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Conformal Transformations on Carnot-Caratheodory Spaces

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Abstract This paper investigates the conformal and projective transformations on Sub-Riemannian manifolds $\{M, Q, g\}$, and obtains some interesting invariants. These results can be regarded as the natural generalizations of those conclusions in Euclidean setting.

Keywords Carnot-Caratheodory Spaces, Sub-Riemannian manifolds, Transformative Groups, Conformal Invariants, Projective Invariants, Non-holonomic Connections

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1. Introduction

A Carnot-Caratheodory space, roughly speaking, is a manifold associated with a distribution and an fibre inner product on the distribution. Carnot-Caratheodory spaces, are also called Sub-Riemannian Manifolds, or Non-holonomic Riemannian spaces.

The study of geometric analysis of sub-Riemannian manifolds has been an active field over the past several decades. In particular, round about 1993, since the formidable papers [1,2,4,15,17,19] were published in succession, these works stimulate such research fields to present a scene of prosperity, and demonstrate the abnormal importance of this topic.

Sub-Riemannian manifolds, on the one hand, are the natural development of Riemannian manifolds, and are the basic metric spaces on which one can consider the geometric analysis problem; On the other hand, Sub-Riemannian manifolds have been found useful in the study of theories and applications of Control theory, PDEs, the calculus of variations, Mechanic, Gauge fields, etc.

The study of geometric analysis of Sub-Riemannian manifolds is carrying on the following two folds. The first fold is describing the geometric properties of

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Sub-Riemannian manifolds [2,5,7,8,9,15]; The second fold is devoted to the analysis problem of Sub-Riemannian manifolds [4,10,11,12,13,14,16].

In this paper we will take the liberty of considering the geometries of Sub-Riemannian manifolds via a point of view of transform groups, our final purpose is to establish the relevant geometries in the sense of transformative theories. As we know that the study of transformative theories over Carnot-Caratheodory spaces is still a gap.

The organization of this paper is as follows. Section 2 is devoted to introducing some Definitions of Sub-riemannian manifolds, and states some interesting results of Sub-Riemannian manifolds; Section 3 studies the corresponding invariants under the conformal transformation and projective transformation.

2. Preliminaries

Definition 2.1 Let M^n be an $n-$dimensional smooth manifold. For each point $p \in M^n$, there assigns a $k(k < n)-$dimensional subspace $D^k(p)$ of the tangent space T_pM, then $D^k = \bigcup\limits_{p\in M} D^k(p)$ forms a tangent sub-bundle of tangent bundles $TM = \bigcup\limits_{p\in M} T_pM$, D^k is called a $k-$dimensional distribution over M^n. For every point p, if there exists a neighbourhood U and k linearly independent vector fields $X_1, \cdots, X_k$ in U such that for each point $q \in U$, $X_1(q), \cdots, X_k(q)$ is a basis of subspace $D^k(q)$, then we call Distribution D^k the $k-$dimensional smooth distribution (called also horizontal bundle), and $X_1, \cdots, X_k$ are called a local basis of D^k in U, or we say that $X_1, \cdots, X_k$ generate D^k in U. We denote by $D^k|_U = \text{Span}\{X_1, \cdots, X_k\}$.

Definition 2.2 Let $\phi : M \to \overline{M}$ be a diffeomorphic mapping. Let $\overline{M}$ be a Riemannian space with metric $\overline{g}_0$. We call $\Phi(\overline{g}_0)$ the induced Riemannian metric via ϕ, where $\Phi(\overline{g}_0)(X, Y) = \overline{g}_0(\phi_*(X), \phi_*(Y))$, $X, Y \in T_pM$. If $g_0 = \Phi(\overline{g}_0)$, then ϕ is called an isometric mapping, M and $\overline{M}$ are called isometric homeomorphic. In particular, if $M = \overline{M}$, ϕ is said to be an isometric transformation.

Definition 2.3 We call $\{M, Q, g\}$ a Sub-Riemannian manifold with the sub-Riemannian structure (Q, g), if Q is a $k-$dimensional smooth Distribution over M^n, and g is a fibre inner product in Q. Here g is called a Sub-Riemannian metric. In general, g is regarded as some Riemannian metric G, defined on tangent bundle TM, restricted to Q. We also denote by $\Gamma(Q)$ the $C^\infty(M)-$module of smooth sections on Q.

