overset{(n+1)}{\sigma_{n+1}}, \overset{(n+2)}{\square]}$$

satisfies the conditions in (b′).

Conversely assume that (a) and (b′) are satisfied. Consider the following equation in L:

$$\overset{(0)}{[\sigma_1\varepsilon^0}, \overset{\cdots}{\cdots}, \overset{(l-2)}{\sigma_{l-1}\varepsilon^0}, \overset{(l-1)}{\square}, \overset{(l)}{\sigma_{l+1}\varepsilon^0}, \overset{\cdots}{\cdots}, \overset{(n)}{\sigma_{n+1}\varepsilon^0}, \overset{(n+1)}{\rho'\varepsilon^0]}$$

and let $\gamma'\in L_{n+1}$ be its solvent. Then a solution $\rho\in K_{n+1}$ of the equation in K:

$$\overset{(0)}{[\gamma'}, \overset{(1)}{\sigma_1}, \overset{\cdots}{\cdots}, \overset{(l-1)}{\sigma_{l-1}}, \overset{(l)}{\square}, \overset{(l+1)}{\sigma_{l+1}}, \overset{\cdots}{\cdots}, \overset{(n+1)}{\sigma_{n+1}}, \overset{(n+2)}{\rho']}$$

is the homotopy loosened in L between σ and τ.

DEFINITION 1.7. Let (K, L) be a Kan pair and $\varphi\in L_0$. For every integer $n\geqq 1$, we define a set $\pi_n(K, L, \varphi)$ as follows. Put $\Gamma_n(K, L, \varphi)=\{\sigma:\sigma\in\Lambda_n(K, L), \sigma\varepsilon^i=\varphi\eta^0\cdots\eta^{n-2}$ *for* $1\leqq i\leqq n\}$. The equivalence relation $\sim$ lsd. L divides $\Gamma_n(K, L, \varphi)$ into classes. Then $\pi_n(K, L, \varphi)$ will be the set of these equivalence classes, i.e.

$$\pi_n(K, L, \varphi)=\Gamma_n(K, L, \varphi)/(\sim\text{lsd. } L).$$

The class containing a simplex σ will be denoted by $\{\sigma\}_{K,L}$.

$\pi_0(K, L, \varphi)$ denotes the set of components of K [5] indexed by L.

For $n\geqq 2$ the set $\pi_n(K, L, \varphi)$ may be converted into a group (the n-th homotopy group of $(K L)$ relative to φ) as follows. Let $\sigma\in a$ and $\tau\in b$ be n-simplices in the classes a and $b\in\pi_n(K, L, \varphi)$. There is the following equation in L:

$$\overset{(0)}{[\varphi\eta^0\cdots\eta^{n-2}}, \overset{\cdots}{\cdots}, \overset{(n-3)}{\varphi\eta^0\cdots\eta^{n-2}}, \overset{(n-2)}{\sigma\varepsilon^0}, \overset{(n-1)}{\square}, \overset{(n)}{\tau\varepsilon^0]}.$$

Let $\gamma\in L_n$ be one of its solvents. We can consider the equation in K;

$$\overset{(0)}{[\gamma}, \overset{(1)}{\varphi\eta^0\cdots\eta^{n-1}}, \overset{\cdots}{\cdots}, \overset{(n-2)}{\varphi\eta^0\cdots\eta^{n-1}}, \overset{(n-1)}{\sigma}, \overset{(n)}{\square}, \overset{(n+1)}{\tau]},$$

and let $\beta \in K_n$ be a solution of this equation. The product $a \cdot b$ is defined by

$$a \cdot b = \{\beta\}_{K,L}.$$

PROPOSITION 1.8. *The multiplication defined above is independent of the choice of σ, τ, γ and β.*

Furthermore

PROPOSITION 1.9. *The multiplication defined above converts $\pi_n(K, L, \varphi)$ into a group for $n \geqq 2$.*

The proofs of PROPOSITIONS 1.8 and 1.9 will be given in §3.

DEFINITION 1.10. Let (K, L, φ) be a c. s. s. pair with base point φ, let $|K|$ denote the geometric realization of K in the sense of J. W. Milnor [7]. Let S be the functor which assings to every topological space its simplicial singular complex. For every integer $n \geqq 0$, we then define $\pi_n(K, L, \varphi)$ by $\pi_n(S|K|, S|L|, i(\varphi))$ (the set in the sense of DEFINITION 1.7 for the Kan pair $(S|K|, S|L|)$ with base point $i(\varphi)$, where $i: K \rightarrow S|K|$ is the natural embedding map [7]).

REMARK 1.11. If (K, L) is a Kan pair, the DEFINITIONS 1.7 and 1.10 of $\pi_n(K, L, \varphi)$ may be seen to be equivalent for all n.

PROPOSITION 1.12. *$\pi_n(K, L, \varphi)$ is abelian for $n \geqq 3$.*

The proof will be given in §4.

§2. The homotopy groups of c.s.s. triads.

If K is a c. s. s. complex and if L, M are subcomplexes of K such that $L \cap M \neq 0$. Then $(K; L, M)$ is called a *c. s. s. traid.* A c. s. s. traid $(K; L, M)$ such that K, L, M and $L \cap M$ are Kan complexes is called a *Kan traid.*

Let $(K; L, M)$ be a c. s. s. triad and put

$$\Lambda_n\{K; L, M) = \{\sigma : \sigma \in K_n, \ \sigma\varepsilon^0 \in L_{n-1}, \ \sigma\varepsilon^1 \in M_{n-1}\}$$

for $n \geqq 2$.

DEFINITION 2.1. Two simplices σ and τ of $\Lambda_n(K; L, M)$ are called *homotopic loosened in L, M* (notation $\sigma \sim \tau$ *lsd. L, M* or $\rho : \sigma \sim \tau$ *lsd. L, M*) if

(a) $\sigma\varepsilon^i = \tau\varepsilon^i$ for all $i \neq 0, 1$

(b) there exists ρ in $\Lambda_{n+1}(K; L, M)$ such that $\rho\varepsilon^n = \sigma$, $\rho\varepsilon^{n+1} = \tau$ and $\rho\varepsilon^i = \sigma\varepsilon^i\eta^{n-1} = \tau\varepsilon^i\eta^{n-1}$ for $2 \leqq i \leqq n-1$.

We call ρ the *homotopy loosened in L, M* between σ and τ.

PROPOSITION 2.2. *Let $(K; L, M)$ be a Kan triad. Then the relation $\sim$ lsd. L, M is an equivalence relation on the simplices of $\Lambda_n(K; L, M)$ for all $n \geqq 2$.*

Proof. It is clear that $\rho = \sigma\eta^n : \sigma \sim \sigma$ lsd. L, M. It remains to show that $\rho_n : \sigma \sim \tau$ lsd. L, M and $\rho_{n+1} : \sigma \sim \omega$ lsd. L, M imply $\tau \sim \omega$ lsd. L, M. Put $\sigma_i = \sigma\eta^n\eta^n\varepsilon^i$, then there exists the following equation in $L \cap M$:

$$\overset{(0)}{[\sigma_2\varepsilon^0\varepsilon^0}, \cdots, \overset{(i)}{\sigma_{i+2}\varepsilon^0\varepsilon^0}, \cdots, \overset{(n-3)}{\sigma_{n-1}\varepsilon^0\varepsilon^0}, \overset{(n-2)}{\rho_n\varepsilon^0\varepsilon^0}, \overset{(n-1)}{\rho_{n+1}\varepsilon^0\varepsilon^0}, \overset{(n)}{\square}].$$

Let $\theta \in (L \cap M)_n$ be one of its solvent. Let $\gamma \in L_{n+1}$ and $\gamma' \in M_{n+1}$ be solvents of the following equations

$$\overset{(0)}{[\theta}, \overset{(1)}{\sigma_2\varepsilon^0}, \cdots, \overset{(i)}{\sigma_{i+1}\varepsilon^0}, \cdots, \overset{(n-2)}{\sigma_{n-1}\varepsilon^0}, \overset{(n-1)}{\rho_n\varepsilon^0}, \overset{(n)}{\rho_{n+1}\varepsilon^0}, \overset{(n+1)}{\square}]$$

and

$$\overset{(0)}{[\theta}, \overset{(1)}{\sigma_2\varepsilon^1}, \cdots, \overset{(i)}{\sigma_{i+1}\varepsilon^1}, \cdots, \overset{(n-2)}{\sigma_{n-1}\varepsilon^1}, \overset{(n-1)}{\rho_n\varepsilon^1}, \overset{(n)}{\rho_{n+1}\varepsilon^1}, \overset{(n+1)}{\square}]$$

respectively.

Then the solution $\beta \in K_{n+1}$ of the equation in K:

$$\overset{(0)}{[\gamma}, \overset{(1)}{\gamma'}, \overset{(2)}{\sigma_2} \cdots, \overset{(i)}{\sigma_i}, \cdots, \overset{(n-1)}{\sigma_{n-1}}, \overset{(n)}{\rho_n}, \overset{(n+1)}{\rho_{n+1}}, \overset{(n+2)}{\square}]$$

gives the required homotopy loosened in L, M between τ and ω.

PROPOSITION 2.3. *Let $(K; L, M)$ be a Kan triad. Two simplices σ and τ of $\Lambda_n(K; L, M)$ are homotopic loosened in L, M if and only if*

(a) $\sigma\varepsilon^i = \tau\varepsilon^i$ *for all* $i \neq 0, 1$

(b′) *there exists $\rho' \varepsilon \Lambda_{n+1}(K; L, M)$ such that $\rho'\varepsilon^{l-1} = \sigma$, $\rho'\varepsilon^l = \tau$ and $\rho'\varepsilon^i = \sigma\eta^{l-1}\varepsilon^i = \tau\eta^{l-1}\varepsilon^i$ for $i \neq 0, 1, l-1, l$, where $3 \leq l \leq n$.*

Proof. Assume that $\rho: \sigma \sim \tau$ lsd. L, M. Put $\sigma_i = \sigma\eta^n\eta^{l-1}\varepsilon^i$, then there exists the following equation in $L \cap M$:

$$\overset{(0)}{[\sigma_2\varepsilon^0\varepsilon^0}, \cdots, \overset{(l-3)}{\sigma_{l-1}\varepsilon^0\varepsilon^0}, \overset{(l-2)}{\rho\varepsilon^0\varepsilon^0}, \overset{(l-1)}{\sigma_{l+1}\varepsilon^0\varepsilon^0}, \cdots, \overset{(n-1)}{\sigma_{n+1}\varepsilon^0\varepsilon^0}, \overset{(n)}{\square}].$$

Let $\omega \in (L \cap M)_n$ be a solvent of this equation. Let $\beta \in L_{n+1}$ and $\gamma \in M_{n+1}$ be solvents of the following equations:

$$\overset{(0)}{[\omega}, \overset{(1)}{\sigma_2\varepsilon^0}, \cdots, \overset{(l-2)}{\sigma_{l-1}\varepsilon^0}, \overset{(l-1)}{\rho\varepsilon^0}, \overset{(l)}{\sigma_{l+1}\varepsilon^0}, \cdots, \overset{(n)}{\sigma_{n+1}\varepsilon^0}, \overset{(n+1)}{\square}]$$

and

$$\overset{(0)}{[\omega}, \overset{(1)}{\sigma_2\varepsilon^1}, \cdots, \overset{(l-2)}{\sigma_{l-1}\varepsilon^1}, \overset{(l-1)}{\rho\varepsilon^1}, \overset{(l)}{\sigma_{l+1}\varepsilon^1}, \cdots, \overset{(n)}{\rho_{n+1}\varepsilon^1}, \overset{(n+1)}{\square}]$$

respectively. Then a solution ρ' of the equation in K:

$$\overset{(0)}{[\beta}, \overset{(1)}{\gamma}, \overset{(2)}{\sigma_2}, \cdots, \overset{(l-1)}{\sigma_{l-1}}, \overset{(l)}{\rho}, \overset{(l+1)}{\sigma_{l+1}}, \cdots, \overset{(n+1)}{\sigma_{n+1}}, \overset{(n+2)}{\square}]$$

satisfies the conditions in (b′).

Conversely assume that (a) and (b′) are satisfied. Let $\omega'\in(L\cap M)_n$ be a solvent of the equation in $L\cap M$:

$$[\overset{(0)}{\sigma_2\varepsilon^0\varepsilon^0}, \overset{\cdots}{\cdots}, \overset{(l-3)}{\sigma_{l-1}\varepsilon^0\varepsilon^0}, \overset{(l-2)}{\square}, \overset{(l-1)}{\sigma_{l+1}\varepsilon^0\varepsilon^0}, \overset{\cdots}{\cdots}, \overset{(n-1)}{\sigma_{n+1}\varepsilon^0\varepsilon^0}, \overset{(n)}{\rho'\varepsilon^0\varepsilon^0}].$$

Let $\beta'\in L_{n+1}$ and $\gamma'\in M_{n+1}$ be solvents of the following equations:

$$[\overset{(0)}{\omega'}, \overset{(1)}{\sigma_2\varepsilon^0}, \overset{\cdots}{\cdots}, \overset{(l-2)}{\sigma_{l-1}\varepsilon^0}, \overset{(l-1)}{\square}, \overset{(l)}{\sigma_{l+1}\varepsilon^0}, \overset{\cdots}{\cdots}, \overset{(n)}{\sigma_{n+1}\varepsilon^0}, \overset{(n+1)}{\rho'\varepsilon^0}]$$

and

$$[\overset{(0)}{\omega'}, \overset{(1)}{\sigma_2\varepsilon^1}, \overset{\cdots}{\cdots}, \overset{(l-2)}{\sigma_{l-1}\varepsilon^1}, \overset{(l-1)}{\square}, \overset{(l)}{\sigma_{l+1}\varepsilon^1}, \overset{\cdots}{\cdots}, \overset{(n)}{\sigma_{n+1}\varepsilon^1}, \overset{(n+1)}{\rho'\varepsilon^1}]$$

respectively. Then a solution $\rho\in K_{n+1}$ of the equation in K:

$$[\overset{(0)}{\beta'}, \overset{(1)}{\gamma'}, \overset{(2)}{\sigma_2}, \overset{\cdots}{\cdots}, \overset{(l-1)}{\sigma_{l-1}}, \overset{(l)}{\square}, \overset{(l+1)}{\sigma_{l+1}}, \overset{\cdots}{\cdots}, \overset{(n+1)}{\sigma_{n+1}}, \overset{(n+2)}{\rho'}]$$

gives the homotopy loosened in L, M between σ and τ.

DEFINITION 2.4. Let $(K;\ L,\ M)$ be a Kan triad and φ be a 0-simplex of $L\cap M$ (the base point).

For every integer $n\geqq 2$, we define a set $\pi_n(K;\ L,\ M,\ \varphi)$ as follows. Put $\Gamma_n(K;\ L,\ M,\ \varphi)=\}\sigma:\sigma\in\Lambda_n(K;\ L,\ M),\ \sigma\varepsilon^i=\varphi\eta^0\cdots\eta^{n-2}$ for $2\leqq i\leqq n\}$. The equivalence relation $\sim$ lsd. L, M divides $\Gamma_n(K;\ L,\ M,\ \varphi)$ into classes. Then $\pi_n(K;\ L,\ M,\ \varphi)$ will be the set of these equivalence classes, i.e. $\pi_n(K;\ L,\ M,\ \varphi)=\Gamma_n(K;\ L,\ M,\ \varphi)/(\sim$ lsd. L, $M)$. The class containing a simplex σ will be denoted by $\{\sigma\}_{K;L,M}$.

