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FINITE OPERATORS AND ORTHOGONALITY

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ABSTRACT. Let H be a separable infinite dimensional complex Hilbert space, and let $\mathcal{L}(H)$ denote the algebra of all bounded linear operators on H. Let $A, B \in \mathcal{L}(H)$ we define the generalized derivation $\delta_{A,B} : \mathcal{L}(H) \mapsto \mathcal{L}(H)$ by

$$\delta_{A,B}(X) = AX - XB,$$

we note $\delta_{A,A} = \delta_A$. If for all $X \in \mathcal{L}(H)$ and for all $T \in \ker \delta_A$ the inequality $||T - (AX - XA)|| \geq ||T||(*)$ holds, then we say that the range of δ_A is orthogonal to kernel δ_A in the sense of Birkhoff. The operator $A \in \mathcal{L}(H)$ is said to be finite [17] if $||I - (AX - XA)|| \geq 1(**)$ for all $X \in \mathcal{L}(H)$, where I is the identity operator. The well-known Inequality (**) due to J.P.Williams [17] is the starting point of the topic of commutator approximation (a topic which has its roots in quantum theory [18]). This topic deals with minimizing the distance, measured by some norms or other, between a varying commutator $XX^* - X^*X$ and some fixed operator [12]. In this paper we prove that a paranormal operator is finite and we present some generalized finite operators. An extension of inequality (*) is also given.

1. Introduction

Let H be a separable infinite dimensional complex Hilbert space, and let $\mathcal{L}(H)$ denote the algebra of all bounded linear operators on H. Let $A, B \in \mathcal{L}(H)$ we define the generalized derivation $\delta_{A,B} : \mathcal{L}(H) \mapsto \mathcal{L}(H)$ by

$$\delta_{A,B}(X) = AX - XB,$$

we note $\delta_{A,A} = \delta_A$. Let E be a complex Banach space. We say that $b \in E$ is orthogonal to $a \in E$ if for all complex λ there holds

$$\|a + \lambda b\| \geq \|a\|. \tag{1.1}$$

This definition has a natural geometric interpretation. Namely, $b \perp a$ if and only if the complex line $\{a + \lambda b \mid \lambda \in \mathbb{C}\}$ is disjoint with the open ball $K(0, \|a\|)$, i.e., iff

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this complex line is a tangent one. Note that if b is orthogonal to a, then a need not be orthogonal to b. If E is a Hilbert space, then from (1.1) follows $\langle a, b \rangle = 0$, i.e., orthogonality in the usual sense.

We say that the operator $A \in \mathcal{L}(H)$ is finite if $||I - (AX - XA)|| \geq 1$ for all $X \in \mathcal{L}(H)$.

Let $A \in \mathcal{L}(H)$, the approximate reduced spectrum of A, $\sigma_{ra}(A)$, is the set of scalars λ for which there exists a normed sequence $\{x_n\}$ in H satisfying

$$(A - \lambda I)x_n \rightarrow 0,\ (A - \lambda I)^* x_n \rightarrow 0.$$

J.P.Williams [17] has shown that the class of finite operators, $\mathcal{F}$, contains every normal, hyponormal operators. In [10], J.P.Williams results are generalized to a more classes of operators containing the classes of normal and hyponormal operators.

An operator $A \in \mathcal{L}(H)$ is said to be normaloid if $||A|| = r(A)$, where $r(A)$ is the spectral radius of A, paranormal if

$$||Ax||^2 \leq ||A^2x|| ||x||, \text{ for all } x \in H,$$

and p-hyponormal if $|A|^{2p} - |A^*|^{2p} \geq 0\,(0 < p \leq 1)$. We have

$$\text{hyponormal} \subset p - \text{hyponormal} \subset \text{ paranormal} \subset \text{normaloid}.$$

A is said to be log-hyponormal if A is invertible and satisfies the following equality

$$\log(A^*A) \geq \log(AA^*).$$

It is known that invertible p-hyponormal operators are *log*-hyponormal operators but the converse is not true [14]. However it is very interesting that we may regard log-hyponormal operators are 0-hyponormal operators [14, 15]. The idea of log-hyponormal operator is due to Ando [3] and the first paper in which log-hyponormality appeared is [6]. For properties of log-hyponormal operators (see [4, 14, 15, 16]).

We say that an operator $A \in \mathcal{L}(H)$ belongs to the class A if $|A^2| \geq |A|^2$. Class A was first introduced by Furuta-Ito-Yamazaki [7] as a subclass of paranormal operators which includes the classes of p-hyponormal and *log*-hyponormal operators. The following theorem is one of the results associated with class A.