Definition 2.4 A non-holonomic connection on sub-bundle $Q \subset TM$ is a binary mapping $\nabla : \Gamma(Q) \times \Gamma(Q) \to \Gamma(Q)$ satisfying the following:

1) $\nabla_X(Y+Z) = \nabla_X Y + \nabla_X Z$

2) $\nabla_X(fY) = X(f)Y + f\nabla_X Y$

3) $\nabla_{fX+gY} Z = f\nabla_X Z + g\nabla_Y Z$

where $X, Y, Z \in \Gamma(Q)$, $f, g \in C^\infty(M)$.

Let $p_0 : TM \to Q$ be a vector bundle homeomorphic generated by the projection from TM onto Q, $q_0 = 1_{TM} - p_0$ be a projection from TM onto the adjoint distribution $\widetilde{Q}$. Then, there holds $\widetilde{Q} \bigoplus Q = TM$.

Let $\pi : TM \to M$ be a canonical projection mapping of tangent bundles. Inclusion mapping $i : Q \to TM$ is a vector bundle homomorphism and satisfies $i \circ \pi = \pi$. We now define the pull-back bundle of Q as follows

$$\pi^* Q = \{(X, Y) \in Q \times Q : \pi(X) = \pi(Y)\}$$

Assume that $\tau_j : \pi^* Q \to Q$, $j = 1, 2$ are projection mappings, which project the pull-back bundle onto the j-th component; $\pi_* : TQ \to TM$ is the induced tangent mapping via π.

Definition 2.5 A smooth bundle mapping $h : \pi^* Q \to TQ$ is called a generalized connection over Q if it satisfies the following conditions:

1) h is linear with respect to the second component;

2) h satisfies: $\pi \circ h = \tau_1$, $\pi_* \circ h = i \circ \tau_2$.

Definition 2.6[3] Let $\{\phi_t\}$ be a canonical dilation vector field over Q, i.e., for the natural coordinate system (x^i, y^A) in Q, there holds $(x^i, y^A) = (x^i, e^t y^A)$. If the generalized connection h satisfies $\forall (e, n) \in \pi^* Q$, $(\phi)_*(h(e, n)) = h(\phi_t(e), n)$, then h is called a generalized linear connection, briefly speaking, $i-$connection.

For the existence of non-holonomic connections on Q, we have the following

Theorem 2.1[3] There exists a non-holonomic connection on a $k-$dimensional smooth distribution Q over M^n.

Assume that $\{e_\lambda\}_{\lambda=1,\cdots,k}$ is a basis of Q, then formulae $\nabla_{e_\lambda} e_\mu = \Gamma^\nu_{\lambda\mu} e_\nu$, $\lambda, \mu, \nu = 1, \cdots, k$ define k^3 functions as $\{\Gamma^\nu_{\lambda\mu}\}$, we call $\{\Gamma^\nu_{\lambda\mu}\}$ the connection coefficients of the non-holomonic connection ∇.

Lemma 2.1 Let $\{e_\lambda\}_{\lambda=1,\cdots,k}$, $\{\widetilde{e}_\lambda\}_{\lambda=1,\cdots,k}$ be two orthogonal bases of Q, respectively, smooth functions $\{\Gamma^l_{ij}\}$ over M be the connection coefficients of a non-holomonic connection ∇ if and only if there holds the following

$$\widetilde{\Gamma}^\nu_{\lambda\mu} = X^i_\lambda X^j_\mu Y^\nu_l \Gamma^l_{ij} - X^i_\lambda e_i(X^l_\nu) Y^\mu_l \tag{2.1}$$

Proof. It is well known that smooth functions $\{\Gamma^l_{ij}\}$ over M are the connection coefficients of an non-holomonic connection ∇ if and only if $\forall\, U, V \in \Gamma(Q)$ there holds

$$U(\widetilde{V}^\mu)\widetilde{e}_\mu + \widetilde{U}^\lambda \widetilde{V}^\mu \widetilde{\gamma}^\nu_{\lambda\mu} \widetilde{e}_\nu = U(V^j) e_j + U^i V^j \Gamma^l_{ij} e_l \tag{2.2}$$

Since the following formulas are tenable

$$U(\widetilde{V}^\mu)\widetilde{e}_\mu = U(Y^\mu_\nu V^\nu) X^\iota_\mu e_\iota = U(V^j) e_j + U(Y^\mu_j) X^i_\mu V^j e_i,$$

$$\widetilde{U}^\lambda \widetilde{V}^\mu \widetilde{\Gamma}^\nu_{\lambda\mu} \widetilde{e}_\nu = \widetilde{U}^\lambda \widetilde{V}^\mu (X^i_\lambda X^j_\mu Y^\nu_l \Gamma^l_{ij} - X^i_\lambda e_i(X^l_\nu) Y^\mu_l) \widetilde{e}_\nu = U^i V^j \Gamma^l_{ij} e_l - U(Y^\mu_i) X^j_\mu V^i e_j.$$