For $n\geqq 3$ the set $\pi_n(K;\ L,\ M,\ \varphi)$ may be converted into a group (the n-th homotopy group of $(K;\ L,\ M)$ relative to φ) as follows. Let $\sigma\in a$ and $\tau\in b$ be n-simplices in the classes a and $b\in\pi_n(K;\ L,\ M,\ \varphi)$. There exists the following equation in $L\cap M$:

$$[\overset{(0)}{\varphi\eta^0\cdots\eta^{n-3}}, \overset{\cdots}{\cdots}, \overset{(n-4)}{\varphi\eta^0\cdots\eta^{n-3}}, \overset{(n-3)}{\sigma\varepsilon^0\varepsilon^0}, \overset{(n-2)}{\square}, \overset{(n-1)}{\tau\varepsilon^0\varepsilon^0}].$$

Let $\theta\in(L\cap M)_{n-1}$ be one of its solvents. Let $\gamma_0\in L_n$ and $\gamma_1\in M_n$ be solvents of the following equations:

$$[\overset{(0)}{\theta}, \overset{(1)}{\varphi\eta^0\cdots\eta^{n-2}}, \overset{\cdots}{\cdots}, \overset{(n-3)}{\varphi\eta^0\cdots\eta^{n-2}}, \overset{(n-2)}{\sigma\varepsilon^0}, \overset{(n-1)}{\square}, \overset{(n)}{\tau\varepsilon^0}]$$

and

$$[\overset{(0)}{\theta}, \overset{(1)}{\varphi\eta^0\cdots\eta^{n-2}}, \overset{\cdots}{\cdots}, \overset{(n-3)}{\varphi\eta^0\cdots\eta^{n-2}}, \overset{(n-2)}{\sigma\varepsilon^1}, \overset{(n-1)}{\square}, \overset{(n)}{\tau\varepsilon^1}]$$

respectively. Then we define the product $a\cdot b$ by

$$a\cdot b=\{\beta\}_{K;L,M}$$

where $\beta\in K_n$ is a solution of the following equation in K:

$$\overset{(0)}{[\gamma_0,}\ \overset{(1)}{\gamma_1,}\ \overset{(2)}{\varphi\eta^{0}\cdots\eta^{n-1},}\ \overset{\cdots}{\cdots,}\ \overset{(n-2)}{\varphi\eta^{0}\cdots\eta^{n-1},}\ \overset{(n-1)}{\sigma,}\ \overset{(n)}{\square,}\ \overset{(n+1)}{\tau].}$$

PROPOSITION 2.5. *The multiplication defined above is independent of the choice of* σ, τ, θ, γ_0, γ_1 *and* β.

Furthermore

PROPOSITION 2.6. *The multiplication defined above converts* $\pi_n(K;\ L,\ M,\ \varphi)$ *into a group for* $n\geqq 3$.

The proofs of PROPOSITIONS 2.5 and 2.6 will be given in §3.

REMARK. The unit element of groups $\pi_n(K,\ \varphi)$, $\pi_n(K,\ L,\ \varphi)$ and $\pi_n(K;\ L,\ M,\ \varphi)$ are $\{\varphi\eta^{0}\cdots\eta^{n-1}\}_K$, $\{\varphi\eta^{0}\cdots\eta^{n-1}\}_{K,L}$ and $\{\varphi\eta^{0}\cdots\eta^{n-1}\}_{K;L,M}$ respectively.

DEFINITION 2.7. For a c. s. s. triad $(K;\ L,\ M)$ with base point φ, we define $\pi_n(K;\ L,\ M,\ \varphi)$ by $\pi_n(S|K|\ ;\ S|L|,\ S|M|,\ i(\varphi))$, the set in the sense of DEFINITION 2.4 for the Kan triad $(S|K|\ ;\ S|L|,\ S|M|,\ i(\varphi))$ with base point $i(\varphi)$.

REMARK. If $(K;\ L,\ M)$ is a Kan triad, the DEFINITIONS 2.4 and 2.7 of $\pi_n(K;\ L,\ M,\ \varphi)$ may be seen to be equivalent for all $n\geqq 2$.

We call DEFINITIONS 1.7 and 2.4 *the first definition,* DEFINITIONS 1.10 and 2.7 *the second definition.* Then in each theorem and proposition, either of these definitions of π_n may be used when we consider only Kan complexes, and only the second definition is used otherwise.

PROPOSITION 2.8. $\pi_n(K;\ L,\ M,\ \varphi)$ *is abelian for* $n\geqq 4$.

The proof will be given in §4.

THEOREM 2.9 *There exists a transformation* u: $\pi_n(K;\ L,\ M,\ \varphi)\rightarrow\pi_n(K;\ M,\ L,\ \varphi)$ *which is isomorphic for* $n\geqq 3$ *and one-to-one onto for* $n=2$.

Proof. It suffices to consider the case of Kan triad using the first definition. Let $(K;\ L,\ M)$ be a Kan triad. For a simplex $\sigma\in\Gamma_n(K;\ L,\ M,\ \varphi)$ consider a solution $\tau\in K_n$ of the following equation in K:

$$\overset{(0)}{[\square,}\ \overset{(1)}{\sigma\varepsilon^{1}\eta^{0},}\ \overset{(2)}{\sigma,}\ \overset{(3)}{\varphi\eta^{0}\cdots\eta^{n-1},}\ \overset{\cdots}{\cdots,}\ \overset{(n+1)}{\varphi\eta^{0}\cdots\eta^{n-1}].}$$

It is easily seen that $\tau\in\Gamma_n(K;\ M,\ L,\ \varphi)$. Then we may define a transformation $u:\pi_n(K;\ L,\ M,\ \varphi)\rightarrow\pi_n(K;\ M,\ L,\ \varphi)$ by $u(\{\sigma\}_{K;L,M})=\{\tau\}_{K;M,L}$ for $n\geqq 2$. This is verified as follows. Let $\rho:\sigma\sim\sigma'$ let L, M, lsd. $\omega\in K_{n+1}$ and $\omega'\in K_{n+1}$ be solvents of the following equations

$$\overset{(0)}{[\square,}\ \overset{(1)}{\sigma\varepsilon^{1}\eta^{0},}\ \overset{(2)}{\sigma,}\ \overset{(3)}{\varphi\eta^{0}\cdots\eta^{n-1},}\ \overset{\cdots}{\cdots,}\ \overset{(n+1)}{\varphi\eta^{0\ldots n-1}]}$$

and

$$\overset{(0)}{[\square}, \overset{(1)}{\sigma'\varepsilon^1\eta^0}, \overset{(2)}{\sigma'}, \overset{(3)}{\varphi\eta^0\cdots\eta^{n-1}}, \cdots, \overset{(n+1)}{\varphi\eta^0\cdots\eta^{n-1}}]$$

respectively, and put $\tau=\omega\varepsilon^0$, $\tau'=\omega'\varepsilon^0$. Let $\rho'\in K_{n+1}$ be a solution of the equation in K:

$$\overset{(0)}{[\square}, \overset{(1)}{\omega_1}, \overset{(2)}{\rho}, \overset{(3)}{\varphi\eta^0\cdots\eta^n}, \cdots, \overset{(n)}{\varphi\eta^0\cdots\eta^n}, \overset{(n+1)}{\omega}, \overset{(n+2)}{\omega'}],$$

where $\omega_1\in M_{n+1}$ is a solvent of the following equation in M:

$$\overset{(0)}{[\square}, \overset{(1)}{\rho\varepsilon^1}, \overset{(2)}{\varphi\eta^0\cdots\eta^{n-1}}, \cdots, \overset{(n-1)}{\varphi\eta^0\cdots\eta^{n-1}}, \overset{(n)}{\sigma\varepsilon^1\eta^0}, \overset{(n+1)}{\sigma'\varepsilon^1\eta^0}].$$

Then it is easily seen that $\rho':\tau\sim\tau'$ lsd. M, L. Thus u is a transformation from $\pi_n(K; L, M, \varphi)$ to $\pi_n(K; M, L, \varphi)$.

Let us now prove that u is a homomorphism for $n\geqq 3$. Put $\{\beta\}_{K:L,M}=\{\sigma\}_{K:L,M}\cdot\{\lambda\}_{K:L,M}$ in $\pi_n(K; L, M, \varphi)$, i.e. $\beta=\Omega_2\varepsilon^n$, where $\Omega_2\in K_{n+1}$ is a solvent of the equation in K:

$$\overset{(0)}{[\gamma_0}, \overset{(1)}{\gamma_1}, \overset{(2)}{\varphi\eta^0\cdots\eta^{n-1}}, \cdots, \overset{(n-2)}{\varphi\eta^0\cdots\eta^{n-1}}, \overset{(n-1)}{\sigma}, \overset{(n)}{\square}, \overset{(n+1)}{\lambda}],$$

($\gamma_0\in L_n$ and $\gamma_1\in M_n$ are defined such as the above equation holds in K, see the definition of product). Put $\{\tau\}_{K:M,L}=u(\{\sigma\}_{K:L,M})$, $\{\mu\}_{K:M,L}=u(\{\lambda\}_{K:L,M})$ and $\{\alpha\}_{K:M,L}=u(\{\beta\}_{K:L,M})$, i.e. $\tau=\Omega_n\varepsilon^0$, $\mu=\Omega_{n+2}\varepsilon^0$ and $\alpha=\Omega_{n+1}\varepsilon^0$, where Ω_n, Ω_{n+2} and Ω_{n+1} are solvents of the following equations in K:

$$\overset{(0)}{[\square}, \overset{(1)}{\sigma\varepsilon^1\eta^0}, \overset{(2)}{\sigma}, \overset{(3)}{\varphi\eta^0\cdots\eta^{n-1}}, \cdots, \overset{(n+1)}{\varphi\eta^0\cdots\eta^{n-1}}],$$

$$\overset{(0)}{[\square}, \overset{(1)}{\lambda\varepsilon^1\eta^0}, \overset{(2)}{\lambda}, \overset{(3)}{\varphi\eta^0\cdots\eta^{n-1}}, \cdots, \overset{(n+1)}{\varphi\eta^0\cdots\eta^{n-1}}]$$

and

$$\overset{(0)}{[\square}, \overset{(1)}{\beta\varepsilon^1\eta^0}, \overset{(2)}{\beta}, \overset{(3)}{\varphi\eta^0\cdots\eta^{n-1}}, \cdots, \overset{(n+1)}{\varphi\eta^0\cdots\eta^{n-1}}]$$

respectively. Let $\Omega_0\in K_{n+1}$ be a solution of the equation in K:

$$\overset{(0)}{[\square}, \overset{(1)}{\Omega_1}, \overset{(2)}{\Omega_2}, \overset{(3)}{\varphi\eta^0\cdots\eta^n}, \cdots, \overset{(n-1)}{\varphi\eta^0\cdots\eta^n}, \overset{(n)}{\Omega_n}, \overset{(n+1)}{\Omega_{n+1}}, \overset{(n+2)}{\Omega_{n+2}}]$$

where $\Omega_1\in M_{n+1}$ is a solvent of the equation in M:

$$\overset{(0)}{[\square}, \overset{(1)}{\gamma_1}, \overset{(2)}{\varphi\eta^0\cdots\eta^{n-1}}, \cdots, \overset{(n-2)}{\varphi\eta^0\cdots\eta^{n-1}}, \overset{(n-1)}{\sigma\varepsilon^1\eta^0}, \overset{(n)}{\beta\varepsilon^1\eta^0}, \overset{(n+1)}{\lambda\varepsilon^1\eta^0}].$$

Then $\Omega_0\varepsilon^0=\Omega_1\varepsilon^0\in M_n$, $\Omega_0\varepsilon^1=\Omega_2\varepsilon^0=\gamma_0\in L_n$, $\Omega_0\varepsilon^i=\varphi\eta^0\cdots\eta^{n-1}$ for $2\leqq i\leqq n-2$, $\Omega_0\varepsilon^{n-1}=$

$\Omega_n\varepsilon^0=\tau$, $\Omega_0\varepsilon^n=\Omega_{n+1}\varepsilon^0=\alpha$, $\Omega_0\varepsilon^{n+1}=\Omega_{n+2}\varepsilon^0=\mu$. Thus $\{\alpha\}_{K;M,L}=\{\tau\}_{K;M,L}\cdot\{\mu\}_{K;M,L}$, namely u is a homomorphism.

We can complete the proof of this PROPOSITION by showing that u is one-to-one onto for $n\geqq 2$.

For a simplex $\tau\in\Gamma_n(K; M, L, \varphi)$, consider a solution $\sigma\in K_n$ of the following equation in K:

$$\overset{(0)}{[\tau},\ \overset{(1)}{\tau\varepsilon^0\eta^0},\ \overset{(2)}{\square},\ \overset{(3)}{\varphi\eta^0\cdots\eta^{n-1}},\ \cdots,\ \overset{(n+1)}{\varphi\eta^0\cdots\eta^{n-1}]}.$$

Then $\sigma\in\Gamma_n(K; L, M, \varphi)$. And we can define a transformation $u':\pi_n(K; M, L, \varphi)\to\pi_n(K; L, M, \varphi)$ by $u'(\{\tau\}_{K;M,L})=\{\sigma\}_{K;L,M}$. Since $u\circ u'=$identity, and $u'\circ u=$identity, u is one-to-one onto.

REMARK. Even in the case $n=2$, $u(\{\varphi\eta^0\eta^1\}_{K;L,M})=\{\varphi\eta^0\eta^1\}_{K;M,L}$.

THEOREM 2.10. *Let $(K; L, M)$ be a c.s.s. triad with base point φ. Moreover assume that $L\supset M$. Then the inclusion map $j:(K; L, N_\varphi, \varphi)\to(K; L, M, \varphi)$ induces the isomorphism $j_*:\pi_n(K, L, \varphi)\to\pi_n(K; L, M, \varphi)$ for $n\geqq 3$, where N_φ means the c.s.s. complex whose only non-degenerate simplex is φ.*

Proof. It is clear that $\pi_n(K, L, \varphi)=\pi_n(K; L, N_\varphi, \varphi)$.

At first, we consider the case of Kan complexes using the first definition. If $\rho:\sigma\sim\tau$ lsd. L where $\sigma, \tau\in\Gamma_n(K, L, \varphi)$, then $j(\rho):j(\sigma)\sim j(\tau)$ lsd. L, M. Therefore we can define a transformation $j_*:\pi_n(K, L, \varphi)\to\pi_n(K; L, M, \varphi)$ by $j^*(\{\sigma\}_{K,L})=\{j(\sigma)\}_{K;L,M}$. It is easily to see that j_* is homomorphic for $n\geqq 3$.

For an arbitrary element $\{\tau\}_{K;L,M}\in\pi_n(K; L, M, \varphi)$, consider a solvent $\rho\in K_{n+1}$ of the equation in K:

$$\overset{(0)}{[\theta},\ \overset{(1)}{\tau\varepsilon^1\eta^0},\ \overset{(2)}{\tau},\ \overset{(3)}{\square},\ \overset{(4)}{\varphi\eta^0\cdots\eta^{n-1}},\ \cdots,\ \overset{(n+1)}{\varphi\eta^0\cdots\eta^{n-1}]},$$

where $\theta\in L_n$ is a solvent of the following equation in L:

$$\overset{(0)}{[\tau\varepsilon^1},\ \overset{(1)}{\tau\varepsilon^0},\ \overset{(2)}{\square},\ \overset{(3)}{\varphi\eta^0\cdots\eta^{n-2}},\ \cdots,\ \overset{(n)}{\varphi\eta^0\cdots\eta^{n-2}]}.$$

Put $\sigma=\rho\varepsilon^3$. Then $\sigma\in\Gamma_n(K, L, \varphi)$. Since $\rho\varepsilon^0=\theta\in L_n$, $\rho\varepsilon^1=\tau\varepsilon^1\eta^0\in M_n$, $\rho\varepsilon^2=\tau$, $\rho\varepsilon^3=\sigma$, $\rho\varepsilon^i=\varphi\eta^0\cdots\eta^{n-i}=\tau\eta^2\varepsilon^i=\sigma\eta^2\varepsilon^i$ for $4\leqq i\leqq n+1$, by PROPOSITION 2.3, we have $\tau\sim\sigma$ lsd. L, M. Thus $j_*(\{\sigma\}_{K,L})=\{\tau\}_{K;L,M}$, namely j_* is onto.

