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K-HOMOLOGY OF CONTINUOUS FIELDS OF NONCOMMUTATIVE TORI

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ABSTRACT. We study K-homology of continuous fields of noncommutative (or commutative) tori, and obtain formulae to count up generators of K-homology for the C^*-algebras of all continuous functions on the usual tori and some continuous field C^*-algebras with fibers noncommutative tori such as the discrete Heisenberg group C^*-algebra and its generalizations.

1. Introduction

C^*-algebras associated with continuous fields of C^*-algebras as fibers over spaces (that we call continuous field C^*-algebras) have been of great interest in the theory of C^*-algebras (see Dixmier [4]). They can be viewed as a noncommutative counterpart to complex vector bundles over spaces by taking as fibers C^*-algebras that are noncommutative in general. C^*-algebras considered as fibers mainly in [4] are elementary C^*-algebras such as the C^*-algebra of all compact operators on a Hilbert space. Beyond this, a typical and important example of continuous field C^*-algebras is given by the group C^*-algebra of the discrete Heisenberg group (of rank 3) that can be viewed as a continuous field of rotation C^*-algebras $\mathfrak{A}_\theta$ (also called noncommutative 2-tori) on the 1-torus $\mathbb{T}$, where $\mathfrak{A}_\theta$ is defined to be the universal C^*-algebra generated by two unitaries U, V such that $VU = e^{2\pi i\theta}UV$ for $\theta \in [0,1]$ (mod 1) identified with $\mathbb{T}$ (for instance, see Anderson and Paschke [1], but this picture is well known).

Among many contributions related to this topic, it is Hadfield [5] who considered its K-homology. K-homology for C^*-algebras (or involutive algebras over $\mathbb{C}$) is a starting point of the quantum calculus in noncommutative geometry of Connes [3]. In this paper, we consider K-homology of continuous field C^*-algebras modifying some methods of Hadfield [5] in more general settings. Our main contribution is to give explicit formulae to count up generators of K-homology for the C^*-algebras of all continuous functions on the usual n-tori $\mathbb{T}^n$ and some continuous field C^*-algebras with fibers noncommutative tori such as the generalized discrete Heisenberg group C^*-algebras and their generalizations, which should have more applications. This result is in fact a K-homology version of the author [7] for K-theory of continuous

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fields of noncommutative (or quantum) tori. It is found that (generalized) Dirac Fredholm modules for K-homology that we define correspond to (generalized) Bott projections for K-theory in these settings. This picture should be of independent interest and might be useful for more studying noncommutative geometry in the future.

2. Fredholm modules as K-homology

Following Hadfield [5] (and Connes [3]), recall that a Fredholm module over an involutive, i.e., $*$-algebra $\mathfrak{A}$ is defined to be a triple (H, π, F), where π is a $*$-representation of $\mathfrak{A}$ to $\mathbb{B}(H)$ the C^*-algebra of bounded operators on a Hilbert space H, and F is a self-adjoint operator on H such that $F^2 = 1$, and the commutators $[F, \pi(a)] = F\pi(a) - \pi(a)F$ for $a \in \mathfrak{A}$ are in $\mathbb{K}(H)$ the C^*-algebra of all compact operators on H. A Fredholm module is even if there exists a grading operator $\gamma \in \mathbb{B}(H)$ such that $\gamma = \gamma^*$, $\gamma^2 = 1$, $[\gamma, \pi(a)] = 0$ for $a \in \mathfrak{A}$, and $F\gamma = -\gamma F$. A Fredholm module is odd if otherwise. It is usual that the $*$-algebra $\mathfrak{A}$ is taken as a C^*-algebra or a dense $*$-subalgebra of a C^*-algebra such that $\mathfrak{A}$ is closed under holomorphic functional calculus.

Example 2.1 Let $\mathfrak{A}$ be a C^*-algebra with a $*$-homomorphism $\varphi : \mathfrak{A} \to \mathbb{C}$. Then we have the even Fredholm module $m_0 = (H, \pi, F)$ over $\mathfrak{A}$ given by

$$H = \mathbb{C}^2, \quad \pi = \varphi \oplus 0, \quad F = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Indeed, we calculate

$$\begin{aligned}
[F, \pi(a)] &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \varphi(a) & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} \varphi(a) & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 \\ \varphi(a) & 0 \end{pmatrix} - \begin{pmatrix} 0 & \varphi(a) \\ 0 & 0 \end{pmatrix} = \varphi(a) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \\
F\gamma &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \\
-\gamma F &= - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = - \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\end{aligned}$$

Note that even Fredholm modules over a C^*-algebra $\mathfrak{A}$ can be viewed as elements (or representatives of classes) of $KK^0(\mathfrak{A}, \mathbb{C})$, i.e., the even K-homology (group) of $\mathfrak{A}$. Moreover, $KK^0(\mathfrak{A}, \mathbb{C})$ is isomorphic to the K_0-group $K_0(\mathfrak{A})$ of $\mathfrak{A}$. See Blackadar [2] for more details in K- and KK-theories.

Example 2.2 Let $\mathfrak{A}$ be a unital C^*-algebra with a nonzero $*$-homomorphism $\varphi : \mathfrak{A} \to \mathbb{C}$. Let $\mathfrak{A} \rtimes_\alpha \mathbb{Z}$ be the crossed product C^*-algebra of $\mathfrak{A}$ by an action α of $\mathbb{Z}$ by automorphisms of $\mathfrak{A}$. Define an odd Fredholm module $m_1 = (H, \pi, F)$ over $\mathfrak{A} \rtimes_\alpha \mathbb{Z}$:

$$H = L^2(\mathbb{T}), \quad \text{and} \quad \pi : \mathfrak{A} \rtimes_\alpha \mathbb{Z} \to \mathbb{B}(L^2(\mathbb{T}))$$

defined by $\pi(a)V^n = \varphi(\alpha^n(a))V^n$ and $\pi(V)V^n = V^{n+1}$ for $a \in \mathfrak{A}$, $n \in \mathbb{Z}$, where V is the implementing unitary for the action α such that $VaV^* = \alpha(a)$, and it is also regarded as a generator of $L^2(\mathbb{T})$, and π is the representation of $\mathfrak{A} \rtimes_\alpha \mathbb{Z}$ induced from the representation φ of $\mathfrak{A}$, and

$$FV^n = \mathrm{sign}(n)V^n = \begin{cases} V^n & n \geq 0, \\ -V^n & n < 0. \end{cases}$$

Indeed, we calculate

$$\begin{aligned} [F, \pi(a)]V^n &= (F\pi(a) - \pi(a)F)V^n \\ &= F\varphi(\alpha^n(a))V^n - \pi(a)\mathrm{sign}(n)V^n = 0, \\ [F, \pi(V)]V^n &= (F\pi(V) - \pi(V)F)V^n \\ &= FV^{n+1} - \pi(V)\mathrm{sign}(n)V^n \\ &= \mathrm{sign}(n+1)V^{n+1} - \mathrm{sign}(n)V^{n+1} \\ &= \begin{cases} 0 & n \geq 0 \text{ and } n \leq -2, \\ 2V^0 = 2 \cdot 1 & n = -1 \end{cases} \end{aligned}$$

Hence, $[F, \pi(V)]$ is a rank one operator, which is compact.

Note that the odd Fredholm module m_1 can be viewed as an element (or a representative of a class) of $KK^1(\mathfrak{A} \rtimes_\alpha \mathbb{Z}, \mathbb{C})$, i.e., the odd K-homology (group) of $\mathfrak{A} \rtimes_\alpha \mathbb{Z}$. Moreover, $KK^1(\mathfrak{B}, \mathbb{C})$ for a C^*-algebra $\mathfrak{B}$ is isomorphic to the K_1-group $K_1(\mathfrak{B})$ of $\mathfrak{B}$. See Blackadar [2] for more details. Note that $\mathbb{C} \rtimes_\alpha \mathbb{Z}$ for α trivial is just the group C^*-algebra $C^*(\mathbb{Z})$ of $\mathbb{Z}$.

In what follows, even or odd Fredholm modules m over a C^*-algebra are often identified with (or distinguished from) their classes $[m]$ of its even or odd K-homology. Emphasized (in part) are Fredholm modules, not their classes.

3. K-homology of $C(\mathbb{T}^n)$

Let $C(\mathbb{T})$ be the C^*-algebra of all continuous functions on the 1-torus $\mathbb{T}$, which is also the universal C^*-algebra generated by a unitary U.

Proposition 3.1 *The even K-homology $KK^0(C(\mathbb{T}), \mathbb{C}) \cong \mathbb{Z}$ is generated by the following even Fredholm module over $C(\mathbb{T})$:*

$$m_{10} = \left(H = \mathbb{C}^2, \pi = \varphi \oplus 0, F = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right), \quad \gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

where $\varphi : C(\mathbb{T}) \to \mathbb{C}$ is the trivial unital $*$-homomorphism (or trivial character) defined by $\varphi(U) = 1$.

Since $C^*(\mathbb{Z})$ is isomorphic to $C(\mathbb{T})$ by the Fourier transform, we have

Proposition 3.2 *The odd K-homology $KK^1(C(\mathbb{T}), \mathbb{C}) \cong \mathbb{Z}$ is generated by the following odd Fredholm module over $C(\mathbb{T})$:*

$$m_{11} = (L^2(\mathbb{T}), \pi_1, F),$$

where π_1 is the (identity) representation defined by $\pi_1(U) = U$ which is regarded as the multiplication operator acting on $L^2(\mathbb{T})$, and $FU^n = \mathrm{sign}(n)U^n$ for $n \in \mathbb{Z}$.

Let $C(\mathbb{T}^2)$ be the C^*-algebra of all continuous functions on the 2-torus $\mathbb{T}^2$, which is also the universal C^*-algebra generated by commuting unitaries U, V.

Proposition 3.3 *The even K-homology $KK^0(C(\mathbb{T}^2), \mathbb{C}) \cong \mathbb{Z}^2$ is generated by the following even Fredholm modules over $C(\mathbb{T}^2)$:*

$$m_{20} = \left(H = \mathbb{C}^2, \pi = \varphi \oplus 0, F = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right), \quad \gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

where $\varphi : C(\mathbb{T}^2) \to \mathbb{C}$ is the trivial unital $*$-homomorphism, and

$$\mathrm{D} = \left(H = L^2(\mathbb{T}^2) \oplus L^2(\mathbb{T}^2), \pi = \pi_0 \oplus \pi_0, F = \begin{pmatrix} 0 & F_0 \\ F_0^* & 0 \end{pmatrix}\right)$$

(which we call the Dirac Fredholm module), where $\pi_0(U)U^n = U^{n+1}$, $\pi_0(U)V^n = V^n$ for $n \in \mathbb{Z}$, and $\pi_0(V)U^n = U^n$, $\pi_0(V)V^n = V^{n+1}$ for $n \in \mathbb{Z}$, and

$$F_0 U^m V^n = (m + in)(m^2 + n^2)^{-1/2} U^m V^n, \quad (m, n) \neq (0, 0),$$

and $F_0 1 = 1$, where U, V are unitary generators of $C(\mathbb{T}^2)$.