Theorem 1.1. [7]

(1) Every log-hyponormal operator is a class A operator.

(2) Every class A operator is a paranormal operator.

J.H.Anderson and C.Foias [2] have shown that if A, B are normal operators, then

$$||T - (AX - XB)|| \geq ||T|| \qquad (1.2),$$

for all $X \in \mathcal{L}(H)$ and for all $T \in \ker \delta_{A,B}$. Hence the range of $\delta_{A,B}$ is orthogonal to the null space of $\delta_{A,B}$. In particular the inequality $||T-(AX-XA)|| \geq ||T||$ means that the range of δ_A is orthogonal to $\ker \delta_A$ in the sense of Birkhoff. It is easy to see that if the range of δ_A is orthogonal to $\ker \delta_A$, then A is finite. Indeed, we have $T = I \in \ker \delta_A$. In this paper we prove that a paranormal operator is finite. An extension of inequality (1.2) is also given.

2. Main results

In the following theorems we will show that a paranormal operator is finite and it remains invariant under compact perturbation.

Lemma 2.1. *Let $A \in \mathcal{L}(H)$ be paranormal. Then $\sigma_{ar}(A) \neq \phi$.*

Proof. If A is paranormal, then A is normaloid. Hence $||A|| = r(A)$. This implies that there exists $\lambda \in \sigma(A)$ such that $|\lambda| = ||A||$. Since λ is in the boundary of $\sigma(A)$, there exist unit vectors x_n such that $(A-\lambda)x_n \to 0$. Then $(A-\lambda)^* \to 0$, because $|\lambda| = ||A||$.

□

Theorem 2.1. *Let $A \in \mathcal{L}(H)$ be paranormal. Then A is finite*

Proof. It is well known [10] if $\sigma_{ar}(A) \neq \phi$, then A is finite. Hence it suffices to apply the previous lemma. □

As a consequence of the previous theorem we obtain.

Corollary 2.1. *The following operators are finite.*
1. *Hyponormal operators,*
2. *p-Hyponormal operators,*
3. *Class A operators,*
4. log-*hyponormal operators.*

Lemma 2.2. *If A is paranormal and if T is a normal operator such that $AT = TA$, then for every $\lambda \in \sigma_p(T)$ (point spectrum of A),*

$$|\lambda| \leq ||T-(AX-XA)||, \text{ for all } X \in \mathcal{L}(H).$$

Proof. Let $\lambda \in \sigma_p(A)$ and M_λ the eigenspace associate to λ. Since $TA = AT$, we have $T^*A = AT^*$ by the Fuglede-Putnam's theorem. Hence M_λ reduces both A and T. According to the decomposition of $H = M_\lambda \oplus M_\lambda^\perp$, we can write A, T and X as follows:

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} T = \begin{bmatrix} \lambda & 0 \\ 0 & T_2 \end{bmatrix} \text{ and } X = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}.$$

Since the restriction of a paranormal operator to an invariant subspace is paranormal, we have

$$||T-(AX-XA)|| = || \begin{bmatrix} \lambda - A_1X_1 + X_1A_1 & * \\ * & * \end{bmatrix} || \geq ||\lambda - A_1X_1 + X_1A_1||$$

$$\geq |\lambda|| 1 - A_1(\frac{X_1}{\lambda}) + (\frac{X_1}{\lambda})A_1|| \geq |\lambda|.$$

□

Proposition 2.1. [5, Berberian technique] *Let H be a complex Hilbert space. Then there exists a Hilbert space $H^{\sim} \supset H$ and $\varphi : \mathcal{L}(H) \mapsto \mathcal{L}(H)\,(A \mapsto A^{\sim})$ satisfying: φ is an *-isometric isomorphism preserving the order such that*

(i) $\varphi(A^*) = \varphi(A)^*$, $\varphi(I) = I^{\sim}$, $\varphi(\alpha A + \beta B) = \alpha\varphi(A) + \beta\varphi(B)$, $\varphi(AB) = \varphi(A)\varphi(B)$, $||\varphi(A)|| = ||A||$, $\varphi(A) \leq \varphi(B)$, if $A \leq B$ for all $A, B \in \mathcal{L}(H)$ and for all $\alpha, \beta \in \mathbb{C}$.

