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# ON A CLASS OF SASAKIAN MANIFOLDS

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ABSTRACT. In the present paper, we shall discuss $C$-Bochner pseudo-symmetric Sasakian manifolds and also Sasakian manifolds satisfying the condition $B \cdot S = 0$ where $B$ and $S$ are the $C$-Bochner curvature tensor and the Ricci tensor of the manifolds respectively.

## 1. Introduction

A Riemannian manifold $(M^n, g)$ is called locally symmetric if its curvature tensor $R$ is parallel i.e., $\nabla R = 0$, where $\nabla$ denotes the Levi Civita connection. As a proper generalization of locally symmetric manifolds the notion of semi-symmetric manifolds was defined by

$$(R(X,Y) \cdot R)(U,V)W = 0, \qquad X, Y, U, V, W \in \chi(M^n)$$

and studied by many authors, e.g. ([13], [14], [20], [19]). A complete intrinsic classification of these spaces was given by Z. I. Szabo [18]. Ryszard Deszcz and others ([6], [7], [5]) weakened the notion of semi-symmetry and introduced the notion of pseudo-symmetric manifolds by

$$(R(X,Y) \cdot R)(U,V)W = L_R[((X \wedge Y) \cdot R)(U,V)W],$$

where $L_R$ is some smooth function on $M^n$ and

$$\begin{aligned}(R(X,Y) \cdot R)(U,V)W &= R(X,Y)R(U,V)W - R(R(X,Y)U,V)W \\ &\quad -R(U,R(X,Y)V)W - R(U,V)R(X,Y)W,\end{aligned}$$

$X \wedge Y$ is an endomorphism defined by

$$(X \wedge Y)Z = g(Y,Z)X - g(X,Z)Y.$$

We refer the reader to R. Deszcz [6] as a general reference for the ideas of pseudo-symmetric manifolds.

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A Riemannian or a semi-Riemannian manifold is said to be $C$-Bochner pseudo-symmetric if

$$(1) \qquad (R(X,Y)\cdot B)(U,V)W = L_B[((X\wedge Y)\cdot B)(U,V)W]$$

holds on the set $U_B = \{x \in M : B \neq 0 \text{ at } x\}$, where $L_B$ is some function on $U_B$ and $B$ is the $C$-Bochner curvature tensor [11]. Recently M. Hotlos [9] has studied Bochner pseudo-symmetric para-Kähler manifold and prove that such a manifold is semi-symmetric. The present paper deals with a Sasakian manifold in which the condition (1) holds. In Section 3, we prove a result ensuring the existence of $n(= 2m+1 \geq 5)$-dimensional $C$-Bochner pseudo-symmetric Sasakian manifolds which are not $C$-Bochner semi-symmetric ones. This result also generalizes the result[3, Theorem 1] and is somewhat connected with the works of [1] and [4]. In the last section, we prove that if a Sasakian manifold $M^n$, $n \geq 5$, is $\eta$-Einstein then the condition $B\cdot S = 0$ holds on $M^n$, where $S$ is the Ricci tensor.

## 2. Preliminaries

Let $(M^n, g)$ be an $n(= 2m+1 \geq 5)$-dimensional contact Riemannian manifold with contact form $\eta$, the associated vector field $\xi$, (1,1)-tensor field $\phi$ and the associated Riemannian metric $g$. If $\xi$ is a Killing vector field then $M^n$ is called a $K$-contact Riemannian manifold ([2], [17]). If in such a manifold the relation

$$(2) \qquad (\nabla_X \phi)Y = g(X,Y)\xi - \eta(Y)X$$

holds, where $\nabla$ denotes the Levi Civita connection of $g$, then $M^n$ is called a *Sasakian manifold.* It is well-known that every Sasakian manifold is $K$-contact but the converse is not true in general. However, a 3-dimensional $K$-contact manifold is Sasakian. On the other hand, the notion of $C$-Bochner curvature tensor on a Sasakian manifold was first introduced by Matsumoto and Chuman [11]. Also, $C$-Bochner curvature tensor has been studied by V. Mihova-Nehmer [12], I. Hasegawa and T. Nakahe [8], T. Ikawa and M. Kon [10], G. Pathak, U. C. De and Y. H. Kim [16].

A contact metric manifold is said to be *$\eta$-Einstein* if its Ricci tensor $S$ is of the form

$$S = ag + b\eta\otimes\eta,$$

where $a, b$ are functions on $M^n$.