Proof. For the Dirac Fredholm module D defined in the statement, we compute that

$$\begin{aligned}
&[F, \pi(U)] \begin{pmatrix} U^m V^n \\ U^s V^t \end{pmatrix} = (F\pi(U) - \pi(U)F) \begin{pmatrix} U^m V^n \\ U^s V^t \end{pmatrix} \\
&= \left(\begin{pmatrix} 0 & F_0 \\ F_0^* & 0 \end{pmatrix} \begin{pmatrix} \pi_0(U) & 0 \\ 0 & \pi_0(U) \end{pmatrix} - \begin{pmatrix} \pi_0(U) & 0 \\ 0 & \pi_0(U) \end{pmatrix} \begin{pmatrix} 0 & F_0 \\ F_0^* & 0 \end{pmatrix} \right) \begin{pmatrix} U^m V^n \\ U^s V^t \end{pmatrix} \\
&= \begin{pmatrix} 0 & F_0\pi_0(U) - \pi_0(U)F_0 \\ F_0^*\pi_0(U) - \pi_0(U)F_0^* & 0 \end{pmatrix} \begin{pmatrix} U^m V^n \\ U^s V^t \end{pmatrix} \\
&= \begin{pmatrix} (F_0\pi_0(U) - \pi_0(U)F_0)U^s V^t \\ (F_0^*\pi_0(U) - \pi_0(U)F_0^*)U^m V^n \end{pmatrix} \\
&= \begin{pmatrix} F_0 U^{s+1} V^t - \pi_0(U)(s + it)(s^2 + t^2)^{-1/2} U^s V^t \\ F_0^* U^{m+1} V^n - \pi(U)(m - in)(m^2 + n^2)^{-1/2} U^m V^n \end{pmatrix} \\
&= \begin{pmatrix} (((s+1) + it)((s+1)^2 + t^2)^{-1/2} - (s + it)(s^2 + t^2)^{-1/2}) U^{s+1} V^t \\ ((m+1) - in)((m+1)^2 + n^2)^{-1/2} - (m - in)(m^2 + n^2)^{-1/2}) U^{m+1} V^n \end{pmatrix}
\end{aligned}$$

for $m, n, s, t \in \mathbb{Z}$.

Furthermore, we compute

$$
\begin{aligned}
&((s+1)+it)((s+1)^2+t^2)^{-1/2}-(s+it)(s^2+t^2)^{-1/2}\\
&=\left(\frac{s+1}{\sqrt{(s+1)^2+t^2}}-\frac{s}{\sqrt{s^2+t^2}}\right)+i\left(\frac{t}{\sqrt{(s+1)^2+t^2}}-\frac{t}{\sqrt{s^2+t^2}}\right).
\end{aligned}
$$

For the imaginary part,

$$
\begin{aligned}
&|t\left(\frac{1}{\sqrt{(s+1)^2+t^2}}-\frac{1}{\sqrt{s^2+t^2}}\right)|\\
&=|t\left(\frac{\sqrt{s^2+t^2}-\sqrt{(s+1)^2+t^2}}{\sqrt{(s+1)^2+t^2}\sqrt{s^2+t^2}}\right)|\\
&=|t\left(\frac{(s^2+t^2)-(s+1)^2-t^2}{\sqrt{(s+1)^2+t^2}\sqrt{s^2+t^2}(\sqrt{s^2+t^2}+\sqrt{(s+1)^2+t^2})}\right)|\\
&\le\frac{|t(-2s-1)|}{\sqrt{(s+1)^2+t^2}(s^2+t^2)}.
\end{aligned}
$$

It is clear that the last expression goes to zero as $s,t\to\infty$. Since

$$
\begin{aligned}
&\frac{s+1}{\sqrt{(s+1)^2+t^2}}-\frac{s}{\sqrt{s^2+t^2}}\\
&=(s+1)\left(\frac{1}{\sqrt{(s+1)^2+t^2}}-\frac{1}{\sqrt{s^2+t^2}}\right)+\frac{1}{\sqrt{s^2+t^2}},
\end{aligned}
$$

the real part also goes to zero as $s,t\to\infty$. Hence, the coefficient for $U^{s+1}V^t$ vanishes as $s,t\to\infty$. Similarly, the coefficient for $U^{m+1}V^n$ vanishes as $m,n\to\infty$. Therefore, $[F,\pi(U)]$ is compact. Similarly, so is $[F,\pi(V)]$. Also,

$$
F\gamma=\begin{pmatrix}0&F_0\\F_0^*&0\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix}=\begin{pmatrix}0&-F_0\\F_0^*&0\end{pmatrix}=-\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}0&F_0\\F_0^*&0\end{pmatrix}=-\gamma F.
$$

□

Remark. The coefficient $(m+in)(m^2+n^2)^{-1/2}$ for F_0 does not vanish as $m,n\to\infty$. For instance, let $m=n>0$. Then $(m+im)(m^2+m^2)^{-1/2}=(1+i)/\sqrt{2}$.

Proposition 3.4 *The odd K-homology $KK^1(C(\mathbb{T}^2),\mathbb{C})\cong\mathbb{Z}^2$ is generated by the following odd Fredholm modules over $C(\mathbb{T}^2)$:*

$$
m_{21}=(L^2(\mathbb{T}),\pi_1,F),\quad m_{22}=(L^2(\mathbb{T}),\pi_2,F)
$$

where $\pi_1(U)=U$, $\pi_1(V)=1$, and $\pi_2(U)=1$, $\pi_2(V)=V$.

Also, the odd Fredholm modules m_{21}, m_{22} *over* $C(\mathbb{T}^2)$ *can be replaced by* $\psi_1^*([m_{11}])$, $\psi_2^*([m_{11}])$ *respectively, where* $\psi_j : C(\mathbb{T}^2) \to C(\mathbb{T})$ $(j = 1, 2)$ *are* $*$*-homomorphisms defined by* $\psi_1(U) = U$, $\psi_1(V) = 1$ *in* $C(\mathbb{T})$, *and* $\psi_2(U) = 1$, $\psi_2(V) = U$ *in* $C(\mathbb{T})$, *and* $\psi_j^* : KK^1(C(\mathbb{T}), \mathbb{C}) \to KK^1(C(\mathbb{T}^2), \mathbb{C})$ $(j = 1, 2)$ *are the induced maps from* ψ_j.

Note that the Baum-Connes assemply map:

$$\mu : KK^j(C(B\mathbb{Z}^2), \mathbb{C}) = KK^j(C(\mathbb{T}^2), \mathbb{C}) \to K_j(C^*(\mathbb{Z}^2)) = K_j(C(\mathbb{T}^2))$$

$(j = 0, 1)$ is an isomorphism, where $B\mathbb{Z}^2 = \mathbb{T}^2$ is the classifying space of $\mathbb{Z}^2$ (see Connes [3]). Without using this map we can obtain the isomorphism identifying the even Fredholm modules m_{20}, D with the K_0-classes [1], $[B]$ in $K_0(C(\mathbb{T}^2))$ and the odd Fredholm modules m_{21}, m_{22} with the K_1-classes $[U]$, $[V]$ in $K_1(C(\mathbb{T}^2))$ respectively, where B is the Bott projection in the 2×2 matrix algebra $M_2(C(\mathbb{T}^2))$ over $C(\mathbb{T}^2)$ (see Anderson and Paschke [1] for B).

Let $C(\mathbb{T}^3)$ be the C^*-algebra of all continuous functions on the 3-torus $\mathbb{T}^3$ generated by mutually commuting unitaries U_1, U_2 and U_3. Then it is known that

$$KK^0(C(\mathbb{T}^3), \mathbb{C}) \cong \mathbb{Z}^4, \quad KK^1(C(\mathbb{T}^3), \mathbb{C}) \cong \mathbb{Z}^4.$$

We can interpret this fact as follows:

Proposition 3.5 *The even K-homology* $KK^0(C(\mathbb{T}^3), \mathbb{C}) \cong \mathbb{Z}^4$ *is generated by the canonical even Fredholm module* m_{30} *over* $C^*(\mathbb{T}^3)$ *that corresponds to the trivial* $*$*-homomorphism:* $C(\mathbb{T}^3) \to \mathbb{C}$ *and the even Fredholm modules* $\varphi_j^*([\mathrm{D}])$ $(1 \le j \le 3)$ *over* $C^*(\mathbb{T}^3)$, *where* $[\mathrm{D}] \in KK^0(C(\mathbb{T}^2), \mathbb{C})$ *is the class of the even Dirac Fredholm module* D *over* $C(\mathbb{T}^2)$ *defined above, and* $*$*-homomorphisms* $\varphi_j : C(\mathbb{T}^3) \to C(\mathbb{T}^2)$ $(1 \le j \le 3)$ *are defined by* $\varphi_j(U_j) = 1$ *and* $\varphi_j(U_k) = U_k$ *for* $k \ne j$, *and* $\varphi_j^* : KK^0(C(\mathbb{T}^2), \mathbb{C}) \to KK^0(C(\mathbb{T}^3), \mathbb{C})$. *Namely,*

$$KK^0(C(\mathbb{T}^3), \mathbb{C}) \cong \mathbb{Z}[m_{30}] \oplus (\oplus_{j=1}^3 \mathbb{Z}\varphi_j^*([\mathrm{D}])).$$

Proposition 3.6 *The odd K-homology* $KK^1(C(\mathbb{T}^3), \mathbb{C}) \cong \mathbb{Z}^4$ *is generated by both the odd Fredholm modules* $\psi_j^*([m_{11}])$ $(1 \le j \le 3)$ *over* $C(\mathbb{T}^3)$, *where* $*$*-homomorphisms* $\psi_j : C(\mathbb{T}^3) \to C(\mathbb{T})$ $(1 \le j \le 3)$ *are defined by* $\psi_j(U_j) = U$ *and* $\psi_j(U_k) = 1$ *for* $k \ne j$, *and the odd Fredholm module* $\varphi_1^*([\mathrm{D}]) \oplus \psi_1^*([m_{11}])$ *over* $C(\mathbb{T}^3)$. *Namely,*

$$KK^1(C(\mathbb{T}^3), \mathbb{C}) \cong (\oplus_{j=1}^3 \mathbb{Z}\psi_j^*([m_{11}])) \oplus \mathbb{Z}(\varphi_1^*([\mathrm{D}]) \oplus \psi_1^*([m_{11}])).$$