(ii) $\sigma(A) = \sigma(A^{\sim}) = \sigma_a(A) = \sigma_a(A^{\sim}) = \sigma_p(A^{\sim})$, where $\sigma_a(A)$ is the approximate spectrum of A and $\sigma_p(A)$ is the point spectrum of A.

Theorem 2.2. *If A is paranormal, then for every normal operator T such that $AT = TA$, we have*

$$||T-(AX-XA)|| \geq ||T||, \text{ for all } X \in \mathcal{L}(H) \qquad (2.1).$$

Proof. Let $\lambda \in \sigma(T) = \sigma_a(T)$ [8], then it follows from Proposition 2.1 that $T^{\sim}$ is normal, $A^{\sim}$ is paranormal, $T^{\sim}A^{\sim} = A^{\sim}T^{\sim}$ and $\lambda \in \sigma_p(A^{\sim})$. By applying Lemma 2.2, we get

$$|\lambda \leq ||T^{\sim} - (A^{\sim}X^{\sim} - X^{\sim}A^{\sim})|| = ||T-(AX-XA)||,$$

for all $X \in \mathcal{L}(H)$. Hence

$$\sup_{\lambda \in \sigma(T^{\sim})} |\lambda| = ||T^{\sim}|| = ||T|| = r(T) \leq ||T-(AX-XA)||,$$

for all $X \in \mathcal{L}(H)$. □

Recall that a paranormal operator on a C^*-algebra $\mathcal{A}$ may be defined as an operator $a \in \mathcal{A}$ satisfying $a^{2*}a^2 - 2ka^*a + k^2 \geq 0$, for all $k > 0$.

Theorem 2.3. *Let $\mathcal{A}$ be a C^*-algebra and let $a \in \mathcal{A}$ be a paranormal operator. Then a is finite.*

Proof. It is known ([9], p.97) that there exists a *-isometric homomorphism φ and a Hilbert space H ($\varphi : \mathcal{A} \mapsto \mathcal{L}(H)$). Then $\varphi(a)$ is paranormal. Since φ is isometric it results from Theorem 2.1 that a is finite. □

Corollary 2.2. *Let $A \in \mathcal{L}(H)$ be paranormal. Then $T = A + K$ is finite, where K is a compact operator.*

Proof. Since the Calkin algebra $\mathcal{L}(H)/K(H)$ is a C^*- algebra, $[A] \in \mathcal{L}(H)/K(H)$ is paranormal. Hence it follows from Theorem 2.3. that $[A] = A + K$ is finite and we have

$$||I - (TX - XT)|| \geq ||[I] - [A][X] - [X][A]|| \geq ||[I]|| = 1.$$

□

In the following theorem we will extend Inequality 2.1 to a more general classes of operators.

Theorem 2.4. *If A is p-hyponormal (resp. $\log$-hyponormal) and if B^* is p-hyponormal (resp. $\log$-hyponormal), then*

$$||T - (AX - XB)|| \geq ||T||,$$

for all $X \in \mathcal{L}(H)$ and for all $T \in \ker \delta_{A,B}$.

Proof. Let $T \in \ker \delta_{A,B}$. Then [15, Theorem 8] implies that $T \in \ker \delta_{A^*,B^*}$. Therefore, $ATT^* = TBT^* = TT^*A$. Since by Corollary 2.1 p-hyponormal or log-hyponormal are finite, Theorem 2.2 implies that

$$||TT^*|| = ||T||^2 \leq ||TT^* - (AXT^* - XT^*A)|| \leq ||TT^* - (AXT^* - XBT^*)||$$

$$\leq ||T^*||||T - (AX - XB)||.$$

Thus

$$||T|| \leq ||T - (AX - XB)||.$$

□

In [10] the author initiates the study of a more general class of finite operators defined by

$$\mathcal{GF}(H) = \{(A, B) \in \mathcal{L}(H) \times \mathcal{L}(H) : ||I - (AX - XB)|| \geq 1, \text{ for each } X \in \mathcal{L}(H)\}.$$

Such operators are called generalized finite operators. In the following theorems we recall some properties of these operators. Let $\mathcal{A}$ be a Banach algebra.

Theorem 2.5. [10] *$\mathcal{GF}(A)$ is closed in $\mathcal{A} \times \mathcal{A}$.*

Theorem 2.6. [10] *For $a, b \in \mathcal{A}$ the following statements are equivalent*
(i) $\|ax - xb - e\| \geq 1$ *for all* $x \in \mathcal{A}$.
(ii) There exists a state f such that $f(ax) = f(xb)$, for all $x \in \mathcal{A}$.
(iii) $0 \in W_0(ax - xb)$, $\forall x \in \mathcal{A}$.