It thus remains to show that j_* is isomorphic. Consider a class $\{\sigma\}_{K,L}$ such that $j_*(\{\sigma\}_{K,L})$ is the unit element of $\pi_n(K; L, M, \varphi)$. Then there exists a homotopy ρ: $\varphi\eta^0\cdots\eta^{n-1}\sim\sigma$ lsd. L, M. Let $\rho''\in K_{n+1}$ be a solution of the equation in K:

$$\overset{(0)}{[\rho'},\ \overset{(1)}{\rho},\ \overset{(2)}{\rho\varepsilon^1\eta^0},\ \overset{(3)}{\varphi\eta^0\cdots\eta^n},\ \cdots,\ \overset{(n+1)}{\varphi\eta^0\cdots\eta^n},\ \overset{(n+2)}{\square]},$$

where $\rho'\in L_{n+1}$ is a solvent of the following equation in L:

$$\overset{(0)}{[\rho\varepsilon^0}, \overset{(1)}{\rho\varepsilon^1}, \overset{(2)}{\varphi\eta^{0}\cdots\eta^{n-1}}, \cdots, \overset{(n)}{\varphi\eta^{0}\cdots\eta^{n-1}}, \overset{(n+1)}{\square}].$$

Then we have $\rho''\varepsilon^0=\rho'\varepsilon^{n+1}\in L_n$, $\rho''\varepsilon^1=\rho\varepsilon^{n+1}=\sigma$, $\rho''\varepsilon^2=\rho\varepsilon^1\eta^0\varepsilon^{n+1}=\rho\varepsilon^{n+1}\varepsilon^1\eta^0=\sigma\varepsilon^1\eta^0=\varphi\eta^0\cdots\eta^{n-1}$, $\rho''\varepsilon^i=\varphi\eta^0\cdots\eta^{n-1}$ for $3\leqq i\leqq n+1$.

Therefore, by PROPOSITION 1.6, we have $\sigma\sim\varphi\eta^0\cdots\eta^{n-1}$ lsd. L, namely $\{\rho\}_{K,L}$ is the unit element of $\pi_n(K, L, \varphi)$. Thus j_* is isomorphic.

Secondly, in the general case j_* is defined by $(S|j|)_*:\pi_n(S|K|, S|L|, i(\varphi))\to\pi_n(S|K|\,;\,S|L|, S|M|, i(\varphi))$ where $|j|$ is the induced continuous map [7]. Therefore j_* is isomorphic.

REMARK 2.11. The inclusion map $j:(K;\,L, N_\varphi, \varphi)\to(K;\,L, M, \varphi)$ induces a transformation $j_*:\pi_2(K;\,L, \varphi)\to\pi_2\,(K;\,L, M, \varphi)$ which is one-to-one onto.

Proof. It suffices to prove the case of Kan triad using the first definition. We may prove that j_* is onto by the method used in the proof of THEOREM 2.10. To show that j_* is one-to-one, consider two simplices σ and $\tau\in\Gamma_2(K, L, \varphi)$ such that $\sigma\sim\tau$ lsd. L, M. Then there exists $\rho\in K_3$ such that $\rho\varepsilon^0\in L_2$, $\rho\varepsilon^1\in M_2$, $\rho\varepsilon^2=\sigma$ and $\rho\varepsilon^3=\tau$. Since $\rho\varepsilon^1\in\Gamma_2(K, L, \varphi)$, $\rho\varepsilon^1\sim\varphi\eta^0\eta^1$ lsd. L and $\{\rho\varepsilon^1\}_{K,L}\cdot\{\tau\}_{K,L}=\{\sigma\}_{K,L}$, we have $\{\tau\}_{K,L}=\{\sigma\}_{K,L}$.

§3. Proofs of PROPOSITIONS 1.8, 1.9, 2.5 and 2.6.

For $n\geqq3$, PROPOSITIONS 1.8 and 1.9 are the special cases of PROPOSITIONS 2.5 and 2.6 respectively, i.e. the case when $M=N_\varphi$. Therefore we give only proofs of PROPOSITIONS 2.5, 2.6 and of the case $n=2$ of PROPOSITIONS 1.8, 1.9.

Proof of PROPOSITION 2.5. Let σ, σ' and τ be simplices of $\Gamma_n(K;\,L, M, \varphi)$ and assume that $\rho:\sigma\sim\sigma'$ lsd. L, M. Let θ, γ_0, γ_1 and β be simplices mentioned in PROPOSITION 2.5 for σ and τ, and let $\alpha\in K_{n+1}$ be a solvent of

$$\overset{(0)}{[\gamma_0}, \overset{(1)}{\gamma_1}, \overset{(2)}{\varphi\eta^0\cdots\eta^{n-1}}, \cdots, \overset{(n-2)}{\varphi\eta^0\cdots\eta^{n-1}}, \overset{(n-1)}{\sigma}, \overset{(n)}{\square}, \overset{(n+1)}{\tau}]$$

such that $\alpha\varepsilon^n=\beta$. Define similarly θ', γ_0', γ_1', β' and α' for σ' and τ. Let $\Theta\in(L\cap M)_n$ be a solvent of the equation in $L\cap M$:

$$\overset{(0)}{[\varphi\eta^0\cdots\eta^{n-2}}, \cdots, \overset{(n-4)}{\varphi\eta^0\cdots\varphi^{n-2}}, \overset{(n-3)}{\rho\varepsilon^0\varepsilon^0}, \overset{(n-2)}{\square}, \overset{(n-1)}{\theta}, \overset{(n)}{\theta'}].$$

Consider a solution $\omega\in K_{n+1}$ of

$$\overset{(0)}{[\omega_0}, \overset{(1)}{\omega_1}, \overset{(2)}{\varphi\eta^0\cdots\eta^n}, \cdots, \overset{(n-2)}{\varphi\eta^0\cdots\eta^n}, \overset{(n-1)}{\rho}, \overset{(n)}{\square}, \overset{(n+1)}{\alpha}, \overset{(n+2)}{\alpha'}],$$

where $\omega_0\in L_{n+1}$ and $\omega_1\in M_{n+1}$ are solvents of the following equations:

$$[\overset{(0)}{\Theta},\ \overset{(1)}{\varphi\eta^{0\cdots}\eta^{n-1}},\ \overset{\cdots}{\cdots},\ \overset{(n-3)}{\varphi\eta^{0\cdots}\eta^{n-1}},\ \overset{(n-2)}{\rho\varepsilon^0},\ \overset{(n-1)}{\square},\ \overset{(n)}{\gamma_0},\ \overset{(n+1)}{\gamma_0'}]$$

and

$$[\overset{(0)}{\Theta},\ \overset{(1)}{\varphi\eta^{0\cdots}\eta^{n-1}},\ \overset{\cdots}{\cdots},\ \overset{(n-3)}{\varphi\eta^{0\cdots}\eta^{n-1}},\ \overset{(n-2)}{\rho\varepsilon^1},\ \overset{(n-1)}{\square},\ \overset{(n)}{\gamma_1},\ \overset{(n+1)}{\gamma_1'}]$$

respectively.

Since $\omega\varepsilon^0=\omega_0\varepsilon^{n-1}\in L_n$, $\omega\varepsilon^1=\omega_1\varepsilon^{n-1}\in M_n$, $\omega\varepsilon^n=\alpha\varepsilon^n=\beta$, $\omega\varepsilon^{n+1}=\alpha'\varepsilon^n=\beta'$ and $\omega\varepsilon^i=\varphi\eta^{0\cdots}\eta^{n-1}=\beta\varepsilon^i\eta^{n-1}=\beta'\varepsilon^i\eta^{n-1}$ for $2\leqq i\leqq n-1$, we have $\omega: \beta\sim\beta'$ lsd. L, M. Thus the product $\{\sigma\}_{K;L,M}\cdot\{\tau\}_{K;L,M}$ is independent of the choice of σ, θ, γ_0, γ_1 and β.

Now it remains to prove that the product $\{\sigma\}_{K;L,M}\cdot\{\tau\}_{K;L,M}$ is independent of the choice of τ. Let σ, τ and τ' be simplices of $\Gamma_n(K;\ L,\ M,\ \varphi)$ and assume that $\tau\sim\tau'$ lsd. L, M. By Proposition 2.3 there exists $\rho'\in\Lambda_{n+1}(K;\ L,\ M)$ such that $\rho'\varepsilon^{n-1}=\tau$, $\rho'\varepsilon^n=\tau'$ and $\rho'\varepsilon^i=\tau\eta^{n-1}\varepsilon^i=\tau'\eta^{n-1}\varepsilon^i$ for $i\neq 0,\ 1,\ n-1,\ n$. Let θ, γ_0, γ_1, α and β be the simplices defined in the preceding part of this proof, and θ', γ_0', γ_1', α' and β' be the simplices defined similarly for σ and τ'. Let $\Theta'\in(L\cap M)_n$ be a solvent of the equation in $L\cap M$:

$$[\overset{(0)}{\varphi\eta^{0\cdots}\eta^{n-2}},\ \overset{\cdots}{\cdots},\ \overset{(n-4)}{\varphi\eta^{0\cdots}\eta^{n-2}},\ \overset{(n-3)}{\theta},\ \overset{(n-2)}{\theta'},\ \overset{(n-1)}{\square},\ \overset{(n)}{\rho'\varepsilon^0\varepsilon^0}].$$

Consider a solution $\omega'\in K_{n+1}$ of

$$[\overset{(0)}{\omega_0'},\ \overset{(1)}{\omega_1'},\ \overset{(2)}{\varphi\eta^{0\cdots}\eta^{n}},\ \overset{\cdots}{\cdots},\ \overset{(n-2)}{\varphi\eta^{0\cdots}\eta^{n}}\ \overset{(n-1)}{\alpha},\ \overset{(n)}{\alpha'},\ \overset{(n+1)}{\square},\ \overset{(n+2)}{\rho'}],$$

where $\omega_0'\in L_{n+1}$ and $\omega_1'\in M_{n+1}$ are solvents of the following equations

$$[\overset{(0)}{\Theta'},\ \overset{(1)}{\varphi\eta^{0\cdots}\eta^{n-1}},\ \overset{\cdots}{\cdots},\ \overset{(n-3)}{\varphi\eta^{0\cdots}\eta^{n-1}},\ \overset{(n-2)}{\gamma_0},\ \overset{(n-1)}{\gamma_0'}\ \overset{(n)}{\square},\ \overset{(n+1)}{\rho'\varepsilon^0}]$$

and

$$[\overset{(0)}{\Theta'},\ \overset{(1)}{\varphi\eta^{0\cdots}\eta^{n-1}},\ \overset{\cdots}{\cdots},\ \overset{(n-3)}{\varphi\eta^{0\cdots}\eta^{n-1}},\ \overset{(n-2)}{\gamma_1},\ \overset{(n-1)}{\gamma_1'},\ \overset{(n)}{\square},\ \overset{(n+1)}{\rho'\varepsilon^1}]$$

respectively. Since $\omega'\varepsilon^0=\omega_0'\varepsilon^n\in L_n$, $\omega'\varepsilon^1=\omega_1'\omega^n\in M_n$, $\omega'\varepsilon^{n-1}=\alpha\varepsilon^n=\beta$, $\omega'\varepsilon^n=\alpha'\varepsilon^n=\beta'$ and $\omega'\varepsilon^i=\varphi\eta^{0\cdots}\eta^{n-1}=\beta\eta^{n-1}\varepsilon^i=\beta'\eta^{n-1}\varepsilon^i$ for $i\neq 0,\ 1,\ n-1,\ n$, we have $\beta\sim\beta'$ lsd. L, M (by Proposition 2.3).

Proof of Proposition 2.6.

Divisibility. Let σ and τ be simplices of $\Gamma_n(K;\ L,\ M,\ \varphi)$. Let $\theta\in(L\cap M)_{n-1}$ be a solvent of the equation in $L\cap M$:

$$[\overset{(0)}{\varphi\eta^{0\cdots}\eta^{n-3}},\ \overset{\cdots}{\cdots},\ \overset{(n-4)}{\varphi\eta^{0\cdots}\eta^{n-3}},\ \overset{(n-3)}{\square},\ \overset{(n-2)}{\tau\varepsilon^0\varepsilon^0}.\ \overset{(n-1)}{\sigma\varepsilon^0\varepsilon^0}].$$

Consider a solution $\alpha\in K_n$ of the equation

$$[\overset{(0)}{\gamma_0},\ \overset{(1)}{\gamma_1},\ \overset{(2)}{\varphi\eta^{0\cdots}\eta^{n-1}},\ \overset{\cdots}{\cdots},\ \overset{(n-2)}{\varphi\eta^{0\cdots}\eta^{n-1}},\ \overset{(n-1)}{\square},\ \overset{(n)}{\tau},\ \overset{(n+1)}{\sigma}],$$

where $\gamma_0 \in L_n$ and $\gamma_1 \in M_n$ are solvents of

$$\overset{(0)}{[\theta}, \overset{(1)}{\varphi\eta^0\cdots\eta^{n-2}}, \cdots, \overset{(n-3)}{\varphi\eta^0\cdots\eta^{n-2}}, \overset{(n-2)}{\square}, \overset{(n-1)}{\tau\varepsilon^0}, \overset{(n)}{\sigma\varepsilon^0}]$$

and

$$\overset{(0)}{[\theta}, \overset{(1)}{\varphi\eta^0\cdots\eta^{n-2}}, \cdots, \overset{(n-3)}{\varphi\eta^0\cdots\eta^{n-2}}, \overset{(n-2)}{\square}, \overset{(n-1)}{\tau\varepsilon^1}, \overset{(n)}{\sigma\varepsilon^1}]$$

respectively. Then we have $\{\alpha\}_{K;L,M}\cdot\{\sigma\}_{K;L,M}=\{\tau\}_{K;L,M}$.

Similarly we may define $\beta\in\Gamma_n(K;\ L,\ M,\ \varphi)$ such that $\{\sigma\}_{K;L,M}\cdot\{\beta\}_{K;L,M}=\{\tau\}_{K;L,M}$.

Associativity. Let σ, τ and υ be simplices of $\Gamma_n(K;\ L,\ M,\ \varphi)$. Put $\alpha_{n-1}=\alpha$ and $\beta_{n-1}=\beta$ where α and β are the simplices defined for σ and τ in the proof of PROPOSITION 2.5, i.e. $\alpha_{n-1}\varepsilon^0\in L_n$, $\alpha_{n-1}\varepsilon^1\in M_n$, $\alpha_{n-1}\varepsilon^i=\varphi\eta^0\cdots\eta^{n-1}$ for $2\leqq i\leqq n-2$, $\alpha_{n-1}\varepsilon^{n-1}=\sigma$, $\alpha_{n-1}\varepsilon^n=\beta_{n-1}$, $\alpha_{n-1}\varepsilon^{n+1}=\tau$. Define similarly α_{n+2} and β_{n+2} for τ and υ, α_{n+1} and β_{n+1} for β and υ.

