On the characterization of semi-groups of linear operators

By

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1. Introduction

In the previous paper [4], we studied the differentiability of semi-groups of linear operators and obtained incidently necessary and sufficient conditions for a given semi-group to be differentiable or to be C^{∞} (Definition). In the above discussions semi-groups once considered were always assumed to be of class (C^0) and of type ω or such type. Our interest and object in the present paper are, however, different from those in the previous paper. We are not to discuss here the differentiability of a semi-group already generated by a linear operator, but to look for necessary and sufficient conditions for a linear operator to generate a semi-group which is of class (C^0), of type ω and differentiable in some sense.

A semi-group T_t, $t \geqq 0$ of continuous linear operators in a locally convex space X is called *of class* (C^0) if

for every $x \in X$, $T_t x$ *is continuous in* $t \geqq 0$,

and it is called *of type* ω if

for some $\omega \geqq 0$, $e^{-\omega t}T_t$, $t \geqq 0$ *is equicontinuous.*

One of our main results is as follows: a linear operator A generates a semi-group which is of class (C^0), of type ω and differentiable (See Theorem 1) if and only if the operator A satisfies among other conditions

$$R(\lambda; A)=\{T+\lambda\int_0^a e^{\lambda(a-t)}S(t)dt\}(\lambda e^{\lambda a}I-S)^{-1}$$

for some continuous linear operators T, S and $S(t)$, $0 \leqq t \leqq a$ and for all λ with $\mathrm{Re}\lambda > C - a^{-1}\log|\lambda|$, $C>0$. This formula resembles to the one frequently used in the previous paper and plays here also essential roles.

Similar considerations yield a new method for caharacterizing a semi-group of class (C^0) and of type ω in terms of the corresponding infinitesimal generator. By this method the well known theorem on the characterization of semi-groups of class (C^0) and of type ω can be now proved without the Laplace transform. To illustrate this situation we will state our assertion in the following form: a necessary and sufficient condition for a linear operator A to be the infinitesimal generator of a semi-group of class (C^0) and of type ω is that A is a densely defined closed linear operator and satisfies

$$R(\lambda; A) = \int_0^a e^{\lambda(a-t)}S(t)dt\,(e^{\lambda a}I-S)^{-1}$$

for some continuous linear operators S, $S(t)$, $0 \leqq t \leqq a$ and for all λ with $\mathrm{Re}\lambda > C \geqq 0$ (Theorem 2). It should be noted that the resolvent of the infinitesimal generator is required to be represented by an intergral only on some finite interval but not by the Laplace transform which always played important roles in the usual theory (See [2] or [5] for example).

2. Preliminaries

Throughout this paper we assume that X is a locally convex, sequentially complete linear topological space. The *infinitesimal generator* A of a semi-group T_t, $t \geqq 0$ of continuous linear operators on X into X is defined as usual by

$$A = \lim_{h \downarrow 0} h^{-1}(T_h - I).$$

As is well known, a semi-group T_t, $t \geqq 0$ of class (C^0) has the property:

for every y in the domain $D(A)$ of A, $T_t y$ is differentiable in $t \geqq 0$ and $(d/dt)T_t y = AT_t y = T_t Ay$.

Definition. A semi-group T_t, $t \geqq 0$ of class (C^0) is called *differentiable* at $t = a$ if there exists an $a > 0$ such that $T_a X$ is included in $D(A)$. It is called C^∞ if it is differentiable at every $t > 0$.

For the later use, we cite here several theorems which have been established in the previous paper [4].

Theorem A. *Let T_t, $t \geqq 0$ be a semi-group of class (C^0) with the infinitesimal generator A, and satisfy the condition:*

for some $b > 0$, T_t, $0 \leqq t \leqq b$ is equicontinuous.

If this is differentiable at $t = a$ for some positive $a \leqq b$ and $(CAT_a)^n$, $n = 1, 2, \ldots$ is equicontinuous for some constant $C > 0$, then the domain

$$\sum = \{\lambda;\ \mathrm{Re}\lambda \geqq a^{-1}\log(2/C) - a^{-1}\log|Im\lambda|\}$$

is included in the resolvent set $\rho(A)$ of A and $\lambda^{-1}R(\lambda; A)$, $\lambda \in \sum$, $\mathrm{Re}\lambda \leqq \gamma$ is equicontinuous for any fixed $\gamma \geqq 0$.