Let $R$, $Q$, $r$ denote respectively the curvature tensor of type (1,3), Ricci operator and scalar curvature of $M^n$. It is known that in a contact manifold $M^n$ the Riemannian metric may be so chosen that the following relations hold [2], [21].

$$(3) \qquad a)\ \ \phi\xi = 0, \quad b)\ \ \eta(\xi) = 1, \quad c)\ \ \eta\circ\phi = 0.$$

$$\phi^2 X = -X + \eta(X)\xi, \tag{4}$$

$$g(X,\xi) = \eta(X), \tag{5}$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y) \tag{6}$$

for any vector fields $X, Y$. If $M^n$ is a Sasakian manifold, then besides (3), (4), (5) and (6) the following relations hold ([2], [21]):

$$\nabla_X \xi = -\phi X, \tag{7}$$

$$\Phi(X,Y) = (\nabla_X \eta)Y, \tag{8}$$

$$\Phi(X,Y) = -\Phi(Y,X), \tag{9}$$

$$\Phi(X,\xi) = 0, \tag{10}$$

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y, \tag{11}$$

$$R(\xi,X)Y = (\nabla_X \phi)Y, \tag{12}$$

$$S(X,\xi) = (n-1)\eta(X). \tag{13}$$

The $C$-Bochner curvature tensor on a Sasakian manifold $M^n (n = 2m+1 \geq 5)$ is defined by [11]

$$\begin{aligned} B(X,Y)Z = R(X,Y)Z + \tfrac{1}{n+3}[&S(X,Z)Y - S(Y,Z)X + g(X,Z)QY \\ &-g(Y,Z)QX + S(\phi X,Z)\phi Y - S(\phi Y,Z)\phi X + g(\phi X,Z)Q\phi Y \\ &-g(\phi Y,Z)Q\phi X + 2S(\phi X,Y)\phi Z + 2g(\phi X,Y)Q\phi Z - S(X,Z)\eta(Y)\xi \\ &+S(Y,Z)\eta(X)\xi - \eta(X)\eta(Z)QY + \eta(Y)\eta(Z)QX] \\ &-\tfrac{k+n-1}{n+3}[g(\phi X,Z)\phi Y - g(\phi Y,Z)\phi X + 2g(\phi X,Y)\phi Z] \\ &-\tfrac{k-4}{n+3}[g(X,Z)Y - g(Y,Z)X] \\ &+\tfrac{k}{n+3}[g(X,Z)\eta(Y)\xi + \eta(X)\eta(Z)Y \\ &-g(Y,Z)\eta(X)\xi - \eta(Y)\eta(Z)X], \end{aligned} \tag{14}$$

where $k = \frac{r+n-1}{n+1}$ and $S(X,Y) = g(QX,Y)$.

From (14), it can be easily verified that in a Sasakian manifold $M^n$, $(n \geq 5)$, the $C$-Bochner curvature tensor satisfies the following properties:

$$B(X,Y)Z = -B(Y,X)Z, \tag{15}$$

$$B(\xi, Y)Z = 0, \tag{16}$$

$$B(X, Y)\xi = 0, \tag{17}$$

$$B(X, Y, Z, \xi) = 0, \tag{18}$$

and

$$B(X, Y, Z, U) = B(Z, U, X, Y), \tag{19}$$

for all vector fields $X, Y, Z, U$ and $B(X, Y, Z, U) = g(B(X, Y)Z, U)$.

The above results will be used in the following sections.

## 3. $C$-Bochner pseudo-symmetric Sasakian manifolds

Let $M^n$ be an $n(= 2m+1 \geq 5)$-dimensional $C$-Bochner pseudo-symmetric Sasakian manifold. Then putting $Y = \xi$ in (1) we have

$$\begin{aligned} (R(X,\xi)\cdot B)(U,V)W &= L_B[((X \wedge \xi)\cdot B)(U,V)W] \\ &= L_B[((X \wedge \xi)(B(U,V)W) - B((X \wedge \xi)U, V)W \\ &\quad -B(U,(X \wedge \xi)V)W - B(U,V)(X \wedge \xi)W]. \end{aligned} \tag{20}$$