Let $\delta\in(L\cap M)_n$ be a solution of the equation in $L\cap M$:

$$\overset{(0)}{[\varphi\eta^0\cdots\eta^{n-2}}, \cdots, \overset{(n-4)}{\varphi\eta^0\cdots\eta^{n-2}}, \overset{(n-3)}{\alpha_{n-1}\varepsilon^0\varepsilon^0}, \overset{(n-2)}{\square}, \overset{(n-1)}{\alpha_{n+1}\varepsilon^0\varepsilon^0}, \overset{(n)}{\alpha_{n+2}\varepsilon^0\varepsilon^0}].$$

Let $\omega\in K_{n+1}$ be a solution of

$$\overset{(0)}{[\omega_0}, \overset{(1)}{\omega_1}, \overset{(2)}{\varphi\eta^0\cdots\eta^n}, \cdots, \overset{(n-2)}{\varphi\eta^0\cdots\eta^n}, \overset{(n-1)}{\alpha_{n-1}}, \overset{(n)}{\square}, \overset{(n+1)}{\alpha_{n+1}}, \overset{(n+2)}{\alpha_{n+2}}],$$

where $\omega_0\in L_{n+1}$ and $\omega_1\in M_{n+1}$ are solvents of the following equations:

$$\overset{(0)}{[\delta}, \overset{(1)}{\varphi\eta^0\cdots\eta^{n-1}}, \cdots, \overset{(n-3)}{\varphi\eta^0\cdots\eta^{n-1}}, \overset{(n-2)}{\alpha_{n-1}\varepsilon^0}, \overset{(n-1)}{\square}, \overset{(n)}{\alpha_{n+1}\varepsilon^0}, \overset{(n+1)}{\alpha_{n+2}\varepsilon^0}]$$

and

$$\overset{(0)}{[\delta}, \overset{(1)}{\varphi\eta^0\cdots\eta^{n-1}}, \cdots, \overset{(n-3)}{\varphi\eta^0\cdots\eta^{n-1}}, \overset{(n-2)}{\alpha_{n-1}\varepsilon^1}, \overset{(n-1)}{\square}, \overset{(n)}{\alpha_{n+1}\varepsilon^1}, \overset{(n+1)}{\alpha_{n+2}\varepsilon^1}]$$

respectively.

$(\{\sigma\}_{K;L,M}\cdot\{\tau\}_{K;L,M})\cdot\{\upsilon\}_{K;L,M}=\{\beta_{n-1}\}_{K;L,M}\cdot\{\upsilon\}_{K;L,M}=\{\beta_{n+1}\}_{K;L,M}$
$=\{\alpha_{n+1}\varepsilon^n\}_{K;L,M}=\{\omega\varepsilon^n\}_{K;L,M}$
Here $\{\omega\varepsilon^n\}_{K;L,M}=\{\sigma\}_{K;L,M}\cdot\{\beta_{n+2}\}_{K;L,M}$, for $\omega\varepsilon^0=\omega_0\varepsilon^{n-1}\in L_n$, $\omega\varepsilon^1=\omega_1\varepsilon^{n-1}\in M_n$, $\omega\varepsilon^i=\varphi\eta^0\cdots\eta^{n-1}$ $(2\leqq i\leqq n-2)$, $\omega\varepsilon^{n-1}=\alpha_{n-1}\varepsilon^{n-1}=\sigma$ and $\omega\varepsilon^{n+1}=\alpha_{n+2}\varepsilon^n=\beta_{n+2}$. Furthermore $\{\beta_{n+2}\}_{K;L,M}=\{\tau\}_{K;L,M}\cdot(\upsilon\}_{K;L,M}$. Thus the associativity holds.

Proof of PROPOSITION 1.8 for $n=2$. Let σ, σ' and τ be simplices of $\Gamma_2(K,\ L,\ \varphi)$ and assume that $\rho;\sigma\sim\sigma'$ lsd. L. Let $\alpha\in K_3$ be a solvent of

$$\overset{(0)}{[\gamma}, \overset{(1)}{\sigma}, \overset{(2)}{\square}, \overset{(3)}{\tau}],$$

where $\gamma \in L_2$ is a solvent of the equation in L:

$$\overset{(0)}{[\sigma\varepsilon^0,}\ \overset{(1)}{\square,}\ \overset{(2)}{\tau\varepsilon^0]}.$$

Put $\beta=\alpha\varepsilon^2$. Define similarly γ', α' and β' for σ' and τ.

Consider a solution $\omega\in K_3$ of

$$\overset{(0)}{[\omega_0,}\ \overset{(1)}{\rho,}\ \overset{(2)}{\square,}\ \overset{(3)}{\alpha,}\ \overset{(4)}{\alpha']},$$

where $\omega_0 \in L_3$ is a solvent of the equation in L:

$$\overset{(0)}{[\rho\varepsilon^0,}\ \overset{(1)}{\square,}\ \overset{(2)}{\gamma,}\ \overset{(3)}{\gamma']}.$$

Then we have $\omega: \beta\sim\beta'$ lsd L.

Next, consider σ, τ and $\tau'\in\Gamma_2(K, L, \varphi)$ and assume that $\tau\sim\tau'$ lsd. L. By PROPOSITION 1.6 there exists $\rho'\in\Lambda_3(K, L)$ such that $\rho'\varepsilon^1=\tau$, $\rho'\varepsilon^2=\tau'$, $\rho'\varepsilon^3=\tau\eta^1\varepsilon^3=\tau'\eta^1\varepsilon^3$.

Let γ, α and β be the simplices defined in the preceding part of this proof, and γ', α' and β' be the simplices defined similarly for σ and τ'.

Consider a solution $\omega'\in K_3$ of

$$\overset{(0)}{[\omega_0',}\ \overset{(1)}{\alpha,}\ \overset{(2)}{\alpha',}\ \overset{(3)}{\square,}\ \overset{(4)}{\rho']},$$

where $\omega_0'\in L_3$ is a solvent of the equation in L:

$$\overset{(0)}{[\gamma,}\ \overset{(1)}{\gamma',}\ \overset{(2)}{\square,}\ \overset{(3)}{\rho']}.$$

Since $\omega'\varepsilon^0=\omega_0'\varepsilon^2\in L_2$, $\omega'\varepsilon^1=\alpha\varepsilon^2=\beta$, $\omega'\varepsilon^2=\alpha'\varepsilon^2=\beta'$ and $\omega'\varepsilon^3=\varphi\eta^0\eta^1=\beta\eta^1\varepsilon^3=\beta'\eta^1\varepsilon^3$, we have $\beta\sim\beta'$ lsd. L (by PROPOSITION 1.6).

Proof of PROPOSITION 1.9 for $n=2$.

Divisibility. Let σ and τ be simplices of $\Gamma_2(K\ L\ \varphi)$. Consider a solution $\alpha\in K_2$ of the equation:

$$\overset{(0)}{[\gamma,}\ \overset{(1)}{\square,}\ \overset{(2)}{\tau,}\ \overset{(3)}{\sigma]},$$

where $\gamma\in L_2$ is a solvent of the equation in L:

$$\overset{(0)}{[\square,}\ \overset{(1)}{\tau\varepsilon^0,}\ \overset{(2)}{\sigma\varepsilon^0]}.$$

Then we have $\{\alpha\}_{K,L}\cdot\{\sigma\}_{K,L}=\{\tau\}_{K,L}$.

Similarly we may define $\beta\in\Gamma_2(K, L, \varphi)$ such that $\{\sigma\}_{K,L}\cdot\{\beta\}_{K,L}=\{\tau\}_{K,L}$.

Associativity. Let σ, τ and υ be simplices of $\Gamma_2(K, L, \varphi)$. Put $\alpha_1=\alpha$ and $\beta_1=\beta$ where α and β are the simplices defined for σ and τ in the proof of PROPOSITION 1.8

for $n=2$, i.e. $\alpha_1\varepsilon^0 \in L_2$, $\alpha_1\varepsilon^1=\sigma$, $\alpha_1\varepsilon^2=\beta$, $\alpha_1\varepsilon^3=\tau$.

Define similarly α_4 and β_4 for τ and υ, α_3 and β_3 for β and υ. Let $\omega \in K_3$ be a solution of the equation:

$$\overset{(0)}{[\omega_0},\ \overset{(1)}{\alpha_1},\ \overset{(2)}{\square},\ \overset{(3)}{\alpha_3},\ \overset{(4)}{\alpha_4}],$$

where $\omega_0 \in L_3$ is a solvent of the equation in L:

$$\overset{(0)}{[\alpha_1\varepsilon^0},\ \overset{(1)}{\square},\ \overset{(2)}{\alpha_3\varepsilon^0},\ \overset{(3)}{\alpha_4\varepsilon^0}].$$

Then we have
$(\{\sigma\}_{K,L}\cdot\{\tau\}_{K,L})\cdot\{\upsilon\}_{K,L}=\{\beta_1\}_{K,L}\cdot\{\upsilon\}_{K,L}=\{\beta_3\}_{K,L}=\{\alpha_3\varepsilon^2\}_{K,L}=\{\omega\varepsilon^2\}_{K,L}$. Here $\{\omega\varepsilon^2\}_{K,L}=\{\sigma\}_{K,L}\cdot\{\beta_4\}_{K,L}$, for $\omega\varepsilon^0=\omega_0\varepsilon^1\in L_2$, $\omega\varepsilon^1=\alpha_1\varepsilon^1=\sigma$ and $\omega\varepsilon^3=\alpha_4\varepsilon^2=\beta_4$. Furthermore $\{\beta_4\}_{K,L}=\{\tau\}_{K,L}\cdot\{\upsilon\}_{K,L}$. Thus the associativity holds.

§4. The proofs of PROPOSITIONS 1.12 and 2.8.

J. C. Moore [8] proved that the n-th homotopy group $\pi_n(K, \varphi)$ of a Kan complex K is abelian for $n\geqq 2$. After the manner of Moore, we prove PROPOSITION 2.8. PROPOSITION 1.12 may be verified by the same method and we omit its proof.

LEMMA 4.1. *Let $(K; L, M)$ be a Kan triad with a base point φ, and a, b and c be elements of $\Gamma_n(K; L, M, \varphi)$, where $n\geqq 4$.*

(A) *If there exists $x_{n+1}\in\Lambda_{n+1}(K; L, M)$ such that $x_{n+1}\varepsilon^{n-2}=a$, $x_{n+1}\varepsilon^{n-1}=b$, $x_{n+1}\varepsilon^n=c$ and $x_{n+1}\varepsilon^i=\varphi\eta^0\cdots\eta^{n-1}$ for $2\leqq i\leqq n-3$ or $i=n+1$, then $\{c\}_{K;L,M}\cdot\{a\}_{K;L,M}=\{b\}_{K;L,M}$.*

(B) *If there exists $x_n\in\Lambda_{n+1}(K; L, M)$ such that $x_n\varepsilon^{n-2}=a$, $x_n\varepsilon^n=b$, $x_n\varepsilon^{n+1}=c$ and $x_n\varepsilon^i=\varphi\eta^0\cdots\eta^{n-1}$ for $2\leqq i\leqq n-3$ or $i=n-1$, then $\{a\}_{K;L,M}\cdot\{b\}_{K;L,M}=\{c\}_{K;L,M}$.*

(C) *If there exists $x_{n+2}\in\Lambda_{n+1}(K; L, M)$ such that $x_{n+2}\varepsilon^{n-2}=a$, $x_{n+2}\varepsilon^{n-1}=b$, $x_{n+2}\varepsilon^n=c$ and $x_{n+2}\varepsilon^i=\varphi\eta^0\cdots\eta^{n-1}$ for $2\leqq i\leqq n-3$ or $i=n+1$, then $\{a\}_{K;L,M}\cdot\{c\}_{K;L,M}=\{b\}_{K;L,M}$.*

Proof of (A). Consider a solvent $\theta\in(L\cap M)_{n-1}$ of the following equation

$$\overset{(0)}{[\varphi\eta^0\cdots\eta^{n-3}},\ \cdots,\ \overset{(n-4)}{\varphi\eta^0\cdots\eta^{n-3}},\ \overset{(n-3)}{\square},\ \overset{(n-2)}{b\varepsilon^0\varepsilon^0},\ \overset{(n-1)}{a\varepsilon^0\varepsilon^0}],$$

and let $x_{n-1}\in K_{n+1}$ be a solvent of

$$\overset{(0)}{[\sigma_0},\ \overset{(1)}{\sigma_1},\ \overset{(2)}{\varphi\eta^0\cdots\eta^{n-1}},\ \cdots,\ \overset{(n-2)}{\varphi\eta^0\cdots\eta^{n-1}},\ \overset{(n-1)}{\square},\ \overset{(n)}{b},\ \overset{(n+1)}{a}]\cdots\cdots\cdots(1)$$

where $\sigma_0\in L_n$ and $\sigma_1\in M_n$ are solvents of

$$[\overset{(0)}{\theta},\ \overset{(1)}{\varphi\eta^{0}\cdots\eta^{n-2}},\ \overset{\cdots}{\cdots},\ \overset{(n-3)}{\varphi\eta^{0}\cdots\eta^{n-2}},\ \overset{(n-2)}{\square},\ \overset{(n-1)}{b\varepsilon^{0}},\ \overset{(n)}{a\varepsilon^{0}}]$$

and

$$[\overset{(0)}{\theta},\ \overset{(1)}{\varphi\eta^{0}\cdots\eta^{n-2}},\ \overset{\cdots}{\cdots},\ \overset{(n-3)}{\varphi\eta^{0}\cdots\eta^{n-2}},\ \overset{(n-2)}{\square},\ \overset{(n-1)}{b\varepsilon^{1}},\ \overset{(n)}{a\varepsilon^{1}}]$$

respectively.

Let $\omega\in(L\cap M)_n$ be a solvent of the following equation

$$[\overset{(0)}{\varphi\eta^{0}\cdots\eta^{n-2}},\ \overset{\cdots}{\cdots},\ \overset{(n-4)}{\varphi\eta^{0}\cdots\eta^{n-2}},\ \overset{(n-3)}{\theta},\ \overset{(n-2)}{\square},\ \overset{(n-1)}{x_{n+1}\varepsilon^{0}\varepsilon^{0}},\ \overset{(n)}{\varphi\eta^{0}\cdots\eta^{n-2}}],$$

and $x_n\in K_{n+1}$ be a solution of

$$[\overset{(0)}{\tau_0},\ \overset{(1)}{\tau_1},\ \overset{(2)}{\varphi\eta^{0}\cdots\eta^{n}},\ \overset{\cdots}{\cdots},\ \overset{(n-3)}{\varphi\eta^{0}\cdots\eta^{n}},\ \overset{(n-2)}{a\eta^{n}},\ \overset{(n-1)}{x_{n-1}},\ \overset{(n)}{\square},\ \overset{(n+1)}{x_{n+1}},\ \overset{(n+2)}{a\eta^{n-2}}]$$

where $\tau^0\in L_{n+1}$ and $\tau_1\in M_{n+1}$ are solvents of

$$[\overset{(0)}{\omega},\ \overset{(1)}{\varphi\eta^{0}\cdots\eta^{n-1}},\ \overset{\cdots}{\cdots},\ \overset{(n-3)}{\varphi\eta^{0}\cdots\eta^{n-1}},\ \overset{(n-2)}{\sigma_0},\ \overset{(n-1)}{\square},\ \overset{(n)}{x_{n+1}\varepsilon^{0}},\ \overset{(n+1)}{\varphi\eta^{0}\cdots\eta^{n-1}}]$$

and

$$[\overset{(0)}{\omega},\ \overset{(1)}{\varphi\eta^{0}\cdots\eta^{n-1}},\ \overset{\cdots}{\cdots},\ \overset{(n-3)}{\varphi\eta^{0}\cdots\eta^{n-1}},\ \overset{(n-2)}{\sigma_1},\ \overset{(n-1)}{\square},\ \overset{(n)}{x_{n+1}\varepsilon^{1}},\ \overset{(n+1)}{\varphi\eta^{0}\cdots\eta^{n-1}}]$$

respectively.