Moreover, for a discrete group Γ, the Baum-Connes assemply map:

$$\mu : KK^j(C_0(B\Gamma), \mathbb{C}) \to K_j(C_r^*(\Gamma))$$

is an isomorphisms in many cases, where $B\Gamma$ is the classifying space of Γ and $C_0(B\Gamma)$ is the C^*-algebra of all continuous functions on $B\Gamma$ vanishing at infinity (see [3]). In particular, let $\Gamma = \mathbb{Z}^3$. Then $B\mathbb{Z}^3 = \mathbb{T}^3$ and

$$\mu : KK^j(C(\mathbb{T}^3), \mathbb{C}) \cong K_j(C^*(\mathbb{Z}^3)) \cong K_j(C(\mathbb{T}^3)).$$

Without using the map we can obtain the isomorphism defining $\mu([m_{30}]) = [1]$, $\mu(\varphi_j^*([\mathrm{D}])) = [B_j]$ $(1 \leq j \leq 3)$, where B_j are the Bott projections that correspond to two variables z_k of $(z_k)_{k=1}^3 \in \mathbb{T}^3$ for $k \neq j$, and $\mu(\psi_j^*([m_{11}])) = [U_j]$ $(1 \leq j \leq 3)$, $\mu(\varphi_1^*([\mathrm{D}]) \oplus \psi_1^*([m_{11}])) = [I_2 + (z_1 - 1) \otimes B_1]$, where $I_2 + (z_1 - 1) \otimes B_1 \in M_2(C(\mathbb{T}^3))$ and I_2 is the 2×2 identity matrix.

Let $C(\mathbb{T}^n)$ be the C^*-algebra of all $\mathbb{C}$-valued continuous functions on the n-torus $\mathbb{T}^n$ generated by the mutually commuting unitaries U_j $(1 \leq j \leq n)$. Then it is known (by Wegge-Olsen [8]) that

$$KK^j(C(\mathbb{T}^n), \mathbb{C}) \cong K_j(C(\mathbb{T}^n)) \cong \mathbb{Z}^{2^{n-1}}, \quad j = 0, 1.$$

Now define the even Fredholm modules over $C(\mathbb{T}^{2k})$ for the even tori $\mathbb{T}^{2k}$:

$$\mathrm{D}_{2k} = \left(H = L^2(\mathbb{T}^{2k}) \oplus L^2(\mathbb{T}^{2k}), \pi_{2k} = \pi_{2k,0} \oplus \pi_{2k,0}, F_{2k} = \begin{pmatrix} 0 & F_{2k,0} \\ F_{2k,0}^* & 0 \end{pmatrix} \right)$$

(which we call the (generalized) Dirac Fredholm modules), where $\pi_{2k,0}(U_j)U_j = U_j^2$ and $\pi_{2k,0}(U_j)U_l = U_l$ for $l \neq j$ $(1 \leq j, l \leq 2k)$, that is, $\pi_{2k,0}$ are the representations of $C(\mathbb{T}^{2k})$ on $L^2(\mathbb{T}^{2k})$ by the multiplication operators, and

$$F_{2k,0} U_{2j-1}^m U_{2j}^n = (m + in)(m^2 + n^2)^{-1/2} U_{2j-1}^m U_{2j}^n, \quad (m, n) \neq (0, 0),$$

and $F_{2k,0}1 = 1$ for $1 \leq j \leq k$, where $\{U_j\}_{j=1}^{2k}$ are the unitary generators of $C(\mathbb{T}^{2k}) \cong \otimes_{j=1}^k C(\mathbb{T}^2)$, which can also be regarded as the generators of $L^2(\mathbb{T}^{2k}) \cong \otimes_{j=1}^k L^2(\mathbb{T}^2)$.

Proposition 3.7 *The (generalized) Dirac Fredholm modules* D_{2k} *defined above are even Fredholm modules over* $C(\mathbb{T}^{2k})$.

Proof. For the Dirac Fredholm modules D_{2k} defined above, we compute that

$$
\begin{aligned}
&[F_{2k}, \pi_{2k}(U_{2j})] \begin{pmatrix} U_1^{m_1}U_2^{m_2}\cdots U_{2k}^{m_{2k}} \\ U_1^{n_1}U_2^{n_2}\cdots U_{2k}^{n_{2k}} \end{pmatrix} \\
&= (F_{2k}\pi_{2k}(U_{2j}) - \pi_{2k}(U_{2j})F_{2k}) \begin{pmatrix} U_1^{m_1}U_2^{m_2}\cdots U_{2k}^{m_{2k}} \\ U_1^{n_1}U_2^{n_2}\cdots U_{2k}^{n_{2k}} \end{pmatrix} = \\
&\left[\begin{pmatrix} 0 & F_{2k,0} \\ F_{2k,0}^* & 0 \end{pmatrix} \begin{pmatrix} \pi_{2k,0}(U_{2j}) & 0 \\ 0 & \pi_{2k,0}(U_{2j}) \end{pmatrix}\right. \\
&\left.- \begin{pmatrix} \pi_{2k,0}(U_{2j}) & 0 \\ 0 & \pi_{2k,0}(U_{2j}) \end{pmatrix}\right] \begin{pmatrix} 0 & F_{2k,0} \\ F_{2k,0}^* & 0 \end{pmatrix} \begin{pmatrix} U_1^{m_1}U_2^{m_2}\cdots U_{2k}^{m_{2k}} \\ U_1^{n_1}U_2^{n_2}\cdots U_{2k}^{n_{2k}} \end{pmatrix} \\
&= \begin{pmatrix} 0 & F_{2k,0}\pi_{2k,0}(U_{2j}) - \pi_{2k,0}(U_{2j})F_{2k,0} \\ F_{2k,0}^*\pi_{2k,0}(U_{2j}) - \pi_{2k,0}(U_{2j})F_{2k,0}^* & 0 \end{pmatrix} \\
&\begin{pmatrix} U_1^{m_1}U_2^{m_2}\cdots U_{2k}^{m_{2k}} \\ U_1^{n_1}U_2^{n_2}\cdots U_{2k}^{n_{2k}} \end{pmatrix} \\
&= \begin{pmatrix} (F_{2k,0}\pi_{2k,0}(U_{2j}) - \pi_{2k,0}(U_{2j})F_{2k,0})U_1^{m_1}U_2^{m_2}\cdots U_{2k}^{m_{2k}} \\ (F_{2k,0}^*\pi_{2k,0}(U_{2j}) - \pi_{2k,0}(U_{2j})F_{2k,0}^*)U_1^{n_1}U_2^{n_2}\cdots U_{2k}^{n_{2k}} \end{pmatrix} \\
&= \begin{pmatrix} x_1\cdots x_{j-1}x_j'x_{j+1}\cdots x_k U_1^{m_1}U_2^{m_2}\cdots U_{2j}^{m_{2j}+1}\cdots U_{2k}^{m_{2k}} \\ y_1\cdots y_{j-1}y_j'y_{j+1}\cdots y_k U_1^{n_1}U_2^{n_2}\cdots U_{2j}^{n_{2j}+1}\cdots U_{2k}^{n_{2k}} \end{pmatrix}
\end{aligned}
$$

for $m_i, n_i \in \mathbb{Z}$ $(1 \le i \le 2k)$, where

$$
\begin{aligned}
x_i &= (m_{2i-1} + i m_{2i})(m_{2i-1}^2 + m_{2i}^2)^{-1/2} \\
&\quad \text{for } 1 \le i \le j-1 \text{ and } j+1 \le i \le k, \\
x_j' &= m_{2j-1} + i(m_{2j}+1))(m_{2j-1}^2 + (m_{2j}+1)^2)^{-1/2} \\
&\quad - (m_{2j-1} + i m_{2j})(m_{2j-1}^2 + m_{2j}^2)^{-1/2}, \\
y_i &= (n_{2i-1} + i n_{2i})(n_{2i-1}^2 + n_{2i}^2)^{-1/2} \\
&\quad \text{for } 1 \le i \le j-1 \text{ and } j+1 \le i \le k, \\
y_j' &= (n_{2j-1} - i(n_{2j}+1))(n_{2j-1}^2 + (n_{2j}+1)^2)^{-1/2} \\
&\quad - (n_{2j-1} - i n_{2j})(n_{2j-1}^2 + n_{2j}^2)^{-1/2}.
\end{aligned}
$$

Furthermore, to show the coefficients vanishing at infinity it suffices to show that x_j', y_j' vanish at infinity as $m_{2j-1}, m_{2j} \to \infty$ and $n_{2j-1}, n_{2j} \to \infty$ respectively. Their computation are the same as given in the proof of Proposition 3.3. Therefore, $[F_{2k}, \pi_{2k}(U_{2j})]$ are compact. Similarly, we can show that $[F_{2k}, \pi_{2k}(U_{2j-1})]$ are compact. Also, we have $F_{2k}\gamma = -\gamma F_{2k}$ as shown in the proof of Proposition 3.3. □

Using the above proposition extensively we obtain fhe following theorems for counting up generators of (even and odd) K-homology for $C(\mathbb{T}^n)$:

Theorem 3.8 *The generators of the even K-homology* $KK^0(C(\mathbb{T}^n), \mathbb{C}) \cong \mathbb{Z}^{2^{n-1}}$ *are determined by the following decompositions*:

$$KK^0(C(\mathbb{T}^{2n}), \mathbb{C}) \cong 2^{2n-1} \cong \mathbb{Z}^{{}_{2n}C_0} \oplus \mathbb{Z}^{{}_{2n}C_2} \oplus \cdots \oplus \mathbb{Z}^{{}_{2n}C_{2n}},$$
$$KK^0(C(\mathbb{T}^{2n+1}), \mathbb{C}) \cong 2^{2n} \cong \mathbb{Z}^{{}_{2n+1}C_0} \oplus \mathbb{Z}^{{}_{2n+1}C_2} \oplus \cdots \oplus \mathbb{Z}^{{}_{2n+1}C_{2n}},$$

for $n \geq 1$, *where the combinations* ${}_{2n}C_{2k}$ *(or* ${}_{2n+1}C_{2k}$*)* $(0 \leq k \leq n)$ *correspond to the canonical even Fredholm module* $m_{2n,0}$ *(or* $m_{2n+1,0}$*) over* $C(\mathbb{T}^{2n})$ *(or* $C(\mathbb{T}^{2n+1})$*) that corresponds to the trivial* $*$*-homomorphism*: $C(\mathbb{T}^{2n})$ *(or* $C(\mathbb{T}^{2n+1})$*)* $\to \mathbb{C}$ *for* $k = 0$, *and to the even Fredholm modules* $\varphi_j^*([\mathrm{D}_{2k}])$ $(1 \leq j \leq {}_{2n}C_{2k}$ *(or* ${}_{2n+1}C_{2k}$*))* *over* $C(\mathbb{T}^{2n})$ *(or* $C(\mathbb{T}^{2n+1})$*) for* $1 \leq k \leq n$ *(respectively), where* $[\mathrm{D}_{2k}] \in KK^0(C(\mathbb{T}^{2k}), \mathbb{C})$ *is the class of the even Dirac Fredholm module* D_{2k} *over* $C(\mathbb{T}^{2k})$ *defined above and* $*$*-homomorphisms* $\varphi_j : C(\mathbb{T}^{2n})$ *(or* $C(\mathbb{T}^{2n+1})$*)* $\to C(\mathbb{T}^{2k})$ $(1 \leq j \leq {}_{2n}C_{2k}$ *(or* ${}_{2n+1}C_{2k}$*))* *are defined by* $\varphi_j(U_{j_1}) = U_{j_1}, \cdots, \varphi_j(U_{j_{2k}}) = U_{j_{2k}}$ *and* $\varphi_j(U_l) = 1$ *for* $l \neq j_1, \cdots, j_{2k}$, *where we identify* $j = \{j_1, \cdots, j_{2k}\}$ *in* $\{1, \cdots, 2n\}$ *(or* $\{1, \cdots, 2n+1\}$*) (respectively), and* $\varphi_j^* : KK^0(C(\mathbb{T}^{2k}), \mathbb{C}) \to KK^0(C(\mathbb{T}^{2n}), \mathbb{C})$ *(or* $KK^0(C(\mathbb{T}^{2n+1}), \mathbb{C})$*) are the induced maps from* φ_j. *Namely,*

$$KK^0(C(\mathbb{T}^{2n}), \mathbb{C}) \cong \mathbb{Z}[m_{2n,0}] \oplus (\oplus_{k=1}^n (\oplus_{1 \leq j \leq {}_{2n}C_{2k}} \mathbb{Z}\varphi_j^*([\mathrm{D}_{2k}]))),$$
$$KK^0(C(\mathbb{T}^{2n+1}), \mathbb{C}) \cong \mathbb{Z}[m_{2n,0}] \oplus (\oplus_{k=1}^n (\oplus_{1 \leq j \leq {}_{2n+1}C_{2k}} \mathbb{Z}\varphi_j^*([\mathrm{D}_{2k}]))).$$

Proof. The statement is explaining our construction for generators of $KK^0(C(\mathbb{T}^n), \mathbb{C})$. Note that the classes of the above generators of $KK^0(C(\mathbb{T}^n), \mathbb{C})$ just correspond to choosing above and not to their permutations since exchanging the unitary generators produces the same class in $KK^0(C(\mathbb{T}^n), \mathbb{C})$. Note that

$$2^{n-1} = {}_nC_0 + {}_nC_2 + {}_nC_4 + \cdots = {}_nC_1 + {}_nC_3 + {}_nC_5 + \cdots .$$

□

Theorem 3.9 *The generators of the odd K-homology* $KK^1(C(\mathbb{T}^n), \mathbb{C}) \cong \mathbb{Z}^{2^{n-1}}$ *are determined by the following decompositions*:

$$KK^1(C(\mathbb{T}^{2n}), \mathbb{C}) \cong 2^{2n-1} \cong \mathbb{Z}^{{}_{2n}C_1} \oplus \mathbb{Z}^{{}_{2n}C_3} \oplus \cdots \oplus \mathbb{Z}^{{}_{2n}C_{2n-1}},$$
$$KK^1(C(\mathbb{T}^{2n+1}), \mathbb{C}) \cong 2^{2n} \cong \mathbb{Z}^{{}_{2n+1}C_1} \oplus \mathbb{Z}^{{}_{2n+1}C_3} \oplus \cdots \oplus \mathbb{Z}^{{}_{2n+1}C_{2n+1}},$$

for $n \geq 1$, *where the combinations* ${}_{2n}C_{2k+1}$ *(or* ${}_{2n+1}C_{2k+1}$*)* $(0 \leq k \leq n)$ *correspond to choosing one of the canonical odd Fredholm modules* $\psi_j^*([m_{11}])$ *over* $C(\mathbb{T}^{2n})$ *(or* $C(\mathbb{T}^{2n+1})$*) that corresponds to the* $*$*-homomorphisms* $\psi_j : C(\mathbb{T}^{2n})$ *(or* $C(\mathbb{T}^{2n+1})$*)* $\to C(\mathbb{T})$ $(1 \leq j \leq 2n$ *(or* $2n+1$*))* *defined by* $\psi_j(U_j) = U$ *and* $\psi_j(U_k) = 1$ *for* $k \neq j$ *and to choosing one of the even Fredholm modules* $\varphi_j^*([\mathrm{D}_{2k}])$ $(1 \leq j \leq {}_{2n}C_{2k}$ *(or* ${}_{2n+1}C_{2k}$*))* *over* $C(\mathbb{T}^{2n})$ *(or* $C(\mathbb{T}^{2n+1})$*) for* $1 \leq k \leq n$ *(respectively), where* $[\mathrm{D}_{2k}] \in KK^0(C(\mathbb{T}^{2k}), \mathbb{C})$ *is the class of the even Dirac Fredholm module* D_{2k} *over* $C(\mathbb{T}^{2k})$

defined above, and $*$-homomorphisms $\varphi_j : C(\mathbb{T}^{2n})$ (or $C(\mathbb{T}^{2n+1})$) $\to C(\mathbb{T}^{2k})$ ($1 \leq j \leq {}_{2n}C_{2k}$ (or ${}_{2n+1}C_{2k}$)) are defined above. Namely,

$$
\begin{aligned}
KK^1(C(\mathbb{T}^{2n}), \mathbb{C}) &\cong (\oplus_{j=1}^{2n} \mathbb{Z}\psi_j^*([m_{11}]))\oplus \\
&\qquad (\oplus_{k=1}^{n-1}(\oplus_{1\leq j\cup j' \leq {}_{2n}C_{2k+1}} \mathbb{Z}(\varphi_j^*([\mathrm{D}_{2k}]) \oplus \psi_{j'}^*([m_{11}])))), \\
KK^1(C(\mathbb{T}^{2n+1}), \mathbb{C}) &\cong (\oplus_{j=1}^{2n+1} \mathbb{Z}\psi_j^*([m_{11}]))\oplus \\
&\qquad (\oplus_{k=1}^{n}(\oplus_{1\leq j\cup j' \leq {}_{2n+1}C_{2k+1}} \mathbb{Z}(\varphi_j^*([\mathrm{D}_{2k}]) \oplus \psi_{j'}^*([m_{11}]))))
\end{aligned}
$$

where $j \cup j'$ means the number that corresponds to the disjoint union $\{j_1, \cdots, j_{2k}\} \cup \{j'\}$ in $\{1, \cdots, 2n\}$ (or $\{1, \cdots, 2n+1\}$).

Proof. The statement is explaining our construction for generators of $KK^1(C(\mathbb{T}^n), \mathbb{C})$. Note that the classes of the above generators of $KK^1(C(\mathbb{T}^n), \mathbb{C})$ just correspond to choosing above and not to their permutations since exchanging the unitary generators produces the same class in $KK^1(C(\mathbb{T}^n), \mathbb{C})$. □

Remark. We can obtain the isomorphisms $KK^j(C(\mathbb{T}^n), \mathbb{C}) \cong K_j(C(\mathbb{T}^n))$ ($j = 0, 1$) by sending the even K-homology classes $\varphi_j^*([\mathrm{D}_{2k}])$ (identified with generalized Dirac Fredholm modules D_{2k}) to the K_0-group classes of the generalized Bott projections B_j in $M_2(C(\mathbb{T}^{2k}))$ and the odd K-homology classes $\varphi_j^*([\mathrm{D}_{2k}]) \oplus \psi_{j'}^*([m_{11}])$ to the K_1-group classes of the unitaries $I_2 + (u_{j'} - 1) \otimes B_j$ in $M_2(C(\mathbb{T}^{2k+1}))$ respectively. See [7] for more details about the generalized Bott projections and the unitaries involving them.

4. K-homology of the discrete Heisenberg group C^*-algebra

The discrete Heisenberg group of rank 3 is defined by

$$
H_{\mathbb{Z}}^3 = \left\{ (c, b, a) = \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \mid a, b, c \in \mathbb{Z} \right\}.
$$

Set $u = (0, 0, 1)$, $v = (0, 1, 0)$, and $w = (1, 0, 0)$. We have $H_{\mathbb{Z}}^3 \cong \mathbb{Z}^2 \rtimes_\alpha \mathbb{Z}$, where $\alpha_a(c, b) = (c + ab, b)$ for $(c, b, a) \in H_{\mathbb{Z}}^3$. Let $C^*(H_{\mathbb{Z}}^3)$ be the group C^*-algebra of $H_{\mathbb{Z}}^3$. Then it is isomorphic to the crossed product C^*-algebra $C^*(\mathbb{Z}^2) \rtimes_\alpha \mathbb{Z}$. By the Fourier transform, $C^*(\mathbb{Z}^2) \rtimes_\alpha \mathbb{Z} \cong C(\mathbb{T}^2) \rtimes_{\alpha^\wedge} \mathbb{Z}$, where $\alpha^\wedge$ is the dual action of α via the dual group $(\mathbb{Z}^2)^\wedge \cong \mathbb{T}^2$. In fact, $C(\mathbb{T}^2) \rtimes_{\alpha^\wedge} \mathbb{Z}$ is generated by the unitaries U, V, W that correspond respectively to u, v, w via the universal representation, where U is the unitary corresponding to the action $\alpha^\wedge$ and $C(\mathbb{T}^2)$ is generated by W, V commuting mutually such that $UV = WVU$ and $UW = WU$.

Note that it can be viewed as the C^*-algebra $\Gamma(\mathbb{T}, \{\mathfrak{A}_\theta\}_{\theta\in\mathbb{T}})$ of a continuous field on the base space $\mathbb{T}$ with fibers noncommutative 2-tori $\mathfrak{A}_\theta$, where we identify $\theta \in [0, 1]$ with $e^{2\pi i\theta} \in \mathbb{T}$, and

$$
C(\mathbb{T}^2) \rtimes_{\alpha^\wedge} \mathbb{Z} \cong \Gamma(\mathbb{T}, \{C(\mathbb{T}) \rtimes_{\alpha_\theta^\wedge} \mathbb{Z}\}_{\theta\in\mathbb{T}})
$$

where $\mathfrak{A}_\theta = C(\mathbb{T}) \rtimes_{\alpha_\theta^\wedge} \mathbb{Z}$ and $\alpha_\theta^\wedge$ is the restriction of $\alpha^\wedge$ to $\{\theta\} \times \mathbb{T}$ in $\mathbb{T}^2$.