Thus we know that formula (2.2) is equivalent to

$$U^i V^j \widetilde{\Gamma}^\nu_{\lambda\mu} Y^\lambda_i Y^\mu_j X^l_\nu e_l + U(Y^\iota_j) X^l_\iota V^j e_l = U^i V^j \Gamma^l_{ij} e_l.$$

That is to say, (2.1) is follows. This completes the proof of Lemma 2.1. □

It is well known that the Lie bracket $[\cdot,\cdot]$ on M is a Lie algebra structure of smooth tangent vector fields $\chi(M)$. Since distribution Q is not integrable, then Lie bracket $[\cdot,\cdot]$ is not a Lie algebra structure on Q. We now project the Lie bracket of tangent vector fields over M onto Q via projection mapping $p_0 : TM \to Q$, then, in the sense of basis, $p_0[e_\lambda, e_\mu] = \Omega^\nu_{\lambda\mu} e_\nu$ determine k^3 functions $\Omega^\nu_{\lambda\mu}$.

Remark 2.1 Let $\{\widetilde{e}_\lambda\}_{\lambda=1,\cdots,k}$ be an new basis of Q, $\widetilde{e}_\lambda = X^\mu_\lambda e_\mu$, where $(X^\mu_\lambda)_{k\times k}$ are non-degenerate, then, for $\Omega^\nu_{\lambda\mu}$, there holds

$$\widetilde{\Omega}^\nu_{\lambda\mu} = X^\kappa_\lambda e_\kappa(X^\omega_\mu) Y^\nu_\omega - X^\omega_\mu e_\omega(X^\kappa_\lambda) Y^\nu_\kappa + X^\kappa_\lambda X^\omega_\mu Y^\nu_\iota \Omega^\iota_{\kappa\omega} \tag{2.3}$$

Definition 2.7 Tensor fields, defined by $T_\nabla(X,Y) = \nabla_X Y - \nabla_Y X - p_0[X,Y]$, $\forall X, Y \in \Gamma(Q)$, are called Torsion tensors of the connection ∇ on Distribution Q.

Similar to the case of Riemannian manifolds, we have the following interesting Theorem

Theorem 2.2[3,18] Given a Sub-Riemannian manifold $\{M, Q, g\}$ and a projection $p_0 : TM \to Q$, then there exists an unique non-holonomic connection ∇ satisfying

1) $\nabla_X g(Y,Z) = X(g(Y,Z)) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z) = 0;$

2) $T_\nabla(X,Y) = \nabla_X Y - \nabla_Y X - p_0[X,Y] = 0.$

Remark 2.2 For the use latter in Section 3, we give a simplificative proof of Theorem 2.2 here.

Proof. Uniqueness. Let ∇ be an admissible non-holonomic connection with torsion-free on M. Without loss of generality, takeing $\{e_\lambda\}_{\lambda=1,\cdots,k}$ to be a basis of Q, then $\nabla_{e_\lambda} e_\mu = \Gamma^\nu_{\lambda\mu} e_\nu$, $\lambda, \mu, \nu = 1, \cdots, k$; let $q_0 = 1_{TM} - p_0$ be a projection onto conjugate distribution $\widetilde{Q}$ over TM, where $\widetilde{Q} \bigoplus Q = TM$. Let $p_0[e_\lambda, e_\mu] = \Omega^\nu_{\lambda\mu} e_\nu$; $q_0[e_\lambda, e_\mu] = M^\alpha_{\lambda\mu} e_\alpha$; $p_0[e_\alpha, e_\mu] = \Lambda^\nu_{\alpha\mu} e_\nu$; where $\alpha = k+1, \cdots, n$. Then we conclude, by using the properties of admissible connections, that

$$e_\lambda(g(e_\mu, e_\nu)) = g(\nabla_{e_\lambda} e_\mu, e_\nu) + g(\nabla_{e_\lambda} e_\nu, e_\mu) \tag{2.4}$$

$$e_\mu(g(e_\lambda, e_\nu)) = g(\nabla_{e_\mu} e_\nu, e_\lambda) + g(\nabla_{e_\mu} e_\lambda, e_\nu) \tag{2.5}$$

$$-e_\nu(g(e_\lambda, e_\mu)) = -g(\nabla_{e_\nu} e_\lambda, e_\mu) - g(\nabla_{e_\nu} e_\mu, e_\lambda) \tag{2.6}$$