This theorem is a consequence of the following

Theorem B. *Under the assumptions of* Theorem A,

i) *A is a densely defined closed linear operator;*

ii) *every complex number* λ *with* $\operatorname{Re}\lambda > a^{-1}\log(1/C) - a^{-1}\log|\lambda|$ *belongs to* $\rho(A)$ *and* $R(\lambda; A)$ *is an everywhere defined continuous linear operator given by*

$$R(\lambda; A) = \left\{T_a + \lambda\int_0^a e^{\lambda(a-t)}T_t\,dt\right\}(\lambda e^{\lambda a}I - AT_a)^{-1}.$$

The following theorem asserts that the semi-group which fulfils the assumptions of Theorem A becomes necessarily of type ω with $\omega e^{\omega a} = 1/C$.

THEOREM C. *Under the assumptions* of Theorem A, $\rho(A)$ *contains the domain*

$$\{\lambda;\ \operatorname{Re}\lambda > \omega\} \text{ where } \omega e^{\omega a} = 1/C$$

and $(\operatorname{Re}\lambda - \omega)^n R(\lambda; A)^n$, $\operatorname{Re}\lambda > \omega$, $n = 1, 2, \ldots$ *is equicontinuous.*

The following theorem is very near to the converse of Theorem A.

THEOREM D. *Let* A *be the infinitesimal generator of a semi-group* T_t, $t \geqq 0$ *of class* (C^0) *and of type* ω. *If for some positive numbers* α, β, $\rho(A)$ *contains the domain*

$$\Sigma = \{\lambda;\ \operatorname{Re}\lambda \geqq \alpha - \beta\log|\operatorname{Im}\lambda|\}$$

and if for some constant $p \geqq 0$, $\lambda^{-p}R(\lambda; A)$, $\lambda \in \Sigma$ *is equicontinuous, then* T_t, $t \geqq 0$ *is differentiable at every* $t > (p+2)/\beta$ and

$$(t - (p+2)/\beta)(CAT_t)^n,\ (p+2)/\beta < t \leqq (p+2)/\beta + 1,\ n = 1, 2, \ldots$$

is equicontinuous for some constant $C > 0$ *independent of* t *and* n.

3. A characterization of differentiable semi-groups

We begin our theory with the characterization of semi-groups, which are of class (C^0) and of type ω and moreover differentiable, in terms of their infinitesimal generators. Arguing Theorems A, B, C and D, we have

THEOREM 1. *A necessary and sufficient condition for a linear operator* A *in* X *into* X *to generate a differentiable* (*at* $t = a$) *semi-group* T_t, $t \geqq 0$ *of class* (C^0) *of type* $\omega > 0$ *satisfying for some positive constants* a *and* C

$$(CAT_a)^n,\ n = 1, 2, \ldots \text{ is equicontinuous}$$

is

a) *A is a densely defined closed linear operator,*

b) *for some* $b > 0$ *there exists a family* T, S; $S(t)$, $0 \leqq t \leqq b$ *of continuous linear operators on* X *into* X *such that for some constant* $C_1 > 0$

$(C_1S)^n$, $n = 1, 2, \ldots$ *is equicontinuos,*

for every $x \in X$, $S(t)x$ *is continuous on* $[0, b]$,

$S(t)$, $0 \leqq t \leqq b$ *is equicontinous,* *and such that every complex number* λ *with* $\operatorname{Re}\lambda > b^{-1}\log(1/C_1) - b^{-1}\log|\lambda|$ *belongs to* $\rho(A)$ *and* $R(\lambda; A)$ *is an everywhere defined continuous linear operator given by*

$$R(\lambda; \mathrm{A}) = \left\{T+\lambda\int_0^b e^{\lambda(b-t)}s(t)dt\right\}(\lambda e^{\lambda b}I-S)^{-1}.$$