Recall that the C^*-algebra $\mathfrak{B} = \Gamma(X, \{\mathfrak{B}_t\}_{t\in X})$ of a continuous field of C^*-algebras $\mathfrak{B}_t$ over a compact Hausdorff space X with the supremum norm is defined by giving a continuous field of certain continuous operator fields $f : t \mapsto f(t) \in \mathfrak{B}_t$ with the norm $\|f\| = \sup_{t\in X} \|f(t)\|$ finite such that the maps $X \ni t \mapsto \|f(t)\|$ are continuous (and vanishing at infinity if X is locally compact and non-compact), where such a continuous field is closed under point-wise operations such as addition, multiplication, and involution, with the local uniform convergence, and has the image dense in $\mathfrak{B}_t$ for each t (see [4]).

For each $\theta \in \mathbb{R}$ (mod 1), we have a surjective $*$-homomorphism $\varphi_\theta : C^*(H_\mathbb{Z}^3) \to \mathfrak{A}_\theta$ given by $\varphi_\theta(U) = U$, $\varphi_\theta(V) = V$, $\varphi_\theta(W) = e^{2\pi i\theta}1$. Then we have the following induced map by φ_θ:

$$\varphi_\theta^* : KK^j(\mathfrak{A}_\theta, \mathbb{C}) \to KK^j(C^*(H_\mathbb{Z}^3), \mathbb{C})$$

which pull back Fredholm modules over $\mathfrak{A}_\theta$ to ones over $C^*(H_\mathbb{Z}^3)$. In particular, for the case $\theta = 0$, let $\varphi_0 : C^*(H_\mathbb{Z}^3) \to \mathfrak{A}_0 = C(\mathbb{T}^2)$ with $\varphi_0(W) = 1$.

It is known (by [8]) that the K-groups of the fibers $\mathfrak{A}_\theta$ $(0 < \theta < 1)$ are given by $K_0(\mathfrak{A}_\theta) \cong \mathbb{Z} + \theta\mathbb{Z}$ generated by the K_0-classes of the identity element of $\mathfrak{A}_\theta$ and the Rieffel projection R_θ of $\mathfrak{A}_\theta$ and $K_1(\mathfrak{A}_\theta) \cong \mathbb{Z}^2$ generated by the K_1-classes of the generating unitaries U_θ, V_θ of $\mathfrak{A}_\theta$ satisfying $V_\theta U_\theta = e^{2\pi i\theta} U_\theta V_\theta$. Since R_θ are not definable at $\theta = 0$ and not continuous at $\theta = 0$ (i.e., not continuous to the Bott projection for $C(\mathbb{T}^2)$ at $\theta = 0$), their classes in $K_0(\mathfrak{A}_\theta)$ do not produce an element for $K_0(C^*(H_\mathbb{Z}^3))$, while U_θ, V_θ are (identically) continuous (at $\theta = 0$ or on $\mathbb{T}$). (This argument was first used in [7]). In fact, the group $\mathbb{Z} + \theta\mathbb{Z}$ converges to $\mathbb{Z}$ as $\theta \to 0$ (or 1 mod 1), not to $\mathbb{Z}^2 \cong K_0(C(\mathbb{T}^2))$. Therefore, it is deduced that we can not have the (non-trivial) Fredholm modules for $KK^0(C^*(H_\mathbb{Z}^3), \mathbb{C})$ by pulling back the ones corresponding to R_θ by the above maps φ_θ^* (in particular, the Dirac Fredholm module over $C(\mathbb{T}^2)$ by φ_0), and it is enough to consider the canonical even Fredholm module in this setting. This interpretation is quite different from that of Hadfield [5], in which it is said that it is sufficient to consider the case $\theta = 0$.

Furthermore, we have $*$-homomorphisms φ_j $(j = 1, 2)$ from $C^*(H_\mathbb{Z}^3)$ to $C(\mathbb{T}^2)$ defined by $\varphi_1(U) = U$, $\varphi_1(V) = 1$, $\varphi_1(W) = W$, and $\varphi_2(U) = 1$, $\varphi_2(V) = V$, $\varphi_2(W) = W$. By pulling back the even Dirac Fredholm module D over $C(\mathbb{T}^2)$ by φ_j we obtain the even Fredholm modules $\varphi_j^*([\mathrm{D}])$ over $C^*(H_\mathbb{Z}^3)$. In these cases, there is no obstruction as explained above. Therefore,

Theorem 4.1 *The generators of the even K-homology $KK^0(C^*(H_\mathbb{Z}^3), \mathbb{C})$ are given by the canonical even Fredholm module h_{30} over $C^*(H_\mathbb{Z}^3)$ that corresponds to the trivial $*$-homomorphism: $C^*(H_\mathbb{Z}^3) \to \mathbb{C}$, and the even Fredholm modules $\varphi_j^*([\mathrm{D}])$ $(1 \leq j \leq 2)$, where* D *is the even Dirac Fredholm module over $C(\mathbb{T}^2)$. Namely,*

$$KK^0(C^*(H_\mathbb{Z}^3), \mathbb{C}) \cong \mathbb{Z}[h_{30}] \oplus \mathbb{Z}\varphi_1^*([\mathrm{D}]) \oplus \mathbb{Z}\varphi_2^*([\mathrm{D}]).$$

Moreover,

Theorem 4.2 *The generators of the odd K-homology $KK^1(C^*(H_{\mathbb{Z}}^3),\mathbb{C})$ are given by the odd Fredholm modules $\psi_3^*([m_{11}])$, $\varphi_1^*([\mathrm{D}])\oplus\psi_2^*([m_{11}])$, and $\varphi_2^*([\mathrm{D}])\oplus\psi_1^*([m_{11}])$, where $*$-homomorphisms $\psi_j : C^*(H_{\mathbb{Z}}^3) \to C(\mathbb{T})$ $(1 \le j \le 3)$ are defined by $\psi_j(U_j) = U$ and $\psi_j(U_k) = 1$ for $k \ne j$ in $C(\mathbb{T})$, where $U_1 = U$, $U_2 = V$, and $U_3 = W$. Namely,*

$$KK^1(C^*(H_{\mathbb{Z}}^3),\mathbb{C}) \cong \mathbb{Z}\psi_3^*([m_{11}]) \oplus \mathbb{Z}(\varphi_1^*([\mathrm{D}]) \oplus \psi_2^*([m_{11}])) \oplus \mathbb{Z}(\varphi_2^*([\mathrm{D}]) \oplus \psi_1^*([m_{11}])).$$

5. K-homology of the generalized discrete Heisenberg group C^*-algebras

The generalized discrete Heisenberg group of rank $2n+1$ is defined by

$$H_{\mathbb{Z}}^{2n+1} = \left\{ (c,b,a) = \begin{pmatrix} 1 & a & c \\ 0_n^t & 1_n & b^t \\ 0 & 0_n & 1 \end{pmatrix} \mid a,b \in \mathbb{Z}^n, c \in \mathbb{Z} \right\},$$

where 1_n is the $n \times n$ identity matrix, $0_n = (0,\cdots,0) \in \mathbb{Z}^n$ (a row vector), and b^t, 0_n^t (column vectors) are the trasposes of $b, 0_n$ (row vectors) respectively. Set $u_j = (0, 0_n, (\delta_{jk}1)_{k=1}^n)$, $v_j = (0, (\delta_{jk}1)_{k=1}^n, 0_n)$, and $w = (1, 0_n, 0_n)$, where $\delta_{jk} = 1$ if $k = j$ and $\delta_{jk} = 0$ if $k \ne j$. We have $H_{\mathbb{Z}}^{2n+1} \cong \mathbb{Z}^{n+1} \rtimes_\alpha \mathbb{Z}^n$, where $\alpha_a(c,b) = (c + \sum_{k=1}^n a_k b_k, b)$ for $(c,b,a) \in H_{\mathbb{Z}}^{2n+1}$ with $a = (a_k)_{k=1}^n$, $b = (b_k)_{k=1}^n$. Let $C^*(H_{\mathbb{Z}}^{2n+1})$ be the group C^*-algebra of $H_{\mathbb{Z}}^{2n+1}$. Then it is isomorphic to the crossed product C^*-algebra $C^*(\mathbb{Z}^{n+1}) \rtimes_\alpha \mathbb{Z}^n$. By the Fourier transform, $C^*(\mathbb{Z}^{n+1}) \rtimes_\alpha \mathbb{Z}^n \cong C(\mathbb{T}^{n+1}) \rtimes_{\alpha^\wedge} \mathbb{Z}^n$, where $\alpha^\wedge$ is the dual action of α defined by $\alpha_a^\wedge(w,v) = (w, (w^{a_k} v_k)_{k=1}^n)$ for $w \in \mathbb{T}$, $v = (v_k)_{k=1}^n \in \mathbb{T}^n$ via the dual group $(\mathbb{Z}^{n+1})^\wedge \cong \mathbb{T}^{n+1}$. Furthermore, $C(\mathbb{T}^{n+1}) \rtimes_{\alpha^\wedge} \mathbb{Z}^n$ is generated by the unitaries U_j, V_j, W $(1 \le j \le n)$ that correspond respectively to u_j, v_j, w $(1 \le j \le n)$ via the universal representation, where U_j are the unitaries corresponding to the action $\alpha^\wedge$ of $\mathbb{Z}^n$, and $C(\mathbb{T}^{n+1})$ is generated by W, V_j $(1 \le j \le n)$ mutually commuting such that $U_j V_j = W V_j U_j$ and $U_j W = W U_j$.

Note that it can be viewed as the C^*-algebra $\Gamma(\mathbb{T}, \{\otimes^n \mathfrak{A}_\theta\}_{\theta \in \mathbb{T}})$ of a continuous field on the base space $\mathbb{T}$ with fibers noncommutative $(2n)$-tori $\otimes^n \mathfrak{A}_\theta$ (the n-fold tensor product), where we identify $\theta \in [0,1]$ with $e^{2\pi i \theta} \in \mathbb{T}$:

$$C(\mathbb{T}^{n+1}) \rtimes_{\alpha^\wedge} \mathbb{Z}^n \cong \Gamma(\mathbb{T}^n, \{C(\mathbb{T}^n) \rtimes_{\alpha_\theta^\wedge} \mathbb{Z}^n\}_{\theta \in \mathbb{T}})$$

where $C(\mathbb{T}^n) \rtimes_{\alpha_\theta^\wedge} \mathbb{Z}^n \cong \otimes^n \mathfrak{A}_\theta$ and $\alpha_\theta^\wedge$ is the restriction of $\alpha^\wedge$ to $\{\theta\} \times \mathbb{T}^n$.