Then $x_n\varepsilon^0=\tau_0\varepsilon^{n-1}\in L$, $x_n\varepsilon^1=\tau_1\varepsilon^{n-1}\in M$, $x_n\varepsilon^{n-1}=x_{n-1}\varepsilon^{n-1}$, $x_n\varepsilon^n=x_{n+1}\varepsilon^n=c$ and $x_n\varepsilon^i=\varphi\eta^0\cdots\eta^{n-1}$ otherwise. Thus we have

$$\{x_{n-1}\varepsilon^{n-1}\}_{K;L,M}=\{c\}_{K;L,M}.$$

On the other hand we have

$$\{x_{n-1}\varepsilon^{n-1}\}_{K;L,M}\cdot\{a\}_{K;L,M}=\{b\}_{K;L,M}$$

by (1). Therefore

$$\{c\}_{K;L,M}\cdot\{a\}_{K;L,M}=\{b\}_{K;L,M}.$$

Proof of (B). Consider a solvent $\theta\in(L\cap M)_{n-1}$ of the following equation

$$[\overset{(0)}{\varphi\eta^{0}\cdots\eta^{n-3}},\ \overset{\cdots}{\cdots},\ \overset{(n-5)}{\varphi\eta^{0}\cdots\eta^{n-3}},\ \overset{(n-4)}{a\varepsilon^{0}\varepsilon^{0}},\ \overset{(n-3)}{\varphi\eta^{0}\cdots\eta^{n-3}},\ \overset{(n-2)}{\square},\ \overset{(n-1)}{\varphi\eta^{0}\cdots\eta^{n-3}}]$$

and let $x_{n-1}\in K_{n+1}$ be a solvent of

$$[\overset{(0)}{\sigma_0},\ \overset{(1)}{\sigma_1},\ \overset{(2)}{\varphi\eta^{0}\cdots\eta^{n-1}},\ \overset{\cdots}{\cdots},\ \overset{(n-3)}{\varphi\eta^{0}\cdots\eta^{n-1}},\ \overset{(n-2)}{a},\ \overset{(n-1)}{\varphi\eta^{0}\cdots\eta^{n-1}},\ \overset{(n)}{\square},\ \overset{(n+1)}{\varphi\eta^{0}\cdots\eta^{n-1}}]\ \cdots\cdots\cdots\cdots\ (2)$$

where $\sigma_0\in L_n$ and $\sigma_1\in M_n$ are solvents of

$$\overset{(0)}{[\theta}, \overset{(1)}{\varphi\eta^0\cdots\eta^{n-2}}, \cdots, \overset{(n-4)}{\varphi\eta^0\cdots\eta^{n-2}}, \overset{(n-3)}{a\varepsilon^0}, \overset{(n-2)}{\varphi\eta^0\cdots\eta^{n-2}}, \overset{(n-1)}{\square}, \overset{(n)}{\varphi\eta^0\cdots\eta^{n-2}]}$$

and

$$\overset{(0)}{[\theta}, \overset{(1)}{\varphi\eta^0\cdots\eta^{n-2}}, \cdots, \overset{(n-4)}{\varphi\eta^0\cdots\eta^{n-2}}, \overset{(n-3)}{a\varepsilon^1}, \overset{(n-2)}{\varphi\eta^0\cdots\eta^{n-2}}, \overset{(n-1)}{\square}, \overset{(n)}{\varphi\eta^0\cdots\eta^{n-2}]}$$

respectively.

Let $\omega\in(L\cap M)_n$ be a solvent of the following equation

$$\overset{(0)}{[\varphi\eta^0\cdots\eta^{n-2}}, \cdots, \overset{(n-5)}{\varphi\eta^0\cdots\eta^{n-2}}, \overset{(n-4)}{a\eta^{n-2}\varepsilon^0\varepsilon^0}, \overset{(n-3)}{\theta}, \overset{(n-2)}{x_n\varepsilon^0\varepsilon^0}, \overset{(n-1)}{\square}, \overset{(n)}{c\eta^n\varepsilon^0\varepsilon^0]}$$

and $x_{n+1}\in K_{n+1}$ be a solution of

$$\overset{(0)}{[\tau_0}, \overset{(1)}{\tau_1}, \overset{(2)}{\varphi\eta^0\cdots\eta^n}, \cdots \overset{(n-3)}{\varphi\eta^0\cdots\eta^n}, \overset{(n-2)}{a\eta^{n-2}}, \overset{(n-1)}{x_{n-1}}, \overset{(n)}{x_n}, \overset{(n+1)}{\square}, \overset{(n+2)}{c\eta^n]}$$

where $\tau_0\in L_{n+1}$ and are $\tau_1\in M_{n+1}$ are solvents of

$$\overset{(0)}{[\omega}, \overset{(1)}{\varphi\eta^0\cdots\eta^{n-1}}, \cdots, \overset{(n-4)}{\varphi\eta^0\cdots\eta^{n-1}}, \overset{(n-3)}{a\eta^{n-2}\varepsilon^0}, \overset{(n-2)}{\sigma_0}, \overset{(n-1)}{x_n\varepsilon^0}, \overset{(n)}{\square}, \overset{(n+1)}{c\eta^n\varepsilon^0]}$$

and

$$\overset{(0)}{[\omega}, \overset{(1)}{\varphi\eta^0\cdots\eta^{n-1}}, \cdots, \overset{(n-4)}{\varphi\eta^0\cdots\eta^{n-1}}, \overset{(n-3)}{a\eta^{n-2}\varepsilon^1}, \overset{(n-2)}{\sigma_1}, \overset{(n-1)}{x_n\varepsilon^1}, \overset{(n)}{\square}, \overset{(n+1)}{c\eta^n\varepsilon^1]}$$

respectively. Then $x_{n+1}\varepsilon^0=\tau_0\varepsilon^n\in L$, $x_{n+1}\varepsilon^1=\tau_1\varepsilon^n\in M$, $x_{n+1}\varepsilon^{n-1}=x_{n-1}\varepsilon^n$, $x_{n+1}\varepsilon^n=b$, $x_{n+1}\varepsilon^{n+1}=c$ and $x_{n+1}\varepsilon^i=\varphi\eta^0\cdots\eta^{n-1}$ otherwise. Thus we have

$$\{x_{n-1}\varepsilon^n\}_{K;L,M}\cdot\{c\}_{K;L,M}=\{b\}_{K;L,M}.$$

On the other hand we have

$$\{x_{n-1}\varepsilon^n\}_{K;L,M}=(\{a\}_{K;L,M})^{-1}$$

by (2). Therefore

$$\{c\}_{K;L,M}=\{a\}_{K;L,M}\cdot\{b\}_{K;L,M}.$$

Proof of (C). Consider a solvent $\theta\in(L\cap M)_{n-1}$ of the following equation

$$\overset{(0)}{[\varphi\eta^0\cdots\eta^{n-3}}, \cdots, \overset{(n-5)}{\varphi\eta^0\cdots\eta^{n-3}}, \overset{(n-4)}{\square}, \overset{(n-3)}{\varphi\eta^0\cdots\eta^{n-3}}, \overset{(n-2)}{\varphi\eta^0\cdots\eta^{n-3}}\ \overset{(n-1)}{a\varepsilon^0\varepsilon^0]}$$

and let $x_{n-2}\in K_{n+1}$ be a solvent of

$$\overset{(0)}{[\sigma_0}, \overset{(1)}{\sigma_1}, \overset{(2)}{\varphi\eta^0\cdots\eta^{n-1}}, \cdots, \overset{(n-3)}{\varphi\eta^0\cdots\eta^{n-1}}, \overset{(n-2)}{\square}, \overset{(n-1)}{\varphi\eta^0\cdots\eta^{n-1}}, \overset{(n)}{\varphi\eta^0\cdots\eta^{n-1}}, \overset{(n+1)}{a]}$$

where $\sigma_0\in L_n$ and $\sigma_1\in M_n$ are solvents of

$$[\overset{(0)}{\theta},\ \overset{(1)}{\varphi\eta^0\cdots\eta^{n-2}},\ \overset{}{\cdots},\ \overset{(n-4)}{\varphi\eta^0\cdots\eta^{n-2}},\ \overset{(n-3)}{\square},\ \overset{(n-2)}{\varphi\eta^0\cdots\eta^{n-2}},\ \overset{(n-1)}{\varphi\eta^0\cdots\eta^{n-2}},\ \overset{(n)}{a\varepsilon^0}]$$

and

$$[\overset{(0)}{\theta},\ \overset{(1)}{\varphi\eta^0\cdots\eta^{n-2}},\ \cdots,\ \overset{(n-4)}{\varphi\eta^0\cdots\eta^{n-2}},\ \overset{(n-3)}{\square},\ \overset{(n-2)}{\varphi\eta^0\cdots\eta^{n-2}},\ \overset{(n-1)}{\varphi\eta^0\cdots\eta^{n-2}},\ \overset{(n)}{a\varepsilon^1}]$$

respectively. Then we have, by (B),

$$\{x_{n-2}\varepsilon^{n-2}\}_{K;L,M}=\{a\}_{K;L,M}. \qquad \cdots\cdots (3)$$

We next consider a solvent $\gamma\in(L\cap M)_{n-1}$ of

$$[\overset{(0)}{\varphi\eta^0\cdots\eta^{n-3}},\ \cdots,\ \overset{(n-5)}{\varphi\eta^0\cdots\eta^{n-3}},\ \overset{(n-4)}{\theta\varepsilon^{n-4}},\ \overset{(n-3)}{\varphi\eta^0\cdots\eta^{n-3}},\ \overset{(n-2)}{\square},\ \overset{(n-1)}{b\varepsilon^0\varepsilon^0}]$$

and let $x_{n-1}\in K_{n+1}$ be a solvent of

$$[\overset{(0)}{\tau_0},\ \overset{(1)}{\tau_1},\ \overset{(2)}{\varphi\eta^0\cdots\eta^{n-1}},\ \cdots,\ \overset{(n-3)}{\varphi\eta^0\cdots\eta^{n-1}},\ \overset{(n-2)}{x_{n-2}\varepsilon^{n-2}},\ \overset{(n-1)}{\varphi\eta^0\cdots\eta^{n-1}},\ \overset{(n)}{\square},\ \overset{(n+1)}{b}]$$

where $\tau_0\in L_n$ and $\tau_1\in M_n$ are solvents of

$$[\overset{(0)}{\gamma},\ \overset{(1)}{\varphi\eta^0\cdots\eta^{n-2}},\ \cdots,\ \overset{(n-4)}{\varphi\eta^0\cdots\eta^{n-2}},\ \overset{(n-3)}{x_{n-2}\varepsilon^{n-2}\varepsilon^0},\ \overset{(n-2)}{\varphi\eta^0\cdots\eta^{n-2}},\ \overset{(n-1)}{\square},\ \overset{(n)}{b\varepsilon^0}]$$

and

$$[\overset{(0)}{\gamma},\ \overset{(1)}{\varphi\eta^0\cdots\eta^{n-2}},\ \cdots,\ \overset{(n-4)}{\varphi\eta^0\cdots\eta^{n-2}},\ \overset{(n-3)}{x_{n-2}\varepsilon^{n-2}\varepsilon^1},\ \overset{(n-2)}{\varphi\eta^0\cdots\eta^{n-2}},\ \overset{(n-1)}{\square},\ \overset{(n)}{b\varepsilon^1}]$$

respectively. Then we have, by (B),

$$\{x_{n-2}\varepsilon^{n-2}\}_{K;L,M}\cdot\{x_{n-1}\varepsilon^{n}\}_{K;L,M}=\{b\}_{K;L,M}. \qquad \cdots\cdots (4)$$

Let $\omega\in(L\cap M)_n$ be a solvent of the following equation

$$[\overset{(0)}{\varphi\eta^0\cdots\eta^{n-2}},\ \cdots,\ \overset{(n-5)}{\varphi\eta^0\cdots\eta^{n-2}},\ \overset{(n-4)}{\theta},\ \overset{(n-3)}{\gamma},\ \overset{(n-2)}{c\eta^n\varepsilon^0\varepsilon^0},\ \overset{(n-1)}{\square},\ \overset{(n)}{x_{n+2}\varepsilon^0\varepsilon^0}]$$

and $x_{n+1}\in K_{n+1}$ be a solution of

$$[\overset{(0)}{\rho_0},\ \overset{(1)}{\rho_1},\ \overset{(2)}{\varphi\eta^0\cdots\eta^{n}},\ \cdots,\ \overset{(n-3)}{\varphi\eta^0\cdots\eta^{n}},\ \overset{(n-2)}{x_{n-2}},\ \overset{(n-1)}{x_{n-1}},\ \overset{(n)}{c\eta^n},\ \overset{(n+1)}{\square},\ \overset{(n+2)}{x_{n+2}}]$$

where $\rho_0\in L_{n+1}$ and $\rho_1\in M_{n+1}$ are solvents of

$$[\overset{(0)}{\omega},\ \overset{(1)}{\varphi\eta^0\cdots\eta^{n-1}},\ \cdots,\ \overset{(n-4)}{\varphi\eta^0\cdots\eta^{n-1}},\ \overset{(n-3)}{\sigma_0},\ \overset{(n-2)}{\tau_0},\ \overset{(n-1)}{c\eta^n\varepsilon^0},\ \overset{(n)}{\square},\ \overset{(n+1)}{x_{n+2}\varepsilon^0}]$$

and

$$[\overset{(0)}{\omega},\ \overset{(1)}{\varphi\eta^0\cdots\eta^{n-1}},\ \cdots,\ \overset{(n-4)}{\varphi\eta^0\cdots\eta^{n-1}},\ \overset{(n-3)}{\sigma_1},\ \overset{(n-2)}{\tau_1},\ \overset{(n-1)}{c\eta^n\varepsilon^1},\ \overset{(n)}{\square},\ \overset{(n+1)}{x_{n+2}\varepsilon^1}]$$

respectively. Then $x_{n+1}\varepsilon^0=\rho^0\varepsilon^n\in L$, $x_{n+1}\varepsilon^1=\rho_1\varepsilon^n\in M$, $x_{n+1}\varepsilon^{n-1}=x_{n-1}\varepsilon^n$, $x_{n+1}\varepsilon^n=c\eta^n\varepsilon^n=c$ and $x_{n+1}\varepsilon^i=\varphi\eta^0\cdots\eta^{n-1}$ for $2\leqq i\leqq n-2$ or $i=n+1$. Thus we have

$$\{x_{n-1}\varepsilon^n\}_{K;L,M}=\{c\}_{K;L,M}. \quad\cdots\cdots (5)$$

By (3), (4) and (5) we have

$$\{a\}_{K;L,M}\cdot\{c\}_{K;L,M}=\{b\}_{K;L,M}.$$

The proof of LEMMA 4.1 is complete.

Proof of PROPOSITION 2.8. It suffices to consider the case of Kan triad using the first definition. Let $(K; L, M)$ be a Kan triad with a base point φ, and a and b be elements of $\Gamma_n(K; L, M)$.

Consider a solvent $\theta\in(L\ M)_{n-1}$ of

$$[\overset{(0)}{\varphi\eta^0\cdots\eta^{n-3}}, \cdots, \overset{(n-5)}{\varphi\eta^0\cdots\eta^{n-3}}, \overset{(n-4)}{a\varepsilon^0\varepsilon^0}, \overset{(n-3)}{b\varepsilon^0\varepsilon^0}, \overset{(n-2)}{\square}, \overset{(n-1)}{\varphi\eta^0\cdots\eta^{n-3}}]$$

and let $c\in K_n$ be a solution of

$$[\overset{(0)}{\sigma_0}, \overset{(1)}{\sigma_1}, \overset{(2)}{\varphi\eta^0\cdots\eta^{n-1}}, \cdots, \overset{(n-3)}{\varphi\eta^0\cdots\eta^{n-1}}, \overset{(n-2)}{a}, \overset{(n-1)}{b}, \overset{(n)}{\square}, \overset{(n+1)}{\varphi\eta^0\cdots\eta^{n-1}}]$$

where $\sigma_0\in L_n$ and $\sigma_1\in M_n$ are solvents of

$$[\overset{(0)}{\theta}, \overset{(1)}{\varphi\eta^0\cdots\eta^{n-2}}, \cdots, \overset{(n-4)}{\varphi\eta^0\cdots\eta^{n-2}}, \overset{(n-3)}{a\varepsilon^0}, \overset{(n-2)}{b\varepsilon^0}, \overset{(n-1)}{\square}, \overset{(n)}{\varphi\eta^0\cdots\eta^{n-1}}]$$

and

$$[\overset{(0)}{\theta}, \overset{(1)}{\varphi\eta^0\cdots\eta^{n-2}}, \cdots, \overset{(n-4)}{\varphi\eta^0\cdots\eta^{n-2}}, \overset{(n-3)}{a\varepsilon^1}, \overset{(n-2)}{b\varepsilon^1}, \overset{(n-1)}{\square}, \overset{(n)}{\varphi\eta^0\cdots\eta^{n-1}}]$$

respectively. Then, by (A) and (C), we have

$$\{c\}_{K;L,M}=\{b\}_{K;L,M}\cdot(\{a\}_{K;L,M})^{-1}$$

and

$$\{c\}_{K;L,M}=(\{a\}_{K;L,M})^{-1}\cdot\{b\}_{K;L,M}.$$

Therefore

$$\{a\}_{K;L,M}\cdot\{b\}_{K;L,M}=\{b\}_{K;L,M}\cdot\{a\}_{K;L,M}.$$

§5. The induced maps.