COROLLARY 1. *A necessary and sufficient condition for a linear operator* A *in* X *into* X *to generate a* C^∞*semi-group* T_t, $t \geqq 0$ *of class* (C^0) *of type* $\omega > 0$ *satisfying for some positive function* $C(t)$ *of* $t > 0$

$$(C(t)AT_t)^n,\ t > 0,\ n = 1, 2, \dots \textit{ is equicontinuous}$$

is

a) *A is a densely defined closed linear operator,*

b) *for every* $b > 0$ *there exists a family* $T, S;\ S(t),\ 0 \leqq t \leqq b$ *of continuous linear operators on* X *into* X *such that for some constant* $C_1 > 0$

$$(C_1S)^n,\ n = 1, 2, \dots \textit{ is equicontinuous,}$$

$$\textit{for every } x \in X,\ S(t)x \textit{ is continuous on } [0, b],$$

$$S(t),\ 0 \leqq t \leqq b \textit{ is equicontinuous,}$$

and such that every complex number λ *with* $\operatorname{Re}\lambda > b^{-1}\log(1/C_1) - b^{-1}\log|\lambda|$ *belongs to* $\rho(A)$ *and* $R(\lambda; A)$ *is an everywhere defined continuous linear operator given by*

$$R(\lambda; A) = \left\{T_t+\lambda\int_0^b e^{\lambda(b-t)}S(t)dt\right\}(\lambda e^{\lambda b}I-S)^{-1}.$$

In the case where X is a Banach space we have

COROLLARY 2. *A necessary and sufficient condition for a linear operator* A *in* X *into* X *to generate a differentiable semi-group of class* (C^0) *is*

a) *A is a densely defined closed linear operator,*

b) *for some* $a > 0$ *there exists a family* $T, S;\ S(t),\ 0 \leqq t \leqq a$ *of bounded linear operators on* X *into* X *such that*

$$\textit{for every } x \in X,\ S(t)x \textit{ is strongly continuous on } [0, a],$$

and such that every complex number λ *with* $\operatorname{Re}\lambda > a^{-1}\log\|S\| - a^{-1}\log|\lambda|$ *belongs to* $\rho(A)$ *and* $R(\lambda; A)$ *is given by*

$$R(\lambda; A) = \left\{T+\lambda\int_0^a e^{\lambda(a-t)}S(t)dt\right\}(\lambda e^{\lambda a}I-S)^{-1}.$$

PROOF OF THEOREM 1. The necessity has been already proved by Theorem B. Defining $b = a$, $C_1 = C$, $T = T_a$, $S = AT_a$ and $S(t) = T_t$, we obtain b).

Sufficiency. Assume that A satisfies a) and b). Our method is quite similar to that for proving Theorems C and A. Differentiating

$$R(\lambda; A) = \left\{T+\lambda\int_0^b e^{\lambda(b-t)}S(t)dt\right\}(\lambda e^{\lambda b}I-S)^{-1}$$

n times with respect to $\lambda > \omega$ with $\omega e^{\omega b} = 1/C_1$, we have

$$(-d/d\lambda)^n R(\lambda;\ A) = T(-d/d\lambda)^n(\lambda e^{\lambda b}I-S)^{-1}$$

$$+\sum_{r=0}^{n} {}_nC_r\int_0^b t^{n-r}e^{-\lambda t}S(t)dt(-d/d\lambda)^r(I-e^{-\lambda b}S/\lambda)^{-1}.$$

Hence for any continuous semi-norm p, there exists a continuous semi-norm q such that for all $x\in X$, $\lambda > \omega$ and $n = 1, 2, \dots$

$$p((-d/d\lambda)^n R(\lambda; A)x) \leqq \left\{(-d/d\lambda)^n(\lambda e^{\lambda b} - C_1^{-1})^{-1}\right.$$

$$\left.+\sum_{r=0}^{n} {}_nC_r\int_0^b t^{n-r}e^{-\lambda t}dt(-d/d\lambda)^r(1-C_1^{-1}e^{-\lambda b}\lambda^{-1})^{-1}\right\} q(x).$$

Thus, making use of Lemma 3, ii) in the next section, we obtain that $\rho(A)$ contains the domain

$$\{\lambda;\ \mathrm{Re}\lambda > \omega\} \text{ where } \omega e^{\omega b} = 1/C_1$$

and that

$$(\lambda-\omega)^n R(\lambda;\ A)^n,\ \lambda > \omega,\ n = 1, 2, \dots \text{ is equicontinuous.}$$