For $\theta \in \mathbb{R}$ (mod 1), we have a surjective $*$-homomorphism $\varphi_\theta : C^*(H_{\mathbb{Z}}^{2n+1}) \to \otimes^n \mathfrak{A}_\theta$ given by $\varphi_\theta(U_j) = U_j$, $\varphi_\theta(V_j) = V_j$, $\varphi_\theta(W) = e^{2\pi i \theta} 1$. Then we have the following induced map by φ_θ:

$$\varphi_\theta^* : KK^j(\otimes^n \mathfrak{A}_\theta, \mathbb{C}) \to KK^j(C^*(H_{\mathbb{Z}}^{2n+1}), \mathbb{C})$$

which pulls back Fredholm modules over $\otimes^n \mathfrak{A}_\theta$ to ones over $C^*(H_{\mathbb{Z}}^{2n+1})$. In particular, for the case $\theta = 0$ let $\varphi_0 : C^*(H_{\mathbb{Z}}^{2n+1}) \to \otimes^n \mathfrak{A}_0 = C(\mathbb{T}^{2n})$ with $\varphi_0(W) = 1$.

By the same reason as given before Theorem 4.1, we have no non-trivial even (and odd) Fredholm modules for $KK^0(C^*(H_{\mathbb{Z}}^{2n+1}), \mathbb{C})$ (and $KK^1(C^*(H_{\mathbb{Z}}^{2n+1}), \mathbb{C})$) by pulling back the ones corresponding to (generalized, or tensor products of) Rieffel projections for $K_0(\otimes^n \mathfrak{A}_\theta)$ (and to unitaries involving (generalized, or tensor products of) Rieffel projections for $K_1(\otimes^n \mathfrak{A}_\theta)$) by the above maps φ_θ^* (respectively) (in particular, the generalized Dirac Fredholm modules over $C(\mathbb{T}^{2n})$ by φ_0), and it is enough to consider the canonical even (and odd) Fredholm module(s) and other even (and odd) Fredholm module(s) corresponding to Dirac Fredholm modules over $C(\mathbb{T}^{2k})$ generated by $2k$ commuting unitaries in $\otimes^n \mathfrak{A}_\theta$ in this setting. In fact, $K_j(\otimes^n \mathfrak{A}_\theta) \cong K_j(C(\mathbb{T}^{2n})) \cong \mathbb{Z}^{2^{2n-1}}$ for $j = 0, 1$ (see [8] and [6]), but for instance, if $n = 2$, then the Künneth formula (see [8]) implies

$$
\begin{aligned}
K_0(\mathfrak{A}_\theta \otimes \mathfrak{A}_\theta) &\cong (K_0(\mathfrak{A}_\theta) \otimes K_0(\mathfrak{A}_\theta)) \oplus (K_1(\mathfrak{A}_\theta) \otimes K_1(\mathfrak{A}_\theta)) \\
&\cong ((\mathbb{Z} + \theta\mathbb{Z}) \otimes (\mathbb{Z} + \theta\mathbb{Z})) \oplus (\mathbb{Z}^2 \otimes \mathbb{Z}^2), \\
K_1(\mathfrak{A}_\theta \otimes \mathfrak{A}_\theta) &\cong (K_0(\mathfrak{A}_\theta) \otimes K_1(\mathfrak{A}_\theta)) \oplus (K_1(\mathfrak{A}_\theta) \otimes K_0(\mathfrak{A}_\theta)) \\
&\cong ((\mathbb{Z} + \theta\mathbb{Z}) \otimes \mathbb{Z}^2) \oplus (\mathbb{Z}^2 \otimes (\mathbb{Z} + \theta\mathbb{Z}))
\end{aligned}
$$

and both converge to $\mathbb{Z} \oplus (\mathbb{Z}^2 \oplus \mathbb{Z}^2) \cong \mathbb{Z}^5$ and to $(\mathbb{Z} \otimes \mathbb{Z}^2) \oplus (\mathbb{Z}^2 \otimes \mathbb{Z}) \cong \mathbb{Z}^4$ as $\theta \to 0$ (or 1 mod 1) respectively. Furthermore, we need to consider the unitary corresponding to the base space $\mathbb{T}$, which also commutes with generating unitaries of $\otimes^n \mathfrak{A}_\theta$. Considering this situation we state the following:

Theorem 5.1 *The generators of the even K-homology $KK^0(C^*(H_{\mathbb{Z}}^{2n+1}), \mathbb{C})$ are determined by the following decompositions*:

$$
\begin{aligned}
KK^0(C^*(H_{\mathbb{Z}}^{2n+1}), \mathbb{C}) &\cong \mathbb{Z} \oplus \mathbb{Z}^{2^2{}_nC_2} \oplus \mathbb{Z}^{2^4{}_nC_4} \oplus \cdots \oplus \mathbb{Z}^{2^{2m}{}_{2m}C_{2m}} \\
&\oplus \mathbb{Z}^{2_nC_1} \oplus \mathbb{Z}^{2^3{}_nC_3} \oplus \mathbb{Z}^{2^5{}_nC_5} \oplus \cdots \oplus \mathbb{Z}^{2^{2m-1}{}_{2m}C_{2m-1}}
\end{aligned}
$$

for $n = 2m$, and

$$
\begin{aligned}
KK^0(C^*(H_{\mathbb{Z}}^{2n+1}), \mathbb{C}) &\cong \mathbb{Z} \oplus \mathbb{Z}^{2^2{}_nC_2} \oplus \mathbb{Z}^{2^4{}_nC_4} \oplus \cdots \oplus \mathbb{Z}^{2^{2m}{}_{2m+1}C_{2m}} \\
&\oplus \mathbb{Z}^{2_nC_1} \oplus \mathbb{Z}^{2^3{}_nC_3} \oplus \mathbb{Z}^{2^5{}_nC_5} \oplus \cdots \oplus \mathbb{Z}^{2^{2m+1}{}_{2m+1}C_{2m+1}},
\end{aligned}
$$

for $n = 2m + 1$, and

$$
KK^0(C^*(H_{\mathbb{Z}}^{2n+1}), \mathbb{C}) \cong \oplus_{k=0}^n \mathbb{Z}^{2^k{}_nC_k} \cong \mathbb{Z}^{3^n},
$$

where $C^*(H_{\mathbb{Z}}^{2n+1}) \cong \Gamma(\mathbb{T}, \{\otimes^n \mathfrak{A}_\theta\}_{\theta \in \mathbb{T}})$ with the fibers $\otimes^n \mathfrak{A}_\theta$ the n-fold tensor products of the rotation algebras $\mathfrak{A}_\theta$ for $\theta \in \mathbb{T} \equiv [0, 1]$ (mod 1), and the combination ${}_nC_{2k}$ corresponds to choosing 2k-fold tensor products $\otimes^{2k} \mathfrak{A}_\theta$ in $\otimes^n \mathfrak{A}_\theta$, and the power 2^{2k} corresponds to choosing commuting 2k unitaries, each of which is chosen from two generating unitaries U_{j1}, U_{j2} of the (different) factors $\mathfrak{A}_\theta = \mathfrak{A}_{\theta,j}$ $(1 \le j \le 2k)$ in $\otimes^{2k} \mathfrak{A}_\theta$, from which we obtain the even Fredholm modules $\varphi_{2k,j}^*([\mathrm{D}_{2k}])$ over $C^*(H_{\mathbb{Z}}^{2n+1})$ $(1 \le j \le 2^{2k}{}_nC_{2k})$, where D_{2k} is the even Dirac Fredholm module *over $C(\mathbb{T}^{2k})$ generated by unitaries U_{jx} $(1 \le j \le 2k, x = 1,2)$ and $*$-homomorphisms $\varphi_{2k,j} : C^*(H_{\mathbb{Z}}^{2n+1}) \to C(\mathbb{T}^{2k})$ are defined by $\varphi_{2k,j}(U_{jx}) = U_{jx}$ and $\varphi_{2k,j}(V) = 1$ for V other generating unitaries, and similarly, the combination ${}_nC_{2k+1}$ corresponds to choosing $(2k+1)$-fold tensor products $\otimes^{2k+1}\mathfrak{A}_\theta$ in $\otimes^n\mathfrak{A}_\theta$, and the power 2^{2k+1} corresponds to choosing commuting $2k+1$ unitaries, each of which is chosen from two generating unitaries U_{j1}, U_{j2} of the (different) factors $\mathfrak{A}_\theta = \mathfrak{A}_{\theta,j}$ $(1 \le j \le 2k+1)$ in $\otimes^{2k+1}\mathfrak{A}_\theta$, from which we obtain the even Fredholm modules $\varphi_{2k+2,j}^*([\mathrm{D}_{2k+2}])$ over $C^*(H_{\mathbb{Z}}^{2n+1})$ $(1 \le j \le 2^{2k+1}{}_nC_{2k+1})$, where D_{2k+2} is the even Dirac Fredholm module over $C(\mathbb{T}^{2k+2})$ generated by unitaries U_{jx} $(1 \le j \le 2k+1, x = 1,2)$ and the generating unitary of $C(\mathbb{T})$, where this $\mathbb{T}$ is the base space, and furthermore, the first $\mathbb{Z}$ corresponds to the canonical even Fredholm module $h_{2n+1,0}$ over $C^*(H_{\mathbb{Z}}^{2n+1})$ that corresponds to the trivial $*$-homomorphism: $C^*(H_{\mathbb{Z}}^{2n+1}) \to \mathbb{C}$. Namely,*

$$KK^0(C^*(H_{\mathbb{Z}}^{2n+1}), \mathbb{C}) \cong \mathbb{Z}[h_{2n+1,0}] \oplus (\oplus_{k=1}^m(\oplus_{1\le j\le 2^{2k}{}_nC_{2k}}\mathbb{Z}\varphi_{2k,j}^*([\mathrm{D}_{2k}]))) \\ \oplus (\oplus_{k=0}^m(\oplus_{1\le j\le 2^{2k+1}{}_nC_{2k+1}}\mathbb{Z}\varphi_{2k+2,j}^*([\mathrm{D}_{2k+2}])))$$

for $n = 2m$ or $n = 2m+1$.