Now we are ready to give a new classes of generalized finite operators. Let $\mathcal{R}_n$ be the set of all $(A, B) \in \mathcal{L}(H) \times \mathcal{L}(H)$ such that A and B have an n-dimensional reducing subspace $\mathcal{M}$ satisfying $A|\mathcal{M} = B|\mathcal{M}$.

By a slight modification in the proof of [17, Theorem 6] we prove the following theorem which is a generalization of Theorem 6 in [17].

Theorem 2.7. *Let $(A, B) \in \mathcal{R}_n$. Then*

$$||AX - XB - I|| \geq 1,$$

that is, (A, B) is generalized finite.

Proof. Let $(e_i)_{i=1}^n$ be an orthonormal basis of $H_1 = \mathcal{M}$. Define the linear form f on $\mathcal{L}(H)$ by $f(X) = \frac{1}{n}\sum_{i=1}^n \langle Xe_i, e_i\rangle$. It is clear that $f(I) = ||f|| = 1$. According to the decomposition of $H = H_1 \oplus H_1^\perp$, we have

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \; B = \begin{bmatrix} A_1 & 0 \\ 0 & B_2 \end{bmatrix} \text{ and } X = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}.$$

□

An easy calculation shows that

$$f(AX - XB) = \frac{1}{n}\sum_{i=1}^{n}\langle (A_1X_1 - X_1A_1)e_i, e_i\rangle$$

$$= \frac{1}{n} tr(A_1X_1 - X_1A_1) = 0,$$

where $tr(A_1X_1 - X_1A_1)$ is the trace of $A_1X_1 - X_1A_1$. Then Theorem 2.6 implies that $||AX - XB - I|| \geq 1$.

Remark 2.1. *It is known* [13] *that there exists a compact operator C such that $\overline{R(\delta_C)} = K(H)$. As a consequence we deduce that $dist(I, K(H)) = 1$, where $dist(I, K(H))$ is the distance from I to $K(H)$. Therefore if A, B are compact operators, then $dist(I, R(\delta_{A,B})) = 1$.*

The previous theorem shows that $\mathcal{R}_n \subset \mathcal{GF}(H)$. Hence it is interesting to ask the following question.

Question. Does $\mathcal{GF}(H) \subset \mathcal{R}_n$?

In the following example we will show that the answer to this question is negative.

Example 2.1. *Let*

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

on $H \oplus H$. Then for every

$$X = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} \in \mathcal{L}(H \oplus H)$$

we have

$$AX - XB - 1 = \begin{bmatrix} -1 & X_2 - X_1 \\ X_3 & -X_3 - 1 \end{bmatrix}.$$

Hence $||AX - XB - 1|| \geq 1$. Thus (A, B) is generalized finite. Clearly (A, B) does not belong to the class $\mathcal{R}_n$.

Remark 2.2. *As I have already mentioned Theorem 2.7 is a generalization of Theorem 6 in* [17]. *By a simple and different technique we will show in the following theorem that the assumption of $\mathcal{R}_n$ that there is a closed subspace $\mathcal{M}$ which reduces A and B such that $A|_{\mathcal{M}} = B|_{\mathcal{M}}$ is rather strong condition for generalized finiteness.*

Theorem 2.8. *Let $\mathcal{RGF}(H)$ be the set of all $(A, B) \in \mathcal{L}(H) \times \mathcal{L}(H)$ such that there is a reducing subspace $\mathcal{M}$ of A such as $\mathcal{M}$ is invariant under B and $(A|_{\mathcal{M}}, B|_{\mathcal{M}}) \in \mathcal{GF}(\mathcal{M})$. Then $\mathcal{RGF}(H) \subset \mathcal{GF}(H)$.*

Proof. Let

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, B = \begin{bmatrix} B_1 & B_2 \\ 0 & B_3 \end{bmatrix}$$

on $\mathcal{M} \oplus \mathcal{M}^{\perp}$. Then for every

$$X = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} \in \mathcal{L}(H \oplus H)$$

we have

$$AX - XB - 1 = \begin{bmatrix} A_1X_1 - X_1B_1 - 1 & * \\ * & * \end{bmatrix}.$$

Hence $||AX - XB - 1|| \geq ||A_1X_1 - X_1B_1 - 1|| \geq 1$ since (A_1, B_1) is generalized finite. Thus $\mathcal{RGF}(H) \subset \mathcal{GF}(H)$. □

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