This implies that A generates a semi-group of class (C^0) and of type ω. On the other hand, because of b), $\rho(A)$ includes the domain

$$\Sigma = \{\lambda;\ \mathrm{Re}\lambda \geqq b^{-1}\log(2/C_1) - b^{-1}\log|\mathrm{Im}\lambda|\}$$

and

$$\lambda^{-1}R(\lambda;\ A),\ \lambda\in\Sigma,\ \mathrm{Re}\lambda \leqq \omega+1 \text{ is equicontinuous.}$$

Thus, by Theorem D, the semi-group $T_t, t\geqq 0$ just generated by A is differentiable at every $t > 3b$ and

$$(CAT_{3b+\varepsilon})^n,\ n = 1, 2, \dots \text{ is equicontinuous } (\varepsilon > 0)$$

for some positive constant C. Q.E.D.

Corollary 1 could be easily proved with the aid of the previous paper [4], Section 4, or directly by arguing precisely and carefully the theorem as well as Theorems A, B, C and D.

In the case where X is a Banach space, a bounded linear operator T on X into X satisfies

$$(CT)^n,\ n = 1, 2, \dots \textit{ is equicontinuous where } 1/C = \|T\|.$$

Hence the proof of Corollary 2 is straightforward.

4. A characterization of semi-groups of class (C^0)

In this section we shall introduce a method for characterizing semi-groups of class (C^0) and of type ω in terms of their infinitesimal generators to obtain Theorem 2 stated below. In this theorem, namely, the Laplace transform does not appear as was mentioned in the introduction. But it will be easy to observe that the method employed in the proof of this theorem is similar to that for proving Theorem 1.

We first collect the lemmas which are useful for the proof of Theorem 2 and also for the completeness of that of Theorem 1. Only i) of Lemma 3 has not been proved yet.

LEMMA 1. *The infinitesimal generator of semi-group of classs (C^0) and of type ω is a densely defined closed linear operator.*

LEMMA 2. *Let B be a densely defined closed linear operator and T, S continuous linear operators on X. Assume that T has the continuous inverse T^{-1} defined on X. If for every $y \in D(B)$, Sy belongs to $D(B)$ and $Ty = BSy = SBy$, then B has the continuous inverse $B^{-1} = ST^{-1}$ defined on X.*

LEMMA 3. *For all $\lambda > \omega \geqq 0$ and integers $n \geqq 0$,*

i) $\sum_{r=0}^{n} {}_nC_r \int_0^b t^{n-r} e^{-\lambda t} dt (-d/d\lambda)^r (1-e^{-\lambda b+\omega b})^{-1} \leqq (-d/d\lambda)^n (\lambda-\omega)^{-1}$,

ii) $(-d/d\lambda)^n (\lambda e^{\lambda b} - \omega e^{\omega b})^{-1} + \sum_{r=0}^{n} {}_nC_r \int_0^b t^{n-r} e^{-\lambda t} dt (-d/d\lambda)^r (1-\omega e^{-\lambda b+\omega b}/\lambda)^{-1}$

$\leqq (-d/d\lambda)^n (\lambda-\omega)^{-1}$.

PROOF OF i). Consider the equality:

$$(\lambda-\omega)^{-1} = \int_0^b e^{\lambda(b-t)} e^{\omega t} dt (e^{\lambda b} - e^{\omega b})^{-1},$$

which is easily verified by calculating the integrating part. Differentiating this n times with respect to $\lambda > \omega$, we have

$$(-d/d\lambda)^n (\lambda-\omega)^{-1} = \sum_{r=0}^{n} {}_nC_r \int_0^b t^{n-r} e^{-\lambda t} e^{\omega t} dt (-d/d\lambda)^r (1-e^{-\lambda b+\omega b})^{-1}.$$

But, $(-d/d\lambda)^n (1-e^{-\lambda b+\omega b})^{-1}$ is positive for every $\lambda > \omega$ and integer $n \geqq 0$, hence the desired inequality follows. Q. E. D.