Proof. The explanation in the statement is just saying our construction for generators of the even K-homology $KK^0(C^*(H_{\mathbb{Z}}^{2n+1}), \mathbb{C})$ of $C^*(H_{\mathbb{Z}}^{2n+1})$. Note that by the binary expansion

$$3^n = (1+2)^n = 1 + 2{}_nC_1 + 2^2{}_nC_2 + \cdots + 2^k{}_nC_k + \cdots + 2^n{}_nC_n.$$

□

Theorem 5.2 *The generators of the odd K-homology $KK^1(C^*(H_{\mathbb{Z}}^{2n+1}), \mathbb{C})$ are determined by the following decompositions:*

$$KK^1(C^*(H_{\mathbb{Z}}^{2n+1}), \mathbb{C}) \cong \mathbb{Z} \oplus \mathbb{Z}^{2^2{}_nC_2} \oplus \mathbb{Z}^{2^4{}_nC_4} \oplus \cdots \oplus \mathbb{Z}^{2^{2m}{}_{2m}C_{2m}} \\ \oplus \mathbb{Z}^{2{}_nC_1} \oplus \mathbb{Z}^{2^3{}_nC_3} \oplus \mathbb{Z}^{2^5{}_nC_5} \oplus \cdots \oplus \mathbb{Z}^{2^{2m-1}{}_{2m}C_{2m-1}}$$

for $n = 2m$, and

$$KK^1(C^*(H_{\mathbb{Z}}^{2n+1}), \mathbb{C}) \cong \mathbb{Z} \oplus \mathbb{Z}^{2^2{}_nC_2} \oplus \mathbb{Z}^{2^4{}_nC_4} \oplus \cdots \oplus \mathbb{Z}^{2^{2m}{}_{2m+1}C_{2m}} \\ \oplus \mathbb{Z}^{2{}_nC_1} \oplus \mathbb{Z}^{2^3{}_nC_3} \oplus \mathbb{Z}^{2^5{}_nC_5} \oplus \cdots \oplus \mathbb{Z}^{2^{2m+1}{}_{2m+1}C_{2m+1}},$$

for $n = 2m+1$, and

$$KK^1(C^*(H_{\mathbb{Z}}^{2n+1}), \mathbb{C}) \cong \oplus_{k=0}^n \mathbb{Z}^{2^k{}_nC_k} \cong \mathbb{Z}^{3^n},$$

where $C^*(H_{\mathbb{Z}}^{2n+1}) \cong \Gamma(\mathbb{T}, \{\otimes^n\mathfrak{A}_\theta\}_{\theta\in\mathbb{T}})$, and the combination ${}_nC_{2k}$ corresponds to choosing $2k$-fold tensor products $\otimes^{2k}\mathfrak{A}_\theta$ in $\otimes^n\mathfrak{A}_\theta$, and the power 2^{2k} corresponds to choosing commuting $2k$ unitaries, each of which is chosen from two generating *unitaries* U_{j1}, U_{j2} *of the (different) factors* $\mathfrak{A}_\theta = \mathfrak{A}_{\theta,j}$ $(1 \le j \le 2k)$ *in* $\otimes^{2k}\mathfrak{A}_\theta$, *from which and the generating unitary of* $C(\mathbb{T})$ *for* $\mathbb{T}$ *the base space we obtain the odd Fredholm modules* $\varphi_{2k,j}^*([\mathrm{D}_{2k}]) \oplus \psi_1^*([m_{11}])$ *over* $C^*(H_{\mathbb{Z}}^{2n+1})$ $(1 \le j \le 2^{2k}{}_nC_{2k})$, *where* D_{2k} *is the even Dirac Fredholm module over* $C(\mathbb{T}^{2k})$ *generated by unitaries* U_{jx} $(1 \le j \le 2k, x = 1, 2)$ *and* m_{11} *is the odd Fredholm module over* $C(\mathbb{T})$ *for* $\mathbb{T}$ *the base space and the* $*$*-homomorphism* $\psi_1 : C^*(H_{\mathbb{Z}}^{2n+1}) \to C(\mathbb{T})$ *is defined by sending the unitary generator for the base space* $\mathbb{T}$ *to itself and other unitary generators to the identity element, and similarly, the combination* ${}_nC_{2k+1}$ *corresponds to choosing* $(2k+1)$*-fold tensor products* $\otimes^{2k+1}\mathfrak{A}_\theta$ *in* $\otimes^n\mathfrak{A}_\theta$, *and the power* 2^{2k+1} *corresponds to choosing commuting* $2k+1$ *unitaries, each of which is chosen from two generating unitaries* U_{j1}, U_{j2} *of the (different) factors* $\mathfrak{A}_\theta = \mathfrak{A}_{\theta,j}$ $(1 \le j \le 2k+1)$ *in* $\otimes^{2k+1}\mathfrak{A}_\theta$, *from which we obtain the odd Fredholm modules* $\varphi_{2k,j}^*([\mathrm{D}_{2k}]) \oplus \psi_{2k+1}^*([m_{11}])$ *over* $C^*(H_{\mathbb{Z}}^{2n+1})$ $(1 \le j \le 2^{2k}{}_nC_{2k})$, *where* D_{2k} *is the even Dirac Fredholm module over* $C(\mathbb{T}^{2k})$ *generated by unitaries* U_{jx} $(1 \le j \le 2k, x = 1, 2)$ *and* m_{11} *is the odd Fredholm module over* $C(\mathbb{T})$ *generated by* $U_{2k+1,x}$ $(x = 1, 2)$ *and* $*$*-homomorphisms* $\psi_{2k+1} : C^*(H_{\mathbb{Z}}^{2n+1}) \to C(\mathbb{T})$ *are defined by* $\psi_{2k+1}(U_{2k+1,x}) = U_{2k+1,x}$ *and* $\psi_{2k+1}(V) = 1$ *for* V *other generating unitaries, and furthermore, the first* $\mathbb{Z}$ *corresponds to the canonical odd Fredholm module* $\psi_1^*([m_{11}])$ *over* $C^*(H_{\mathbb{Z}}^{2n+1})$ *that corresponds to the* $*$*-homomorphism* $\psi_1 : C^*(H_{\mathbb{Z}}^{2n+1}) \to C(\mathbb{T})$. *Namely,*

$$\begin{aligned} &KK^1(C^*(H_{\mathbb{Z}}^{2n+1}), \mathbb{C}) \cong \\ &\mathbb{Z}\psi_1^*([m_{11}]) \oplus (\oplus_{k=1}^m(\oplus_{1\le j\le 2^{2k}{}_nC_{2k}}\mathbb{Z}\varphi_{2k,j}^*([\mathrm{D}_{2k}]) \oplus \psi_1^*([m_{11}]))) \\ &\oplus (\oplus_{k=0}^m(\oplus_{1\le j\le 2^{2k+1}{}_nC_{2k+1}}\mathbb{Z}(\varphi_{2k,j}^*([\mathrm{D}_{2k}])) \oplus \psi_{2k+1}^*([m_{11}])) \end{aligned}$$

for $n = 2m$ *or* $n = 2m+1$.

Proof. The explanation in the statement is just saying our construction for generators of the odd K-homology $KK^1(C^*(H_{\mathbb{Z}}^{2n+1}), \mathbb{C})$ of $C^*(H_{\mathbb{Z}}^{2n+1})$. □

6. K-homology of certain continuous fields of noncommutative tori

Let $\mathfrak{A}_{\Theta_n}$ be a noncommutative n-torus generated by n unitaries U_j $(1 \le j \le n)$ such that $U_jU_i = e^{2\pi i\theta_{ij}}U_iU_j$ for $1 \le i, j \le n$, where $\theta_{ij} \in \mathbb{R}$ and $\Theta_n = (\theta_{ij})_{i,j=1}^n$ is an $n \times n$ skew-adjoint matrix over $\mathbb{R}$. See Rieffel [6].

Let $\Gamma(\mathbb{T}^m, \{\mathfrak{A}_{\Theta_n(z)}\}_{z\in\mathbb{T}^m})$ be a continuous field C^*-algebra over the m-torus $\mathbb{T}^m$ with the fibers $\mathfrak{A}_{\Theta_n(z)}$ noncommutative n-torus generated by n unitaries U_j $(1 \le j \le n)$ such that $U_jU_i = z_1z_2\cdots z_mU_iU_j$ for $1 \le i < j \le n$ and $z = (z_1, z_2, \cdots, z_m) \in \mathbb{T}^m$.

See [7] for an explicit description for the generalized Rieffel projections and the associated unitaries for $K_0(\mathfrak{A}_{\Theta_n})$ and $K_1(\mathfrak{A}_{\Theta_n})$ of a general noncommutative n-torus $\mathfrak{A}_{\Theta_n}$ respectively, and we omit the details. This part is quite crucial, but our principle is that by the same reasons as given before Theorems 4.1 and 5.1 we do not need to consider those projections and unitaries in caluculating K-theory groups or K-homology groups in these settings as well. Note that $K_j(\mathfrak{A}_{\Theta_n}) \cong K_j(C(\mathbb{T}^n)) \cong \mathbb{Z}^{2^{n-1}}$ by Pimsner-Voiculescu exact sequece for crossed product C^*-algebras by $\mathbb{Z}$ (see [6] and [2]), and $\mathfrak{A}_{\Theta_n}$ can be viewed as a successive crossed product C^*-algebra by $\mathbb{Z}$, i.e., $C(\mathbb{T}) \rtimes \mathbb{Z} \cdots \rtimes \mathbb{Z}$ (($n-1$)-times). Then we have the following:

Theorem 6.1 *The generators of the even K-homology for* $\Gamma(\mathbb{T}^m, \{\mathfrak{A}_{\Theta_n(z)}\}_{z\in\mathbb{T}^m})$ *are determined by the following decompositions*:

$$\begin{aligned}
&KK^0(\Gamma(\mathbb{T}^m, \{\mathfrak{A}_{\Theta_n(z)}\}_{z\in\mathbb{T}^m}), \mathbb{C}) \cong \\
&KK^0(C(\mathbb{T}^m), \mathbb{C}) \oplus (\sum_{j=1}^n KK^0(C(\mathbb{T}^{m+1}), \mathbb{C})/KK^0(C(\mathbb{T}^m), \mathbb{C})) \cong \\
&\mathbb{Z}^{2^{m-1}} \oplus (\oplus_{j=1}^n \mathbb{Z}^{2^m}/\mathbb{Z}^{2^{m-1}}) \cong \mathbb{Z}^{(n+1)2^{m-1}},
\end{aligned}$$

where $KK^0(C(\mathbb{T}^m), \mathbb{C})$ *is the even K-homology for* $C(\mathbb{T}^m)$ *for* $\mathbb{T}^m$ *the base space, and* $C(\mathbb{T}^{m+1})$ *for each* $KK^0(C(\mathbb{T}^{m+1}), \mathbb{C})$ *is the* C^**-algebra generated by one of unitary generators of* $\mathfrak{A}_{\Theta_n(z)}$ *and the unitary generators of* $C(\mathbb{T}^m)$ *for* $\mathbb{T}^m$ *the base space, and* $KK^0(C(\mathbb{T}^{m+1}), \mathbb{C})/KK^0(C(\mathbb{T}^m), \mathbb{C})$ *means a quotient group.*