Finding the summation of three formulas above, and considering the properties of torsion-free tensors of nonholonomic connections, then we get, by a direct computation, that

$$\begin{aligned} 2g(\nabla_{e_\lambda} e_\mu, e_\nu) &= e_\lambda(g(e_\mu, e_\nu)) + e_\mu(g(e_\lambda, e_\nu)) - e_\nu(g(e_\lambda, e_\mu)) \\ &\quad + g(e_\nu, p_0[e_\lambda, e_\mu]) + g(e_\mu, p_0[e_\nu, e_\lambda]) - g(e_\lambda, p_0[e_\mu, e_\nu]). \end{aligned}$$

Furthermore, we know that

$$\begin{aligned} \Gamma^\nu_{\lambda\mu} &= \frac{1}{2} g^{\nu\kappa}\{e_\lambda(g(e_\mu, e_\kappa)) + e_\mu(g(e_\lambda, e_\kappa)) - e_\kappa(g(e_\lambda, e_\mu)) \\ &\quad + g(e_\kappa, p_0[e_\lambda, e_\mu]) + g(e_\mu, p_0[e_\kappa, e_\lambda]) - g(e_\lambda, p_0[e_\mu, e_\kappa])\} \\ &= \frac{1}{2} g^{\nu\kappa}\{e_\lambda(g_{\mu\kappa}) + e_\mu(g_{\lambda\kappa}) - e_\kappa(g_{\lambda\mu}) + g_{\kappa\omega}\Omega^\omega_{\lambda\mu} + g_{\mu\omega}\Omega^\omega_{\kappa\lambda} - g_{\lambda\omega}\Omega^\omega_{\mu\kappa}\} \end{aligned} \tag{2.7}$$

Thus, the admissible connection with torsion-free is determined uniquely by the metric tensor g and the projection $p_0 : TM \to Q$.

Conversely, the non-holonomic connection ∇ with connection coefficients defined by (2.7) is independent of the choice of basis. In fact, we might as well assume that $\{e_\lambda\}_{\lambda=1,\cdots,k}$ is an unit orthogonal basis of Q, then (2.7) turns into $\Gamma^\nu_{\lambda\mu} = \frac{1}{2}(\Omega^\nu_{\lambda\mu} + \Omega^\mu_{\nu\lambda} - \Omega^\lambda_{\mu\nu})$.

Let $\{\widetilde{e}_\lambda\}_{\lambda=1,\cdots,k}$ be an new unit orthogonal basis of Q. Then we have $\widetilde{e}_\lambda = X^j_\lambda e_j$, where matrix $(X^j_\lambda)_{k\times k}$ is non-degenerate, $(Y^\mu_i)_{k\times k}$ is the inverse matrix of $(X^j_\lambda)_{k\times k}$, so one arrives, by using (2.3), at the following

$$
\begin{aligned}
\widetilde{\Gamma}^{\nu}_{\lambda\mu} &= \frac{1}{2}(\widetilde{\Omega}^{\nu}_{\lambda\mu} + \widetilde{\Omega}^{\mu}_{\nu\lambda} - \widetilde{\Omega}^{\lambda}_{\mu\nu}) \\
&+ \frac{1}{2}(X^i_\lambda X^j_\mu Y^\nu_l \Omega^l_{ij} + X^i_\lambda X^l_\nu Y^\mu_j \Omega^j_{li} - X^l_\nu X^j_\mu Y^\lambda_i \Omega^i_{jl}) \\
&+ \frac{1}{2}[X^i_\lambda e_i(X^j_\mu)Y^\nu_j - X^j_\mu e_j(X^i_\lambda)Y^\nu_i] + \frac{1}{2}[X^l_\nu e_l(X^i_\lambda)Y^\mu_i - X^i_\lambda e_i(X^l_\nu)Y^\mu_l] \\
&- \frac{1}{2}[X^j_\mu e_j(X^l_\nu)Y^\lambda_l - X^l_\nu e_l(X^j_\mu)Y^\lambda_j].
\end{aligned}
$$