Let $f:(K, \varphi)\to(K', \varphi')$ be a c. s. s. map where K and K' are Kan complexes and φ and $\varphi'$are 0-simplices of K and K' respectively.

If ρ is a homotopy between simplices σ and τ of $\Gamma_n(K, \varphi)$, the it is clear that

$f(\rho): f(\sigma) \sim f(\tau)$. Therefore we may define the induced map $f_*: \pi_n(K, \varphi) \to \pi_n(K', \varphi')$ by $f_*(\{\sigma\}_K) = \{f(\sigma)\}_{K'}$. It is easy to see that f_* is homomorphic when $n \geqq 1$.

If f is a c. s. s. map between two Kan pairs (K, L, φ) and (K', L', φ'), then f may induce also a transformation $f_*: \pi_n(K, L, \varphi) \to \pi_n(K', L', \varphi')$ for $n \geqq 1$, and f_* is homomorphic when $n \geqq 2$.

If $f: (K; L, M, \varphi) \to (K'; L', M', \varphi')$ is a c. s. s. map where $(K; L, M)$ and $(K'; L', M')$ are Kan triads and φ and φ' are base points of these triads, then the induced transformation $f_*: \pi_n(K; L, M, \varphi) \to \pi_n(K'; L', M', \varphi')$ may be defined for $n \geqq 2$, and it is homomorphic when $n \geqq 3$.

When f is a map of arbitrary c. s. s. complexes, we define f_* by $(S|f|)_*$ in each case considered above. Then f_* is homomorphic for the same n as before. If f is a map of Kan complexes, it is easily seen that these two definitions of f_* may be indentified. It is easy to see that the following two theorems hold for absolute, relative and triadic homotopy theories.

THEOREM 5.1. *If $f: (K; L, M, \varphi) \to (K; L, M, \varphi)$ is the identity, then f_* is the identity.*

THEOREM 5.2. *$(g \circ f)_* = g_* \circ f_*$ where $f: (K; L, M, \varphi) \to (K'; L', M', \varphi')$ and g: $(K'; L', M', \varphi') \to (K''; L'', M'', \varphi'')$ are maps.*

Now we define two boundary operators $\delta_0: \pi_n(K; L, M, \varphi) \to \pi_{n-1}(L, L \cap M, \varphi)$ and $\delta_1: \pi_n(K; L, M, \varphi) \to \pi_{n-1}(M, L \cap M, \varphi)$.

In the case of Kan complexes using the first definition of π_n, define δ_0 and δ_1 by $\delta_0(\{\sigma\}_{K;L,M}) = \{\sigma\varepsilon^0\}_{L,L\cap M}$ and $\delta_1(\{\sigma\}_{K;L,M}) = \{\sigma\varepsilon^1\}_{M;L\cap M}$ where $\sigma \in \Gamma_n(K; L, M, \varphi)$ and $n \geqq 2$.

In the general case, for $\tau \in \Gamma_n(S|K|; S|L|, S|M|, i(\varphi))$ define δ_0 and δ_1 by

$$\delta_0(\{\tau\}_{S|K|;\, S|L|,\, S|M|}) = \{\tau\varepsilon^0\}_{S|L|,\, S|L| \cap S|M|}$$

and

$$\delta_1(\{\tau\}_{S|K|;S|L|,\, S|M|}) = \{\tau\varepsilon^1\}_{S|M|,\, S|L| \cap S|M|}.$$

For the case of Kan complexes the following diagrams are commutative

$$\begin{array}{ccc} \pi_n(K; L, M, \varphi) & \xrightarrow{\delta_0} & \pi_{n-1}(L, L \cap M, \varphi) \\ \downarrow i_* & & \downarrow i_* \\ \pi_n(S|K|; S|L|, S|M|, i(\varphi)) & \xrightarrow{\delta_0} & \pi_{n-1}(S|L|, S|L| \cap S|M|, i(\varphi)) \end{array}$$

and

$$\begin{array}{ccc} \pi_n(K; L, M, \varphi) & \xrightarrow{\delta_1} & \pi_{n-1}(M, L \cap M, \varphi) \\ \downarrow i_* & & \downarrow i_* \\ \pi_n(S|K|; S|L|, S|M|, i(\varphi)) & \xrightarrow{\delta_1} & \pi_{n-1}(S|M|, S|L| \cap S|M|, i(\varphi)) \end{array}$$

where π_n and π_{n-1} mean those of the first definition. Since i_* is an isomorphism, we may identify two δ_0's and two δ_1's respectively.

PROPOSITION 5.3. *δ_0 and δ_1 are homomorphic for $n \geqq 3$.*

Proof. It suffices to consider the case of Kan complexes using the first definition. Let σ and τ be simplices of $\Gamma_n(K; L, M, \varphi)$ and $\beta \in \Gamma_n(K; L, M, \varphi)$ be a representative of $\{\sigma\}_{K;L,M} \cdot \{\tau\}_{K;L,M}$, then there exists $\alpha \in K_{n+1}$ such that $\alpha\varepsilon^0 \in L_n$, $\alpha\varepsilon^1 \in M_n$, $\alpha\varepsilon^i = \varphi\eta^0 \cdots \eta^{n-1}$ for $2 \leqq i \leqq n-2$, $\alpha\varepsilon^{n-1} = \sigma$, $\alpha\varepsilon^n = \beta$ and $\alpha\varepsilon^{n+1} = \tau$.

Since $\alpha\varepsilon^0\varepsilon^0 \in (L \cap M)_{n-1}$, $\alpha\varepsilon^i = \varphi\eta^0 \cdots \eta^{n-2}$ for $1 \leqq i \leqq n-3$, $\alpha\varepsilon^0\varepsilon^{n-2} = \sigma\varepsilon^0$, $\alpha\varepsilon^0\varepsilon^{n-1} = \beta\varepsilon^0$ and $\alpha\varepsilon^0\varepsilon^n = \tau^0$, we have $\{\beta\varepsilon^0\}_{L,\, L\cap M} = \{\sigma\varepsilon^0\}_{L,\, L\cap M} \cdot \{\tau\varepsilon^0\}_{L,\, L\cap M}$. Thus δ_0 is homomorphic. It may be proved similarly that δ_1 is homomorphic.

REMARK 5.4. $\delta: \pi_n(K, L, \varphi) \to \pi_{n-1}(L, \varphi)$ defined by $\delta(\{\sigma\}_{K,L}) = \{\sigma\varepsilon^0\}_L$ is homomorphic when $n \geqq 2$.

Since c. s. s. map f commutes with the face operators ε^0 and ε^1, we have

THEOREM 5.5. *The following diagrams are commutative:*

$$\begin{array}{ccc} \pi_n(K; L, M, \varphi) & \xrightarrow{\delta_0} & \pi_{n-1}(L, L\cap M, \varphi) \\ \downarrow f_* & & \downarrow (f|L)_* \\ \pi_n(K'; L', M', \varphi') & \xrightarrow{\delta_0} & \pi_{n-1}(L', L'\cap M', \varphi'), \end{array}$$

$$\begin{array}{ccc} \pi_n(K; L, M, \varphi) & \xrightarrow{\delta_1} & \pi_{n-1}(M, L\cap M, \varphi) \\ \downarrow f_* & & \downarrow (f|M)_* \\ \pi_n(K'; L', M', \varphi') & \xrightarrow{\delta_1} & \pi_{n-1}(M', L'\cap M', \varphi'). \end{array}$$

§6. The homotopy of c. s. s. maps.

Let $I = \Delta[1]$ (the standard 1-simplex defined in DEFINITION 2.2 of [6], this is the c. s. s. complex $K[1]$ defined in [1, p. 508]), $\varepsilon_1 \in I$ be the only non-degenerate 1-simplex. Let K be a c. s. s. complex and $I \times K$ be the cartesian product ([6], Definition 2.1) of I and K.

For every simplex $\sigma \in K_n$ consider $n+2$ simplices of $(I \times K)_n$:

$$\tau_0(\sigma) = (\varepsilon_1\varepsilon^0\eta^0\eta^1 \cdots \eta^{n-1}, \sigma),$$

$$\tau_i(\sigma) = (\varepsilon_1\underbrace{\eta^0 \cdots \eta^0}_{i-1}\eta^i \cdots \eta^{n-1}, \sigma) \text{ for } 1 \leqq i \leqq n,$$

$$\tau_{n+1}(\sigma) = (\varepsilon_1\varepsilon^1\underbrace{\eta^0 \cdots \eta^0}_{n}, \sigma).$$

($\tau_0(\sigma)$ and $\tau_{n+1}(\sigma)$ are the simplices $0_K\sigma$ and $1_K\sigma$ respectively defined in [6].)

Kan [6] defined a homotopy relation relative to the base point for maps of c. s. s. complexes with base point as follows.

Let (K, φ) and (K', φ') be two c. s. s. complexes with base points φ and φ', then two c. s. s. maps $f, g: (K, \varphi)\to(K', \varphi')$ are called *homotopic relative to* φ if there exists a c. s. s. map $f_I: I\times K\to K'$ such that

(i) $f_I: f\simeq g$, i.e. $f_I(\tau_0(\sigma))=f(\sigma)$ and $f_I(\tau_{\dim.\sigma+1}(\sigma))=g(\sigma)$ for every $\sigma\in K$,

(ii) $f_I(\varepsilon_1, \varphi\eta^0)=\varphi'\eta^0$.

We denote this homotopy relation by $f_I: f\simeq g$ *rel.* φ. By the condition (ii) we have $f_I(\varepsilon_1\varepsilon^0, \varphi)=f_I(\varepsilon_1\varepsilon^1, \varphi)=\varphi'$.

Now we extend this definition for the case of maps of c. s. s. pairs and triads with base point as follows.

DEFINITION 6.1. Let $(K; L, M, \varphi)$ and $(K'; L', M', \varphi')$ be two c. s. s. triads with base points φ and φ'. Two c. s. s. maps $f, g: (K; L, M, \varphi)\to(K'; L', M', \varphi')$ are called *homotopic on* $(K; L, M, \varphi)$ if there exists a c. s. s. map $f_I: (I\times K; I\times L, I\times M) \to (K'; L', M')$ such that

(i) $f_I(\tau_0(\sigma))=f(\sigma)$ and $f_I(\tau_{\dim.\sigma+1}(\sigma))=g(\sigma)$ for every $\sigma\in K$,

(ii) $f_I(\varepsilon_1, \varphi\eta^0)=\varphi'\eta^0$.

We denote this relation by $f_I: f\simeq g$ *on* $(K; L, M, \varphi)$

When $M=N_\varphi$ and $M'=N_{\varphi'}$, we denote this relation by $f_I: f\simeq g$ *on* (K, L, φ).

We may prove the following Lemma by simple computation.

LEMMA 6.2. *Let* $(K; L, M, \varphi)$ *and* $(K'; L', M', \varphi')$ *be c. s. s. triads with base point,* $F: (I\times K; I\times L, I\times M)\to(K'; L', M')$ *be a c. s. s. map such that* $F(\varepsilon_1, \varphi\eta^0)=\varphi'\eta^0$. *If* σ *is a simplex of* $\Gamma_n(K; L, M, \varphi)$ *then* $\omega_i=F(\tau_i(\sigma))$ *is a simplex of* $\Gamma_n(K'; L', M', \varphi')$ $(0\leqq i\leqq n+1)$.

Now we have

LEMMA 6.3. *Let* $(K; L, M, \varphi)$ *be a c. s. s. triad with base point* φ, $(K'; L', M', \varphi')$ *be a Kan triad with base point* φ', F *and* ω_i *be those considered in* LEMMA 6.2. *Then we have*

$$\omega_i\sim\omega_{i+1} \textit{ lsd. } L', M' \textit{ for } 0\leqq i\leqq n.$$

Proof. Put

$$\rho_i=F(\varepsilon_1\underbrace{\eta^0\cdots\eta^0}_{i}\eta^{i+1}\cdots\eta^n, \sigma\eta^i) \text{ for } 0\leqq i\leqq n.$$

(1) We have

$$\rho_0\varepsilon^0=\omega_0,\ \rho_0\varepsilon^1=\omega_1,\ \rho_0\varepsilon^2=F(\varepsilon_1\eta^1\cdots\eta^{n-1}, \sigma\varepsilon^1\eta^0)\in M'_n,$$

$$\rho_0\varepsilon^i=\rho'\eta^0\cdots\eta^{n-1}\ (3\leqq i\leqq n+1).$$

Let $\xi\in M'_{n+1}$ be a solvent of the following equation in M'

$$\begin{matrix} (0) & (1) & (2) & (3) & (4) & \cdots & (n+1) \\ [\square, & \omega_1\eta^2\varepsilon^1, & \omega_1\eta^0\varepsilon^2, & \rho_0\varepsilon^2, & \varphi'\eta^0\cdots\eta^{n-1}, & \cdots, & \varphi'\eta^0\cdots\eta^{n-1}]. \end{matrix}$$

And consider a solution $\eta\in K'_{n+1}$ of the following equation

$$\begin{matrix} (0) & (1) & (2) & (3) & (4) & (5) & \cdots & (n+2) \\ [\square, & \omega_1\eta^2, & \xi, & \omega_1\eta^0, & \rho_0, & \varphi'\eta^0\cdots\eta^n, & \cdots, & \varphi'\eta^0\cdots\eta^n]. \end{matrix}$$

Then we have

$$\eta\varepsilon^0=\omega_1\eta^2\varepsilon^0\in L'_n,\ \eta\varepsilon^1=\xi\varepsilon^0\in M'_n,\ \eta\varepsilon^2=\omega_1,\ \eta\varepsilon^3=\omega_0,$$

$$\eta\varepsilon^i=\varphi'\eta^0\cdots\eta^{n-1}\ (4\leqq i\leqq n+1).$$

Therefore $\omega_0\sim\omega_1$ lsd, L', M'.

(2) Since

$$\omega_1\varepsilon^1=F(\varepsilon_1\eta^1\cdots\eta^{n-2},\ \sigma\varepsilon^1)\ \text{and}\ \omega_2\varepsilon^1=F(\varepsilon_1\eta^1\cdots\eta^{n-2},\ \sigma\varepsilon^1)$$

we have $\omega_1\varepsilon^i=\omega_2\varepsilon^i$ for $1\leqq i\leqq n$.

Moreover

$\rho_1\varepsilon^0=F(\varepsilon_1\eta^1\cdots\eta^{n-1},\ \sigma\varepsilon^0\eta^0)\in L'_n$, $\rho_1\varepsilon^1=\omega_1$, $\rho_1\varepsilon^2=\omega_2$ and $\rho^1\varepsilon^i=\varphi'\eta^0\cdots\eta^{n-1}$ for $3\leqq i\leqq n+1$. Therefore we have $\omega_1\sim\omega_2$ lsd. L', and consequently $\omega_1\sim\omega_2$ lsd. L', M'.