The above equation can be written as

$$\begin{aligned} &R(X,\xi)B(U,V)W - B(R(X,\xi)U,V)W - B(U,R(X,\xi)V)W \\ &-B(U,V)R(X,\xi)W = L_B[B(U,V,W,\xi)X - B(U,V,W,X)\xi \\ &\quad -\eta(U)B(X,V)W + g(X,U)B(\xi,V)W \\ &\quad -\eta(V)B(U,X)W + g(X,V)B(U,\xi)W \\ &\quad -\eta(W)B(U,V)X + g(X,W)B(U,V)\xi]. \end{aligned} \tag{21}$$

Now using (11), (16), (17) and (18) into (21) it follows that

$$\begin{aligned} &-B(U,V,W,X)\xi - \eta(V)B(U,X)W - \eta(W)B(U,V)X \\ &-\eta(U)B(X,V)W = -L_B[B(U,V,W,X)\xi \\ &+\eta(V)B(U,X)W + \eta(W)B(U,V)X + \eta(U)B(X,V)W]. \end{aligned}$$

Putting $V = \xi$ in the last equation and using (17) and (18) we obtain

$$(L_B - 1)B(U,X)W = 0. \tag{22}$$

From (22), we have easily the following theorem.

**Theorem 3.1** *Let $M^n$ be an $n(= 2m + 1 \geq 5)$-dimensional $C$-Bochner pseudo-symmetric Sasakian manifold. Then, either $B \neq 0$ and $L_B = 1$ or $B = 0$ holds at each point of $M^n$.*

Since $C$-Bochner semi-symmetric Sasakian manifold can be regarded as a special $C$-Bochner pseudo-symmetric Sasakian manifold, from the above Theorem 3.1, we have immediately the following.

**Corollary 3.2** *An $n(= 2m+1 \geq 5)$-dimensional $C$-Bochner semi-symmetric Sasakian manifold is C-Bochner flat.*

The above corollary was already proved in [3].

## 4. Sasakian manifolds satisfying $B \cdot S = 0$

Let $M^n$ be an $n(= 2m + 1 \geq 5)$-dimensional $\eta$-Einstein Sasakian manifold. Then we can write

$$S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y), \tag{23}$$

where $a$ and $b$ are constants.

Putting $X = Y = e_i$ in (23), where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold and taking summation over $i$, $1 \leq i \leq n$ we obtain

$$r = na + b. \tag{24}$$

On the other hand, putting $X = Y = \xi$ in (23) and using (13) we also have

$$n - 1 = a + b. \tag{25}$$

Hence it follows from (24) and (25) that

$$a = \tfrac{r}{n-1} - 1 \quad , \quad b = n - \tfrac{r}{n-1}.$$

So the Ricci tensor $S$ of an $\eta$-Einstein Sasakian manifold is given by

$$S(X, Y) = \left(\frac{r}{n-1} - 1\right) g(X, Y) + \left(n - \frac{r}{n-1}\right) \eta(X)\eta(Y). \tag{26}$$

Now

$$\begin{aligned} &(B(U, X) \cdot S)(Y, Z) = -S(B(U, X)Y, Z) - S(Y, B(U, X)Z) \\ &= \left(1 - \tfrac{r}{n-1}\right) B(U, X, Y, Z) + \left(\tfrac{r}{n-1} - n\right) \eta(B(U, X)Y)\eta(Z) \\ &+ \left(1 - \tfrac{r}{n-1}\right) B(U, X, Z, Y) + \left(\tfrac{r}{n-1} - n\right) \eta(B(U, X)Z)\eta(Y). \end{aligned} \tag{27}$$

Using (19) and (18) in (27) we obtain $B \cdot S = 0$. Thus we can state the following:

**Theorem 4.1** *Let $(M^n, g)$ be an $n(= 2m+1 \geq 5)$-dimensional $\eta$-Einstein Sasakian manifold. Then the condition $B \cdot S = 0$ holds on $M^n$.*

**Remark.** It is known that an $n(= 2m + 1 \geq 5)$-dimensional Sasakian manifold of constant $\phi$-sectional curvature(namely, a Sasakian space form) is $C$-Bochner flat and $\eta$-Einstein and also that an $n(= 2m + 1 \geq 5)$- dimensional $C$-Bochner flat Sasakian manifold is $\eta$-Einstein if and only if it is a Sasakian space form ([11], Theorem 2.4, Corollary 2.5). From these observations, it seems that the converse of the above Theorem 4.1 is not necessarily valid in general.

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