By virtue of $e_\lambda(X^i_\mu Y^\nu_i) = 0$ and $X^i_\mu = Y^\mu_i$ again, we find $\widetilde{\Gamma}^{\nu}_{\lambda\mu} = X^i_\lambda X^j_\mu Y^\nu_l \Gamma^l_{ij} - X^i_\lambda e_i(X^l_\nu)Y^\mu_l$. In addition, by using Lemma 2.1, we get that the connection coefficients defined by (2.7) determine uniquely an admissible non-holonomic connection ∇ with torsion-free. This ends the proof of Theorem 2.2. □

Remark 2.3 Similar to Riemannian manifolds, we also say that the non-holonomic connections with property 1), respectively, 2) are metric, respectively, torsion-free. The non-holonomic connections satisfying 1) and 2) are called Sub-Riemannian connections. For Theorem 2.2, [18] gives a proof with a approach of projecting Riemannian connection onto the distribution. The method proposed here is a direct proof for the existence of Sub-Riemannian connections.

We now give some properties of nonholonomic connections as follows. The proofs are simple. We omit them here.

Properties 2.1 If $\Gamma^{\nu}_{\lambda\mu}$ is a non-holonomic connection of $\{M, Q, g\}$, T is a (1,2)-tensor field, then $L^{\nu}_{\lambda\mu} = \Gamma^{\nu}_{\lambda\mu} + T^{\nu}_{\lambda\mu}$ is also a non-holonomic connection of $\{M, Q, g\}$.

Properties 2.2 Every non-holonomic connection can be decomposed into the following two parts. The first part is a multiplier of torsion tensors. The second part is a connection with vanishing torsion tensor.

Properties 2.3 The non-holonomic connections and the corresponding connections with vanishing torsion tensors have the same self-parallel vector fields.

Theorem 2.3 Under the basis $\{e_\lambda\}_{\lambda=1,\cdots,k}$, the connection coefficients of non-holonomic connections with vanishing torsion tensor are symmetric with respect to the subscript if and only if $[e_\lambda, e_\mu] \in \widetilde{Q}$, i.e., $\Omega^{\nu}_{\lambda\mu} = 0$, $\lambda, \mu, \nu = 1, \cdots, k$.

Proof. Since $\Gamma^{\nu}_{\lambda\mu} - \Gamma^{\nu}_{\mu\lambda} = \Omega^{\nu}_{\lambda\mu}$. This ends the proof of Theorem 2.3. □

For Sub-Riemannian manifolds, Shouten first considered the curvature problem of non-holonomic connections (see [3,6]), he defined the curvature tensors as follows:

Definition 2.8 A *Shouten* tensor is a mapping $K : \Gamma(Q)\times\Gamma(Q) \to \Gamma(Q)$ defined by

$$K(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{p_0[X,Y]}Z - p_0[q_0[X,Y],Z]$$

where $X, Y, Z \in \Gamma(Q)$.

Remark 2.4 It is easy to check that Definition 2.8 is well defined. In fact, we know that the following formulas are tenable.

$$\begin{aligned} K(fX,Y)Z &= fK(X,Y)Z; \\ K(X,fY)Z &= fK(X,Y)Z; \\ K(X,Y)(fZ) &= fK(X,Y)Z. \end{aligned}$$

For *Shouten* tensors, we have:

Properties 2.4 $K(X,Y)Z = -K(Y,X)Z$.

Proof. Since $[X,Y] = -[Y,X]$, so there holds $p_0[X,Y] = -p_0[Y,X]$, $p_0[q_0[X,Y],Z] = -p_0[q_0[Y,X],Z]$. Then we get, by a direct computation, that

$$\begin{aligned} K(X,Y)Z + K(Y,X)Z &= \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z\nabla_{p_0[X,Y]}Z - p_0[q_0[X,Y],Z] \\ &+ \nabla_Y\nabla_X Z - \nabla_X\nabla_Y Z\nabla_{p_0[Y,X]}Z - p_0[q_0[Y,X],Z] \\ &= 0 \end{aligned}$$

By using Jacobi identity of Poisson bracket and Definition 2.8, we also have

Properties 2.5 $K(X,Y)Z + K(Y,Z)X + K(Z,X)Y = 0$.