(3) For $2\leqq i\leqq n$, we have

$$\rho_i\varepsilon^0=F(\varepsilon_1\underbrace{\eta^0\cdots\eta^0}_{i}\eta^{i+1}\cdots\eta^n\varepsilon^0,\ \sigma\eta^i\varepsilon^0)$$

$$=F(\varepsilon_1\eta^0\cdots\eta^0\eta^{i+1}\cdots\eta^n\varepsilon^0,\ \sigma\varepsilon^0\eta^{i-1})\in L'_n,$$

$$\rho_i\varepsilon^1=F(\varepsilon_1\eta^0\cdots\eta^0\eta^{i+1}\cdots\eta^n\varepsilon^1,\ \sigma\eta^i\varepsilon^1)\in M'_n,$$

$$\rho_i\varepsilon^i=F(\tau_i(\sigma))=\omega_i,\ \rho_i\varepsilon^{i+1}=F(\tau_{i+1}(\sigma))=\omega_{i+1}.$$

Moreover if $2\leqq j\leqq i-1$

$$\rho_i\varepsilon^j=F(\varepsilon_1\underbrace{\eta^0\cdots\eta^0}_{i}\eta^{i+1}\cdots\eta^n\varepsilon^j,\ \sigma\eta^i\varepsilon^j)$$

$$=F(\varepsilon_1\underbrace{\eta^0\cdots\eta^0}_{i-1}\eta^i\cdots\eta^{n-1},\ \varphi\eta^0\cdots\eta^{n-1})$$

$$=F(\varepsilon_1,\underbrace{\eta^0\cdots\eta^0}_{i-1}\eta^i\cdots\eta^{n-1},\ \varphi\underbrace{\eta^0\cdots\eta^0}_{i}\eta^i\cdots\eta^{n-1})$$

$$=(F(\varepsilon_1,\ \varphi\eta^0))\underbrace{\eta^0\cdots\eta^0}_{i-1}\eta^i\cdots\eta^{n-1}$$

$$=\varphi'\eta^0\underbrace{\eta^0\cdots\eta^0}_{i-1}\eta^i\cdots\eta^{n-1}=\varphi'\eta^0\eta^1\cdots\eta^{n-1}.$$

Similarly if $i+2\leqq j\leqq n+1$, $\rho_i\varepsilon^j=\varphi'\eta^0\eta^1\cdots\eta^{n-1}$.

Thus we have $\omega_i\sim\omega_{i+1}$ lsd. L', M'.

THEOREM 6.4. *If f and $g:(K: L, M, \varphi)\rightarrow(K'; L', M', \varphi')$ are homotopic on $(K; L, M, \varphi)$, then*

$$f_*=g_*: \pi_n(K: L, M, \varphi)\rightarrow\pi_n(K'; L', M', \varphi')$$

for $n\geqq 2$.

If $f\simeq g$ on (K, L, φ), then

$$f_*=g_*: \pi_n(K, L, \varphi)\rightarrow\pi_n(K', L', \varphi') \text{ for } n\geqq 1.$$

If $f\simeq g$ rel. φ, then

$$f_*=g_*: \pi_n(K, \varphi)\rightarrow\pi_n(K,' \varphi') \text{ for } n\geqq 0.$$

Proof. When we use the first definitition, this theorem follows from LEMMAS 6.2, 6.3 and the tranisitivity of homotopy relation of c. s. s. complexes (PROPOSITIONS 1.5 and 2.2).

In the general case, if $f\simeq g:(K; L, M, \varphi)\rightarrow(K'; L', M', \varphi')$ on $(K; L, M \varphi)$, then it is easy to see that $S|f|\simeq S|g|:(S|K|; S|L|, S|M|, i(\varphi))\rightarrow(S|K'|; S|L'|, S|M'|, i(\varphi'))$ on $(S|K|; S|L|, S|M|, i(\varphi))$. Therefore $f_*=(S|f|)_*=(S|g|)_*=g_*$.

§7. The exact sequences.

In this section we prove the exactness of the following two sequences, considering the homotopy class $\{\varphi\eta^0\}$ or $\{\varphi\eta^0\eta^1\}$ as the zero element in the case where the set of homotopy classes is not group.

$$\text{(I)}\cdots\rightarrow\pi_n(L, L\cap M, \varphi)\xrightarrow{i_{0*}}\pi_n(K, M, \varphi)\xrightarrow{j_{0*}}\pi_n(K; L, M, \varphi)\xrightarrow{\delta_0}\pi_{n-1}(L, L\cap M, \varphi)\longrightarrow\cdots$$
$$\cdots\rightarrow\pi_2(K; L M, \varphi)\xrightarrow{\delta_0}\pi_1(L, L\cap M, \varphi)\xrightarrow{i_{0*}}\pi_1(K, M, \varphi),$$

$$\text{(II)}\cdots\pi_n(K, L\cap M, \varphi)\xrightarrow{i_{1*}}\pi_n(K, L, \varphi)\xrightarrow{j_{1*}}(K; L, M, \varphi)\xrightarrow{\delta_1}\pi_{n-1}(M, L\cap M, \varphi)\longrightarrow$$
$$\cdots\rightarrow\pi_2(K; L, M, \varphi)\xrightarrow{\delta_1}\pi_1(M, L\cap M, \varphi)\xrightarrow{i_{1*}}\pi_1(K, L, \varphi),$$

where

$$i_0:(L, L\cap M)\rightarrow(K, M),\ i_1:(M, L\cap M)\rightarrow(K, L),$$
$$j_0:(K; N_\varphi, M)\rightarrow(K; L, M,) \text{ and } j_1:(K; L, N_\varphi)\rightarrow(K; L, M)$$

are inclusion maps and i_{0*}, i_{1*}, j_{0*} and j_{1*} are the induced maps, identifying $\pi_n(K, M, \varphi)=\pi_n(K; M, N_\varphi, \varphi)$ with π_n $(K; N_\varphi, M, \varphi)$.

In the special case $M=N_\varphi$, the sequence (I) reduces to the following sequence:

$$\text{(III)}\cdots\rightarrow\pi_n(L, \varphi)\xrightarrow{i_*}\pi_n(K, \varphi)\xrightarrow{j_*}\pi_n(K, L, \varphi)\xrightarrow{\delta}\pi_{n-1}(L, \varphi)\longrightarrow\cdots\longrightarrow\pi_1(L, \varphi)$$
$$\xrightarrow{i_*}\pi_1(K, \varphi).$$

We will prove also the exactness of

$$\text{(III}'\text{)}\quad \pi_1(L, \varphi)\xrightarrow{i_*}\pi_1(K, \varphi)\xrightarrow{j_*}\pi_1(K, L, \varphi)\xrightarrow{\delta}\pi_0(L, \varphi)\xrightarrow{i_*}\pi_0(K, \varphi)\xrightarrow{j_*}\pi_0(K, L, \varphi)\longrightarrow 0,$$

considering the component $\{\varphi\}$ as the zero element of π_0.

THEOREM 7.1. *The sequence* (I) *is exact.*

Proof. It suffices to consider the case of Kan complexes using the first definition.
1°. Let $\{\sigma\}_{L,L\cap M}$ be an element of $\pi_n(L, L\cap M, \varphi)$. Then $i_{0*}(\{\sigma\}_{L,L\cap M})=\{\sigma\}_{K,M}$. Since $\sigma\in\Gamma_n(L, L\cap M, \varphi)$, we have the following equation in L:

$$\overset{(0)}{[\square}, \overset{(1)}{\sigma\varepsilon^1\eta^0}, \overset{(2)}{\sigma}, \overset{(3)}{\varphi\eta^0\cdots\eta^{n-1}}, \cdots, \overset{(n+1)}{\varphi\eta^0\cdots\eta^{n-1}}].$$

Let $\tau\in L_n$ be a solution of this equation. Then $\tau\varepsilon^1=\sigma\varepsilon^0\in(L\cap M)_{n-1}$ and $\tau\varepsilon^i=\varphi\eta^0\cdots\eta^{n-2}$ for each $i\neq 1$. Therefore the following equation holds in L:

$$\overset{(0)}{[\square}, \overset{(1)}{\tau\varepsilon^1\eta^0}, \overset{(2)}{\tau}, \overset{(3)}{\varphi\eta^0\cdots\eta^{n-1}}, \cdots, \overset{(n+1)}{\varphi\eta^0\cdots\eta^{n-1}}]$$

and we have $\{\tau\}_{K;L,M}=0$ (by THEOREM 2.3). Thus the image of i_{0*} is contained in the kernel of j_{0*}.
2°. Let $\{\sigma\}_{K,M}$ be an element of $\pi_n(K, M, \varphi)$ and $\tau\in K_n$ be a solution of the equation

$$\overset{(0)}{[\square}, \overset{(1)}{\sigma\varepsilon^1\eta^0}, \overset{(2)}{\sigma}, \overset{(3)}{\varphi\eta^0\cdots\eta^{n-1}}, \cdots, \overset{(n+1)}{\varphi\eta^0\cdots\eta^{n-1}}].$$

Then $\tau\varepsilon^1=\sigma\varepsilon^0\in M_{n-1}$ and $\tau\varepsilon^i=\varphi\eta^0\cdots\eta^{n-2}$ for $i\neq 1$, and $\delta_0 j_{0*}(\{\sigma\}_{K,M})=\delta_0(\{\tau\}_{K;L,M})=\{\tau\varepsilon^0\}_{L,L\cap M}=0$. Thus the image of j_{0*} is contained in the kernel of δ_0.
3°. Let $\{\sigma\}_{K;L,M}$ be an element of $\pi_n(K; L, M, \varphi)$. Then $i_{0*}\delta_0(\{\sigma\}_{K;L,M})=i_{0*}(\{\sigma\varepsilon^0\}_{L,L\cap M})=\{\sigma\varepsilon^0\}_{K,M}$. Let $\tau\in K_n$ be a solution of

$$\overset{(0)}{[\square}, \overset{(1)}{\sigma\varepsilon^1\eta^0}, \overset{(2)}{\sigma}, \overset{(3)}{\varphi\eta^0\cdots\eta^{n-1}}, \cdots, \overset{(n+1)}{\varphi\eta^0\cdots\eta^{n-1}}].$$

Then $\tau\varepsilon^0=\sigma\varepsilon^1\in M_{n-1}$, $\tau\varepsilon^1=\sigma\varepsilon^0$ and $\tau\varepsilon^i=\varphi\eta^0\cdots\eta^{n-2}$ for $i\neq 0, 1$. Therefore $\{\sigma\varepsilon^0\}_{K,M}=0$ (by PROPOSITION 1.6). Thus the image of δ_0 is contained in the kernel of i_{0*}.
4°. Let $\{\sigma\}_{K,M}$ be an element of $\pi_n(K, M, \varphi)$ such that $j_{0*}(\{\sigma\}_{K,M})=0$. It is clear that the following diagram is commutative:

$$\begin{array}{ccc} \pi_n(K, M, \varphi) & \xrightarrow{j_{0*}} & \pi_n(K; L, M, \varphi) \\ \| & & \downarrow u \\ \pi_n(K; M, N_\varphi, \varphi) & \xrightarrow{k_*} & \pi_n(K; M, L, \varphi) \end{array}$$

where k_* is induced by the inclusion map $k:(K; M, N_\varphi)\to(K; M, L)$ and u is the transformation defined in THEOREM 2.9. Therefore we have $\{\sigma\}_{K;M,L}=0$, namely there exists $\omega\in K_{n+1}$ such that $\omega\varepsilon^0\in M_n$, $\omega\varepsilon^1\in L_n$, $\omega\varepsilon^2=\sigma$ and $\omega\varepsilon^i=\varphi\eta^0\cdots\eta^{n-1}$ for $i\neq 0$, 1, 2. Then we have $\{\sigma\}_{K,M}=\{\omega\varepsilon^1\}_{K,M}$ by PROPOSITION 1.6. On the other hand $\omega\varepsilon^1\in\Gamma_n(L, L\cap M, \varphi)$, hence $\{\sigma\}_{K,M}=i_{0*}(\{\omega\varepsilon^1\}_{L,L\cap M})$. Thus the kernel of j_{0*} is contained in the image of i_{0*}.

5°. Let $\{\sigma\}_{K;L,M}$ be an element of $\pi_n(K; L, M, \varphi)$ such that $\delta_0(\{\sigma\}_{K;L,M})=0$. i.e. $\{\sigma\varepsilon^0\}_{L,L\cap M}=0$. By PROPOSITION 1.6, there is $\omega\in L_n$ such that $\omega\varepsilon^0\in(L\cap M)_{n-1}$, $\omega\varepsilon^1=\sigma\varepsilon^0$ and $\omega\varepsilon^i=\varphi\eta^0\cdots\eta^{n-2}$ for $i\neq 0, 1$. Let $\beta\in K_n$ be a solution of

$$\overset{(0)}{[\omega}, \overset{(1)}{\alpha}, \overset{(2)}{\sigma}, \overset{(3)}{\square}, \overset{(4)}{\varphi\eta^0\cdots\eta^{n-1}}, \overset{\cdots}{\cdots}, \overset{(n+1)}{\varphi\eta^0\cdots\eta^{n-1}]},$$

where $\alpha\in M_n$ be a solvent of the equation

$$\overset{(0)}{[\omega\varepsilon^0}, \overset{(1)}{\sigma\varepsilon^1}, \overset{(2)}{\square}, \overset{(3)}{\varphi\eta^0\cdots\eta^{n-2}}, \overset{\cdots}{\cdots}, \overset{(n)}{\varphi\eta^0\cdots\eta^{n-2}]}.$$

Then $\{\sigma\}_{K;L,M}=\{\beta\}_{K;L,M}$ and $\beta\in\Gamma_n(K; N_\varphi, M, \varphi)$. Let $a\in\pi_n(K, M, \varphi)=\pi_n(K; M, N_\varphi, \varphi)$ be the element corresponding to $\{\beta\}_{K;N_\varphi,M}$ by the transformation given in THEOREM 2.9. By the definition of j_{0*}, we have $\{\sigma\}_{K;L,M}=j_{0*}(a)$. Thus the kernel of δ_0 is contained in the image of j_{0*}.

6°. Let $\{\sigma\}_{L,L\cap M}$ be an element of $\pi_{n-1}(L, L\cap M, \varphi)$ such that $i_{0*}(\{\sigma\}_{L,L\cap M})=0$. Then there exists $\omega\in K_n$ such that $\omega\varepsilon^0\in M_{n-1}$, $\omega\varepsilon^1=\sigma$ and $\omega\varepsilon^i=\varphi\eta^0\cdots\eta^{n-2}$ for $i\neq 0, 1$. Let $\omega'\in K_n$ be a solution of

$$\overset{(0)}{[\square}, \overset{(1)}{\omega\varepsilon^1\eta^0}, \overset{(2)}{\omega}, \overset{(3)}{\varphi\eta^0\cdots\eta^{n-1}}, \overset{\cdots}{\cdots}, \overset{(n+1)}{\varphi\eta^0\cdots\eta^{n-1}]}.$$

Then $\omega'\in\Gamma_n(K; L, M, \varphi)$ and $\delta^0(\{\omega'\}_{K;L,M})=\{\omega'\varepsilon^0\}_{L,L\cap M}=\{\omega\varepsilon^1\}_{L;L\cap M}=\{\sigma\}_{L,L\cap M}$. Thus the kernel of i_{0*} is contained in the image of δ_0.