Similarly, we obtain the following:

Theorem 6.2 *The generators of the odd K-homology for* $\Gamma(\mathbb{T}^m, \{\mathfrak{A}_{\Theta_n(z)}\}_{z\in\mathbb{T}^m})$ *are determined by the following decompositions*:

$$\begin{aligned}
&KK^1(\Gamma(\mathbb{T}^m, \{\mathfrak{A}_{\Theta_n(z)}\}_{z\in\mathbb{T}^m}), \mathbb{C}) \cong \\
&KK^1(C(\mathbb{T}^m), \mathbb{C}) \oplus (\sum_{j=1}^n KK^1(C(\mathbb{T}^{m+1}), \mathbb{C})/KK^1(C(\mathbb{T}^m), \mathbb{C})) \cong \\
&\mathbb{Z}^{2^{m-1}} \oplus (\oplus_{j=1}^n \mathbb{Z}^{2^m}/\mathbb{Z}^{2^{m-1}}) \cong \mathbb{Z}^{(n+1)2^{m-1}}.
\end{aligned}$$

More generally, let $\Gamma(\mathbb{T}^m, \{\otimes_{k=1}^l \mathfrak{A}_{\Theta_n(z)}\}_{z\in\mathbb{T}^m})$ be a continuous field C^*-algebra over the m-torus $\mathbb{T}^m$ with the fibers $\otimes_{k=1}^l \mathfrak{A}_{\Theta_n(z)}$ l-fold tensor products of noncommutative n-tori generated by n unitaries U_j ($1 \leq j \leq n$) such that $U_jU_i = z_1z_2\cdots z_mU_iU_j$ for $1 \leq i < j \leq n$ and $z = (z_1, z_2, \cdots, z_m) \in \mathbb{T}^m$. We first consider the case $m = 1$.

Theorem 6.3 *The generators of the even K-homology for the continuous field* C^**-algebra* $\Gamma(\mathbb{T}, \{\otimes_{k=1}^l \mathfrak{A}_{\Theta_n(z)}\}_{z\in\mathbb{T}})$ *are determined by the following*:

$$\begin{aligned}
&KK^0(\Gamma(\mathbb{T}, \{\otimes_{k=1}^l \mathfrak{A}_{\Theta_n(z)}\}_{z\in\mathbb{T}}), \mathbb{C}) \cong \\
&\mathbb{Z} \oplus \mathbb{Z}^{{}_lC_1 n} \oplus \cdots \oplus \mathbb{Z}^{{}_lC_{2k} n^{2k}} \oplus \mathbb{Z}^{{}_lC_{2k+1} n^{2k+1}} \oplus \cdots \oplus \mathbb{Z}^{{}_lC_l n^l} \\
&\cong \mathbb{Z}^{\sum_{k=0}^l {}_lC_k n^k} = \mathbb{Z}^{(1+n)^l},
\end{aligned}$$

where the combination ${}_lC_k$ *corresponds to choosing* k*-fold tensor products* $\otimes^k\mathfrak{A}_{\Theta_n(z)}$ *in* $\otimes_{k=1}^l\mathfrak{A}_{\Theta_n(z)}$*, and the power* n^k *corresponds to choosing* k *commuting unitaries, each of which is chosen from the generating unitaries of the (different) factors* $\mathfrak{A}_{\Theta_n(z)}$ *of* $\otimes^k\mathfrak{A}_{\Theta_n(z)}$*, from which we obtain the even Fredholm modules by pulling back Dirac Fredholm modules over* $C(\mathbb{T}^k)$ *generated by either those unitaries if* k *is even or those unitaries and the generating unitary of* $C(\mathbb{T})$ *for* $\mathbb{T}$ *the base space if* k *is odd, and the first* $\mathbb{Z}$ *corresponds to the canonical even Fredholm module.*

Proof. By the binary expansion, we have

$$(1+n)^l = 1 + {}_lC_1 n + {}_lC_2 n^2 + \cdots + {}_lC_k n^k + \cdots + {}_lC_l n^l.$$

□

Similarly, we obtain the following:

Theorem 6.4 *The generators of the odd K-homology for the continuous field* C^**-algebra* $\Gamma(\mathbb{T}, \{\otimes_{k=1}^l\mathfrak{A}_{\Theta_n(z)}\}_{z\in\mathbb{T}})$ *are determined by the following decompositions:*

$$\begin{aligned} &KK^1(\Gamma(\mathbb{T}, \{\otimes_{k=1}^l\mathfrak{A}_{\Theta_n(z)}\}_{z\in\mathbb{T}}), \mathbb{C}) \cong \\ &\mathbb{Z} \oplus \mathbb{Z}^{{}_lC_1 n} \oplus \cdots \oplus \mathbb{Z}^{{}_lC_{2k} n^{2k}} \oplus \mathbb{Z}^{{}_lC_{2k+1} n^{2k+1}} \oplus \cdots \oplus \mathbb{Z}^{{}_lC_l n^l} \\ &\cong \mathbb{Z}^{\sum_{k=0}^l {}_lC_k n^k} = \mathbb{Z}^{(1+n)^l}, \end{aligned}$$

where the combination ${}_lC_k$ *corresponds to choosing* k*-fold tensor products* $\otimes^k\mathfrak{A}_{\Theta_n(z)}$ *in* $\otimes_{k=1}^l\mathfrak{A}_{\Theta_n(z)}$*, and the power* n^k *corresponds to choosing* k *commuting unitaries, each of which is chosen from the generating unitaries of the (different) factors* $\mathfrak{A}_{\Theta_n(z)}$ *of* $\otimes^k\mathfrak{A}_{\Theta_n(z)}$*, from which if* k *is even we obtain the odd Fredholm modules by pulling back Dirac Fredholm modules over* $C(\mathbb{T}^k)$ *generated by those unitaries plus the odd Fredholm module over* $C(\mathbb{T})$ *for* $\mathbb{T}$ *the base space, and if* k *is odd we obtain the odd Fredholm modules by pulling back Dirac Fredholm modules over* $C(\mathbb{T}^{k-1})$ *generated by those unitaries plus the odd Fredholm module over* $C(\mathbb{T})$ *generated by the other unitary, and the first* $\mathbb{Z}$ *corresponds to the canonical odd Fredholm module.*

Furthermore, we obtain the following two theorems:

Theorem 6.5 *The generators of the even K-homology for the continuous field* C^**-algebra* $\Gamma(\mathbb{T}^m, \{\otimes_{k=1}^l\mathfrak{A}_{\Theta_n(z)}\}_{z\in\mathbb{T}})$ *are determined by the following:*

$$KK^0(\Gamma(\mathbb{T}^m, \{\otimes_{k=1}^l\mathfrak{A}_{\Theta_n(z)}\}_{z\in\mathbb{T}}), \mathbb{C}) \cong \mathbb{Z}^{2^{m-1}\sum_{k=0}^l {}_lC_k n^k} = \mathbb{Z}^{2^{m-1}(1+n)^l},$$

where we have

$${}_mC_0 + {}_mC_2 + {}_mC_4 + \cdots = 2^{m-1} = {}_mC_1 + {}_mC_3 + {}_mC_5 + \cdots.$$

so that the power 2^{m-1} *corresponds to considering how many unitary generators of* $C(\mathbb{T}^m)$ *are involved in constructing the even Fredholm modules by pulling back and using these unitaries and the other commuting unitaries which are taken from the fibers.*

Theorem 6.6 *The generators of the odd K-homology for the continuous field C^*-algebra $\Gamma(\mathbb{T}^m, \{\otimes_{k=1}^l \mathfrak{A}_{\Theta_n(z)}\}_{z\in\mathbb{T}})$ are determined by the following:*

$$KK^1(\Gamma(\mathbb{T}^m, \{\otimes_{k=1}^l \mathfrak{A}_{\Theta_n(z)}\}_{z\in\mathbb{T}}), \mathbb{C}) \cong \mathbb{Z}^{2^{m-1}\sum_{k=0}^l {}_lC_k n^k} = \mathbb{Z}^{2^{m-1}(1+n)^l}.$$

More generally, let $\Gamma(\mathbb{T}^m, \{\otimes_{k=1}^l \mathfrak{A}_{\Theta_{n_k}(z)}\}_{z\in\mathbb{T}^m})$ be a continuous field C^*-algebra over the m-torus $\mathbb{T}^m$ with the fibers $\otimes_{k=1}^l \mathfrak{A}_{\Theta_{n_k}(z)}$ l-fold tensor products of noncommutative n_k-tori ($1 \leq k \leq l$) generated by n unitaries U_j ($1 \leq j \leq n_k$) such that $U_jU_i = z_1z_2\cdots z_mU_iU_j$ for $1 \leq i < j \leq n_k$ and $z = (z_1, z_2, \cdots, z_m) \in \mathbb{T}^m$.

Theorem 6.7 *The generators of the even and odd K-homology for the continuous field C^*-algebra $\Gamma(\mathbb{T}^m, \{\otimes_{k=1}^l \mathfrak{A}_{\Theta_{n_k}(z)}\}_{z\in\mathbb{T}})$ are determined by the following:*

$$KK^*(\Gamma(\mathbb{T}^m, \{\otimes_{k=1}^l \mathfrak{A}_{\Theta_{n_k}(z)}\}_{z\in\mathbb{T}}), \mathbb{C}) \cong \mathbb{Z}^{2^{m-1}\sum_{k=0}^l {}_lC_k n_{k_1}n_{k_2}\cdots n_{k_k}}$$

for $* = 0, 1$, where the subset $\{k_1, k_2, \cdots, k_k\}$ of $\{1, 2, \cdots, l\}$ corresponds to one of ${}_lC_k$ combinations.

Remark. Summing up our whole argument, our K-homology formulae for the continuous field C^*-algebras with fibers noncommutative tori say that it is enough to count the Fredholm modules coming from commuting unitaries from the base spaces and fibers inside the continuous field C^*-algebras, and not to count those coming from uncommuting unitaries in the fibers since they are not continuous at the end points or on the base spaces.

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