We now denote by $K(X,Y;Z,W) = g(K(Z,W)Y,X)$. Then, for the arbitrary $X = X^\lambda e_\lambda, Y = Y^\mu e_\mu, Z = Z^\nu e_\nu, W = W^\kappa e_\kappa, K(e_\lambda, e_\mu)e_\nu = K^\kappa{}_{\nu\lambda\mu}e_\kappa$, we get

$$K(X,Y;Z,W) = K_{\lambda\mu\nu\kappa}X^\lambda Y^\mu Z^\nu W^\kappa.$$

In particular,

$$K_{\lambda\mu\nu\kappa} = K(e_\lambda, e_\mu; e_\nu, e_\kappa).$$

Take a basis of distribution Q as $\{e_\lambda\}_{\lambda=1,\cdots,k}$, by using Properties 2.4 and 2.5 of Shouten tensors, one can write down

Proposition 2.6 1) $K^\lambda{}_{\mu\nu\kappa} = -K^\lambda{}_{\mu\nu\kappa}$; 2) $K^\lambda{}_{\mu\nu\kappa} + K^\lambda{}_{\nu\kappa\mu} + K^\lambda{}_{\kappa\mu\nu} = 0$.

It is well known that there hold the following formulas for the curvature tensor R over Riemannian manifolds:

*3) $R(X,Y;Z,W) = -R(Y,X;Z,W) = -R(X,Y;W,Z)$;

*4) $R(X,Y;Z,W) = R(Z,W;X,Y)$.

Since Distribution Q is not involutive, so the curvature tensor K does not satisfy the properties $*3) * 4)$. We only obtain:

$$\begin{aligned} K(X,Y;Z,W) &= -K(Y,X;Z,W) - g(p_0[q_0[Z,W],X],Y) \\ &\quad -g(p_0[q_0[Z,W],Y],X) + q_0[Z,W]g(X,Y). \end{aligned}$$

Of course, when Q is involutive, $q_0[Z,W] = 0$. In this setting, we have the analogue similar to Riemannian curvature tensors.

Remark 2.5 Since the curvature tensor K does not satisfy properties $*3) * 4)$, so we can not give out the second *Bianchi* identity of Shouten curvature tensors similar to Riemannian curvature tensors.

Let $K(e_\lambda, e_\mu)e_\nu = K^\kappa{}_{\nu\lambda\mu}e_\kappa$, by using $\nabla_{e_\lambda}e_\mu = \Gamma^\nu_{\lambda\mu}e_\nu$, $p_0[e_\lambda, e_\mu] = \Omega^\nu_{\lambda\mu}e_\nu$, $q_0[e_\lambda, e_\mu] = M^\alpha_{\lambda\mu}e_\alpha$, $p_0[e_\alpha, e_\mu] = \Lambda^\nu_{\alpha\mu}e_\nu$, $\lambda, \mu, \nu = 1, \cdots, k$; $\alpha = k+1, \cdots, n$, then we know that

$$K^\kappa{}_{\nu\lambda\mu} = e_\lambda(\Gamma^\kappa_{\mu\nu}) - e_\mu(\Gamma^\kappa_{\lambda\nu}) + \Gamma^\kappa_{\lambda\omega}\Gamma^\omega_{\mu\nu} - \Gamma^\kappa_{\mu\omega}\Gamma^\omega_{\lambda\nu} - \Omega^\omega_{\lambda\mu}\Gamma^\kappa_{\omega\nu} - M^\alpha_{\mu\nu}\Lambda^\kappa_{\alpha\nu} \tag{2.8}$$

According to Proposition 2.6, we get

$$K^\omega{}_{\nu\omega\mu} - K^\omega{}_{\mu\omega\nu} = K^\omega{}_{\omega\nu\mu} \tag{2.9}$$

Thus, $g^{\nu\mu}(K^\omega{}_{\nu\omega\mu} - K^\omega{}_{\mu\omega\nu}) = 0$. Similar to the case of Riemannian manifolds, We call

$$K = g^{\nu\mu}K^\omega{}_{\nu\omega\mu} \tag{2.10}$$

the scalar curvature of Shouten curvature tensors.

3. The Invariants of Some Transformations

For the transformative theories on Riemannian manifolds, there are many interesting research results. The authors had been studied this class problems and obtained some conclusions (see [21, 22, 23]). But, as we know, there is not any research results related to transformative theories over Sub-Riemannian manifolds. In view of this fact, we will consider the transformative theories over Carnot-Caratheodory spaces.

3.1 Conformal Transformations

Definition 3.1.1 Let $g, \overset{*}{g}$ be two bundle metrics on an non-integrable smooth distribution Q over M^n. If there exists a function ρ such that

$$\overset{*}{g}= e^{2\rho}g \tag{3.1.1}$$

then we say that $\overset{*}{g}$ and g are conformal (briefly, $\overset{*}{g}$ is conformal to g). In particular, if ρ=constant, we say that $\overset{*}{g}$ and g are similitude (briefly, $\overset{*}{g}$ is similar to g).