THEOREM 7.2. *The sequence* (II) *is exact.*

Proof. Consider the following commutative diagram

$$\begin{array}{ccccccccc}
\to\cdots\pi_n(M, L\cap M, \varphi) & \xrightarrow{i_{1*}} & \pi_n(K, L, \varphi) & \xrightarrow{j_{1*}} & \pi_n(K; L, M, \varphi) & \xrightarrow{\delta_1} & \pi_{n-1}(M, L\cap M, \varphi) & \to\cdots \\
\| & & \| & & \downarrow u & & \| & \\
\cdots\to\pi_n(M, L\cap M, \varphi) & \xrightarrow{i_{1*}} & \pi_n(K, L, \varphi) & \xrightarrow{h_*} & \pi_n(K; M, L, \varphi) & \xrightarrow{\delta_0} & \pi_{n-1}(M, L\cap M, \varphi) & \to\cdots
\end{array}$$

where h_* is the injection induced by the inclusion map $h:(K; N_\varphi, L)\to(K; M, L)$, identifying $\pi_n(K, L, \varphi)=\pi_n(K; L, N_\varphi, \varphi)$ with $\pi_n(K; N_\varphi, L, \varphi)$, δ_0 is the boundary operator induced by the face operator ε^0, and u is the transformation defined in THEOREM 2.9.

Since the lower sequence is exact by THEOREM 7.1, we have

$\operatorname{Im} i_{1*}=\operatorname{Ker} h_*=\operatorname{Ker}(u\circ j_{1*})=\operatorname{Ker} j_{1*}$,

$\operatorname{Im} j_{1*}=\operatorname{Im}(u^{-1}\circ h_*)=u^{-1}(\operatorname{Im} h_*)=u^{-1}(\operatorname{Ker}\delta_0)=u^{-1}(\operatorname{Ker}(\delta_1\circ u^{-1}))$

$\quad=u^{-1}(u(\operatorname{Ker}\delta_1))=\operatorname{Ker}\delta_1$,

$\mathrm{Im}\ \delta_1 = \mathrm{Im}\ (\delta_0 \circ u) = \mathrm{Im}\ \delta_0 = \mathrm{Ker}\ i_{1*}$.

Thus the upper sequence, i.e. (II), is exact.

THEOREM 7.3. *The sequence* (III) *and* (III′) *are exact.*

Proof. Since the sequence (III) is a special case of (I), the sequence (III) is exact. In order to show the exactness of (III′), it suffices to consider the case of Kan complexes using the first definition.

1°. For $\{\sigma\}_L \in \pi_1(L, \varphi)$, $j_* i_*(\{\sigma\}_L) = \{\sigma\}_{K,L}$. Let $\rho \in L_2$ be a solvent of the following equation in L:

$$\overset{(0)}{[\square},\ \overset{(1)}{\sigma},\ \overset{(2)}{\varphi\eta^0}].$$

Then $\rho : \sigma \sim \varphi\eta^0$ lsd. L, i.e. $\{\sigma\}_{K,L} = 0$.

2°. For $\{\sigma\}_K \in \pi_1(K, \varphi)$ such that $j_*(\{\sigma\}_K) = 0$, i.e. $\{\sigma\}_{K,L} = 0$, there exists $\rho \in K_2$ such that $\rho\varepsilon^0 \in L_1$, $\rho\varepsilon_1 = \sigma$ and $\rho\varepsilon^2 = \varphi\eta^0$. Then $\rho : \rho\varepsilon^0 \sim \sigma$, i.e. $i_*(\{\rho\varepsilon^0\}_L) = \{\sigma\}_K$.

3°. For $\{\sigma\}_K \in \pi_1(K, \varphi)$, $\delta j_*(\{\sigma\}_K) = \{\sigma\varepsilon^0\}_L = \{\varphi\}_L = 0$.

4°. For $\{\sigma\}_{K,L} \in \pi_1(K, L, \varphi)$ such that $\delta(\{\sigma\}_{K,L}) = 0$, i.e. $\{\sigma\varepsilon^0\}_L = 0$, there exists $\rho \in L_1$ such that $\rho\varepsilon^0 = \varphi$ and $\rho\varepsilon^1 = \sigma\varepsilon^0$. Let $\omega \in K_1$ be a solution of

$$\overset{(0)}{[\rho},\ \overset{(1)}{\square},\ \overset{(2)}{\sigma}].$$

Then $j_*(\{\omega\}_K) = \{\sigma\}_{K,L}$.

5°. For $\{\sigma\}_{K,L} \in \pi_1(K, L, \varphi)$, $i_*\delta(\{\sigma\}_{K,L}) = \{\sigma\varepsilon^0\}_K$. Since $\sigma\varepsilon^1 = \varphi$, we have $\sigma : \sigma\varepsilon^0 \sim \varphi$, i.e. $\{\sigma\varepsilon^0\}_K = 0$.

6°. For $\{\sigma\}_L \in \pi_0(L, \varphi)$ such that $i_*(\{\sigma\}_L) = 0$, i.e. $\{\sigma\}_K = 0$, there exists $\rho \in K_1$ such that $\rho\varepsilon^0 = \sigma$ and $\rho\varepsilon^1 = \varphi$. Therefore $\delta(\{\rho\}_{K,L}) = \{\sigma\}_L$.

7°. For $\{\sigma\}_L \in \pi_0(L, \varphi)$, $j_* i_*(\{\sigma\}_L) = \{\sigma\}_{K,L}$. From the definition of the zero element of $\pi_0(K, L, \varphi)$, we have $\{\sigma\}_{K,L} = 0$.

8°. For $\{\sigma\}_K \in \pi_0(K, \varphi)$ such that $j_*(\{\sigma\}_K) = 0$, i.e. $\{\sigma\}_{K,L} = 0$, there exists $\rho \in K_1$ such that $\rho\varepsilon^0 \in L$ and $\rho\varepsilon^1 = \sigma$. Then $i_*(\{\rho\varepsilon^0\}_L) = \{\sigma\}_K$.

9°. By the definition, $j_* : \pi_0(K, \varphi) \to \pi_0(K, L, \varphi)$ is onto.

§8. The fibering theorem.

D. M. Kan [5] defined a fibre map as follows:

DEFINITION 8.1. For c.s.s. complexes E and B, a c.s.s. map $p : E \to B$ is called a *fibre map* if for every pair of integers (k, n) such that $0 \leqq k \leqq n$, for every equation in E:

$$\overset{(0)}{[\sigma_0},\ \cdots,\ \overset{(k-1)}{\sigma_{k-1}},\ \overset{(k)}{\square},\ \overset{(k+1)}{\sigma_{k+1}},\ \cdots,\ \overset{(n)}{\sigma_n}]$$

and for every solvent $\tau\in B_n$ of the equation

$$\overset{(0)}{[p(\sigma_0)}, \cdots, \overset{(k-1)}{p(\sigma_{k-1})}, \overset{(k)}{\Box}, \overset{(k+1)}{p(\sigma_{k+1})}, \cdots, \overset{(n)}{p(\sigma_n)]},$$

there is a solvent $\sigma\in E_n$ of the former equation such that $p(\sigma)=\tau$. The complex E is called the *total complex*, B the *base*. Let $\varphi\in B_0$ be a 0-simplex. Then we mean by the *fibre* of p over φ the subcomplex $F\subset E$ such that $F_n=p^{-1}(\varphi\eta^{0\cdots}\eta^{n-1})$ for all n.

PROPOSITION 8.2. *$S|\ |$ is a fibre-presesving functor where $|\ |$ is the geometric realization functor in the sense of J. Milnor* [7] *and S is the simplicial singular functor, i.e. if $p: E\to B$ is a fibre map, thee $S|p|: S|E|\to S|B|$ is also a fibre map.*

Proof. Consider an arbitrary equation in $S|E|$:

$$\overset{(0)}{[\sigma_0}, \cdots, \overset{(k-1)}{\sigma_{k-1}}, \overset{(k)}{\Box}, \overset{(k+1)}{\sigma_{k+1}}, \cdots, \overset{(n)}{\sigma_n]},$$

and assume that there is a solvent $\tau\in(S|B|)_n$ of the equation in $S|B|$:

$$\overset{(0)}{[S|p|(\sigma_0)}, \cdots, \overset{(k-1)}{S|p|(\sigma_{k-1})}, \overset{(k)}{\Box}, \overset{(k+1)}{S|p|(\sigma_{k+1})}, \cdots, \overset{(n)}{S|p|(\sigma_n)]}.$$

$S|p|(\sigma_i)$ is the composition $|p|\circ\sigma_i$ of maps σ_i and $|p|$, and $|p|$ is homeomorphic on the realization of each simplex of E, and p is a fibre map. Therefore we may choice a solvent $\sigma\in(S|E|)_n$ of the former equation such that $S|p|(\sigma)=\tau$.

THEOREM 8.3. *Let $p: E\to B$ be a fibre map, B' and B'' be subcomplexes of B and $E'=p^{-1}(B')$, $E''=p^{-1}(B'')$, Then* 1) *$p_*: \pi_n((E;\ E',\ E'',\ \psi)\to\pi_n(B;\ B',\ B'',\ \varphi)$ is isomorphic for $n\geqq 3$, one-to-one onto for $n=2$, where $\varphi\in(B'\cap B'')_0$ and $\psi\in(E'\cap E'')_0$ such that $p(\psi)=\varphi$,* 2) *$p_*: \pi_n(E,\ E',\ \psi)\to\pi_n(B,\ B',\ \varphi)$ is isomorphic for $n\geqq 2$, one-to-one onto for $n=1$ and $p_*^{-1}(0)=0$ for $n=0$, where $\varphi\in B'_0$ and $\psi\in E'_0$ such that $p(\psi)=\varphi$.*

Proof. It suffices to prove the THEOREM for the case E, E', E'', B, B' and B'' are Kan complexes using the first definition (Cf. PROPOSITION 8.2).

Proof of 1). It is clear that p_* is homomorphic for $n\geqq 3$. Let σ and $\tau\in\Gamma_n(E;\ E',\ E'',\ \psi)$ be simplices such that $p(\sigma)\sim p(\tau)$ lsd. B', B'', and $\rho\in B_{n+1}$ be the homotopy between $p(\sigma)$ and $p(\tau)$. Since p is a fibre map, there exists a solvent $\theta\in E'_n$ of the equation

$$\overset{(0)}{[\Box}, \overset{(1)}{\psi\eta^{0\cdots}\eta^{n-2}}, \cdots, \overset{(n-2)}{\psi\eta^{0\cdots}\eta^{n-2}}\ \overset{(n-1)}{\sigma\varepsilon^0}, \overset{(n)}{\tau\varepsilon^0]}$$

such that $p(\theta)=\rho\varepsilon^0$. Furthermore there exists a solvent $\rho'\in E_{n+1}$ of the equation

$$\overset{(0)}{[\theta}, \overset{(1)}{\Box}, \overset{(2)}{\psi\eta^{0\cdots}\eta^{n-1}}, \cdots, \overset{(n-1)}{\psi\eta^{0\cdots}\eta^{n-1}}, \overset{(n)}{\sigma}, \overset{(n+1)}{\tau]}$$

such that $p(\rho')=\rho$. Since $\rho'\varepsilon^1\in p^{-1}(\rho\varepsilon^1)\subset E''$, we have $\rho': \sigma\sim\tau$ lsd. E', E''. Thus p_* is one-to-one for $n\geqq 2$.

To prove that p_* is onto for $n\geqq 2$. consider $\sigma\in\Gamma_n(B; B', B'', \varphi)$. Since $\sigma\varepsilon^0$ is a solvent of the equation in B':

$$\overset{(0)}{[\square}, \overset{(1)}{\varphi\eta^0\cdots\eta^{n-3}}, \cdots, \overset{(n-1)}{\varphi\eta^0\cdots\eta^{n-3}}]$$

there exists a simplex $\omega\in E'_{n-1}$ which is a solvent of

$$\overset{(0)}{[\square}, \overset{(1)}{\phi\eta^0\cdots\eta^{n-3}}, \cdots, \overset{(n-1)}{\phi\eta^0\cdots\eta^{n-3}}]$$

and $p(\omega)=\sigma\varepsilon^0$. Since σ is a solvent of

$$\overset{(0)}{[\sigma\varepsilon^0}, \overset{(1)}{\square}, \overset{(2)}{\varphi\eta^0\cdots\eta^{n-2}}, \cdots, \overset{(n)}{\varphi\eta^0\cdots\eta^{n-2}}]$$

there exists a sinmplex $\tau\in\Gamma_n(E; E', E'', \phi)$ such that $p(\tau)=\sigma$, i.e. $p_*(\{\tau\}_{E;E',E''}) =\{\sigma\}_{B;B',B''}$. Thus p_* is onto for $n\geqq 2$,

Proof of 2). We can prove that p_* is one-to-one onto for $n\geqq 1$ similarly to the proof of 1).

To show that $p_*^{-1}(0)=0$ for $n=0$, consider a simplex $\sigma\in E_0$ such that $\{p(\sigma)\}_{B,B'}$ is the zero element of $\pi_0(B, B', \varphi)$. Then there exists $\rho\in B_1$ such that $\rho\varepsilon^0\in B'$ $\rho\varepsilon^1=p(\sigma)$. Since p is a fibre map, there exists $\rho'\in E_1$ such that $p(\rho')=\rho$ and $\rho'\varepsilon^1=\sigma$. Then $\rho'\varepsilon^0\in p^{-1}(\rho\varepsilon^0)\subset E'$. Thus $\{\sigma\}_{E,E'}=0$, i.e. $p_*^{-1}(0)=0$ for $n=0$.

THEOREM 8.4. *Let $p: E\to B$ be a fibre map, B' be a subcomplex of B and $E'=p^{-1}(B')$. Then the sequence*

$$\text{(IV)}\cdots\to\pi_n(E', \phi)\xrightarrow{i_*}\pi_n(E, \phi)\xrightarrow{\bar{j}}\pi_n(B, B', \varphi)\xrightarrow{\bar{\delta}}\pi_{n-1}(E', \phi)\longrightarrow\cdots\xrightarrow{\bar{j}}\pi_0(B, B', \varphi)$$

is exact, where i_* is the injection, $\bar{j}=p_*\circ j_*$, j_* being the injection, and $\bar{\delta}=\delta\circ p_*^{-1}$, $\delta=(\varepsilon^0)_*$. Moreover if p maps E onto B, then $\bar{j}: \pi_0(E, \phi)\to\pi_0(B, B', \varphi)$ is onto.

Proof. It suffices to prove the THEOREM for the case E, E', B and B' are Kan complexes using the first definition. The exactness of sequences (IV) is an immediate consequence of THEOREM 8.3 and the exactness of sequences (III), (III′).

To complete this proof, assume that $p: E\to B$ is onto. Then $p_*: \pi_0(E, E', \phi)\to \pi_0(B, B', \varphi)$ is onto. Since $j_*: \pi_0(E, \phi)\to\pi_0(E, E', \phi)$ is onto, $\bar{j}: \pi_0(E, \phi)\to\pi_0(B, B, \varphi)$ is onto.

References

[1] S. Eilenberg and J. A. Zilber: *Semi-simplicial complex and singular homology*, Ann. of Math., 51 (1950), 499–513.

[2] S. T. Hu: *Axiomatic approach to the homotopy groups*, Bull. Amer. Math. Soc., 62 (1956), 490–504.

[3] D. M. Kan: *Abstract homotopy III*, Proc. Nat. Acad. Sci U.S.A., 42 (1956), 419-421.
[4] D. M. Kan: *On c.s.s. complexes*, Amer. Journ. of Math., 79 (1957), 449-476.
[5] D. M. Kan: *A combinatorial definition of homotopy groups*, Ann. of Math., 67 (1958), 282-312.
[6] D. M. Kan: *On homotopy theory and c.s.s. groups*, Ann. of Math., 68(1958), 38-53.
[7] J. W. Milnor: *The geometric realization of a semi-simplicial complex*, Ann. of Math., 65 (1957), 357-362.
[8] J. C. Moore: *Semi-simplicial complexes*, Lecture notes, Princeton, 1955 (mimeographed).