We can now prove

THEOREM 2. *A necessary and sufficient condition for a linear operator A in X into X to generate a semi-group of class (C^0) and of type ω is*

a) *A is a densely defined closed linear operator,*

b) *for some $a > 0$ there exists a family S, $S(t)$, $0 \leqq t \leqq a$ of continuous linear operators* *on X into X such that for some constat $C > 0$*

$$(CS)^n,\ n = 1, 2, \ldots \text{ is equicontinuous,}$$

$$\text{for every } x \in X,\ S(t)x \text{ is continuous on } [0, a],$$

$$S(t),\ 0 \geqq t \geqq a \text{ is equicontinuous,}$$

and such that every complex number λ with $\mathrm{Re}\lambda > a^{-1}\log(1/C)$ *belongs to $\rho(A)$ and $R(\lambda; A)$ is an everywhere defined continuous linear operator given by*

$$R(\lambda; A) = \int_0^a e^{\lambda(a-t)} s(t)dt(e^{\lambda a}I - S)^{-1}.$$

Corollary. *Let X be a Banach space. A necessary and sufficient condition for a linear operator A in X into X to generate a semi-group of class (C^0) is*

a) *A is a densely defined closed linear operator,*

b) *for some $a > 0$ there exists a family S, $S(t)$, $0 \leqq t \leqq a$ of bounded linear operators on X such that*

$$\text{for every } x \in X,\ S(t)x \text{ is strongly continuous on } [0, a],$$

and such that every complex number λ with $\mathrm{Re}\lambda > a^{-1}\log\|S\|$ *belongs to $\rho(A)$ and $R(\lambda; A)$ is given by*

$$R(\lambda; A) = \int_0^a e^{\lambda(a-t)} S(t)dt(e^{\lambda a}I - S)^{-1}.$$

Remark. Cf. [1], proposition 1. 3. 10. The author does not know whether or not the semi-group uniquely determined by A coincides with $S(t)$ on $[0, a]$.

Proof of Theorem 2. Necessity. Let T_t, $t \geqq 0$ be a semi-group of class (C^0) of type ω with the infinitesimal generator A. Lemma 1 says that a) is valid. For every complex number λ and $y \in D(A)$,

$$(e^{\lambda a}I - T_a)y = (\lambda I - A)\int_0^a e^{\lambda(a-t)} T_t y dt = \int_0^a e^{\lambda(a-t)} T_t y dt(\lambda I - A)y,\ a > 0.$$

Here $(e^{\lambda a}I - T_a)$ and $\int_0^a e^{\lambda(a-t)} T_t dt$ are continuous linear operators on X, and if $\mathrm{Re}\lambda > \omega$, the former has the continuous inverse everywhere defined on X:

$$(e^{\lambda a}I - T_a)^{-1} = \sum_{k=0}^{\infty} e^{-(k+1)\lambda a} T_a{}^k.$$

Applying Lemma 2, we have

$$R(\lambda; A) = \int_0^a e^{\lambda(a-t)} T_t dt\,(e^{\lambda a}I - T^a)^{-1},$$

Thus, defining $S(t) = T_t$, $S = T_a$ and $C = e^{-\omega a}$, we obtain b).

Sufficiency. Assume that A satisfies a) and b). Differentiating

$$R(\lambda; A) = \int_0^a e^{\lambda(a-t)} S(t) dt\, (e^{\lambda a} I - S)^{-1}$$

n times with respect to $\lambda > \omega$, $\omega = a^{-1} \log(1/C)$, we have

$$(-d/d\lambda)^n R(\lambda; A) = \sum_{r=0}^{n} {}_nC_r \int_0^a t^{n-r} e^{-\lambda t} S(t) dt\, (-d/d\lambda)^r (I - e^{-\lambda a} S)^{-1}.$$

Hence for any continuous semi-norm p, there exists a continuous semi-norm q such that for all $x \in X$, $\lambda > \omega$ and $n = 1, 2, \ldots$

$$p((-d/d\lambda)^n R(\lambda; A)x) \leqq \sum_{r=0}^{n} {}_nC_r \int_0^a t^{n-r} e^{-\lambda t} dt (-d/d\lambda)^r (1 - e^{-\lambda a + \omega a})^{-1} q(x),$$

that is,

$$p((-d/d\lambda)^n R(\lambda; A)x) \leqq (-d/d\lambda)^n (\lambda - \omega)^{-1} q(x),$$

because of Lemma 3, i). Thus we obtain that

$$(\lambda - \omega)^n R(\lambda; A)^n,\ \lambda > \omega,\ n = 1, 2, \ldots \text{ is equicontinuous,}$$

which implies that A is the infinitesimal generator of a semi-group of class (C^0) and of type ω. Q. E. D.

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