Since the Sub-Riemannian metric is positive definite, so the fact $\overset{*}{g}$ is conformal to g is equivalent to that there exists a positive function σ such that $\overset{*}{g}= \sigma g$. It is obvious that if $\overset{*}{g}$ is conformal to g, then g is conformal to $\overset{*}{g}$. In addition, if $\overset{*}{g}$ is conformal to g', then g is conformal to g'.

Let $\{e_\lambda\}_{\lambda=1,\cdots,k}$ be a basis of Q, then (3.1.1) can be written as

$$\overset{*}{g}_{\lambda\mu}= e^{2\rho}g_{\lambda\mu} \tag{3.1.2}$$

Denote by $\|X\|, \|X\|^*, \theta, \theta^*$ the length of $X \in \Gamma(Q)$ and the included angle of $X, Y \in \Gamma(Q)$ corresponding to g and $\overset{*}{g}$, respectively, then we have $\|X\| = e^\rho\|X\|, \theta = \theta^*$.

Assume that g is conformal to $\overset{*}{g}$ and $\dim(Q) > 2$. We will study the relationships of Sub-Riemannian connection coefficients, curvature tensors corresponding to g and $\overset{*}{g}$. At first, it is easy to see that $\overset{*}{g}{}^{\lambda\mu} = e^{-2\rho}g^{\lambda\mu}$. By a direct computation in terms of (2.7), we find the Sub-Riemannian connection corresponding to $\overset{*}{g}$ can be written as

$$\begin{aligned}
\overset{*}{\Gamma}{}^{\nu}_{\lambda\mu} &= \Gamma^\nu_{\lambda\mu} + \frac{1}{2}e^{-2\rho}g^{\nu\kappa}\{g_{\mu\kappa}e_\lambda(e^{2\rho}) + g_{\kappa\lambda}e_\mu(e^{2\rho}) - g_{\lambda\mu}e_\kappa(e^{2\rho})\} \\
&= \Gamma^\nu_{\lambda\mu} + g^{\nu\kappa}\{g_{\mu\kappa}e_\lambda(\rho) + g_{\kappa\lambda}e_\mu(\rho) - g_{\lambda\mu}e_\kappa(\rho)\} \\
&= \Gamma^\nu_{\lambda\mu} + \delta^\nu_\mu e_\lambda(\rho) + \delta^\nu_\lambda e_\mu(\rho) - g_{\lambda\mu}g^{\nu\kappa}e_\kappa(\rho)
\end{aligned}$$

that is

$$\overset{*}{\Gamma}{}^{\nu}_{\lambda\mu} = \Gamma^\nu_{\lambda\mu} + \delta^\nu_\mu e_\lambda(\rho) + \delta^\nu_\lambda e_\mu(\rho) - g_{\lambda\mu}g^{\nu\kappa}e_\kappa(\rho) \tag{3.1.3}$$

Substituting (3.1.3) into (2.8), we have

$$\begin{aligned}
\overset{*}{K}{}^{\kappa}{}_{\nu\lambda\mu} &= e_\lambda(\overset{*}{\Gamma}{}^{\kappa}_{\mu\nu}) - e_\mu(\overset{*}{\Gamma}{}^{\kappa}_{\lambda\nu}) + \overset{*}{\Gamma}{}^{\kappa}_{\lambda\omega}\overset{*}{\Gamma}{}^{\omega}_{\mu\nu} - \overset{*}{\Gamma}{}^{\kappa}_{\mu\omega}\overset{*}{\Gamma}{}^{\omega}_{\lambda\nu} - \Omega^{\omega}_{\lambda\mu}\overset{*}{\Gamma}{}^{\kappa}_{\omega\nu} - M^{\alpha}_{\lambda\mu}\Lambda^{\kappa}_{\alpha\nu} \\
&= K^{\kappa}{}_{\nu\lambda\mu} + e_\lambda(\delta^\kappa_\mu e_\nu(\rho) + \delta^\kappa_\nu e_\mu(\rho) - g_{\mu\nu}g^{\kappa\omega}e_\omega(\rho)) \\
&- e_\mu(\delta^\kappa_\lambda e_\nu(\rho) + \delta^\kappa_\nu e_\lambda(\rho) - g_{\lambda\nu}g^{\kappa\omega}e_\omega(\rho)) \\
&+ (\delta^\kappa_\lambda e_\omega(\rho) + \delta^\kappa_\omega e_\lambda(\rho) - g_{\lambda\omega}g^{\kappa\iota}e_\iota(\rho))(\delta^\omega_\mu e_\nu(\rho) + \delta^\omega_\nu e_\mu(\rho) - g_{\mu\nu}g^{\omega\iota}e_\iota(\rho)) \\
&- (\delta^\kappa_\mu e_\omega(\rho) + \delta^\kappa_\omega e_\mu(\rho) - g_{\mu\omega}g^{\kappa\iota}e_\iota(\rho))(\delta^\omega_\lambda e_\nu(\rho) + \delta^\omega_\nu e_\lambda(\rho) - g_{\lambda\nu}g^{\omega\iota}e_\iota(\rho)) \\
&- \Omega^{\omega}_{\lambda\mu}(\delta^\kappa_\omega e_\nu(\rho) + \delta^\kappa_\nu e_\omega(\rho) - g_{\omega\nu}g^{\kappa\iota}e_\iota(\rho)) \\
&+ \Gamma^{\kappa}_{\lambda\omega}(\delta^\omega_\mu e_\nu(\rho) + \delta^\omega_\nu e_\mu(\rho) - g_{\mu\nu}g^{\omega\iota}e_\iota(\rho)) \\
&+ \Gamma^{\omega}_{\mu\nu}(\delta^\kappa_\lambda e_\omega(\rho) + \delta^\kappa_\omega e_\lambda(\rho) - g_{\lambda\omega}g^{\kappa\iota}e_\iota(\rho)) \\
&- \Gamma^{\kappa}_{\mu\omega}(\delta^\omega_\lambda e_\nu(\rho) + \delta^\omega_\nu e_\lambda(\rho) - g_{\lambda\nu}g^{\omega\iota}e_\iota(\rho)) \\
&- \Gamma^{\omega}_{\lambda\nu}(\delta^\kappa_\mu e_\omega(\rho) + \delta^\kappa_\omega e_\mu(\rho) - g_{\mu\omega}g^{\kappa\iota}e_\iota(\rho))
\end{aligned}$$

By using $\Gamma^\nu_{\lambda\mu} - \Gamma^\nu_{\mu\lambda} = \Omega^\nu_{\lambda\mu}$, $[e_\lambda, e_\mu] - \Omega^\kappa_{\lambda\mu}e_\kappa = M^\alpha_{\lambda\mu}e_\alpha$, and putting formula above in order, then one arrives at

$$\begin{aligned}
\overset{*}{K}{}^{\kappa}{}_{\nu\lambda\mu} &= K^{\kappa}{}_{\nu\lambda\mu} + \delta^\kappa_\mu\{e_\lambda \circ e_\nu(\rho) + e_\lambda(\rho)e_\nu(\rho) + \frac{1}{2}g_{\lambda\nu}g^{\omega\iota}e_\omega(\rho)e_\iota(\rho) - \Gamma^{e}_{\lambda\nu}e_\omega(\rho)\} \\
&- \delta^\kappa_\lambda\{e_\mu \circ e_\nu(\rho) + e_\mu(\rho)e_\nu(\rho) + \frac{1}{2}g_{\mu\nu}g^{\omega\iota}e_\omega(\rho)e_\iota(\rho) - \Gamma^{\omega}_{\mu\nu}e_\omega(\rho)\} \\
&+ g_{\lambda\nu}\{e_\mu(g^{\kappa\omega}e_\omega(\rho)) + \Gamma^{\kappa}_{\mu\iota}g^{\omega\iota}e_\omega(\rho) - g^{\kappa\iota}e_\iota(\rho)e_\mu(\rho)\frac{1}{2}\delta^\kappa_\mu g^{\omega\iota}e_\omega(\rho)e_\iota(\rho)\} \\
&- g_{\mu\nu}\{e_\lambda(g^{\kappa\omega}e_\omega(\rho)) + \Gamma^{\kappa}_{\lambda\iota}g^{\omega\iota}e_\omega(\rho) - g^{\kappa\iota}e_\iota(\rho)e_\lambda(\rho)\frac{1}{2}\delta^\kappa_\lambda g^{\omega\iota}e_\omega(\rho)e_\iota(\rho)\} \\
&+ \delta^\kappa_\nu M^{\alpha}_{\lambda\mu}e_\alpha
\end{aligned} \tag{3.1.4}$$

Denote by $\nabla_{e_\lambda}e_\nu(\rho) = e_\lambda \circ e_\nu(\rho) - \Gamma^\omega_{\lambda\nu}e_\omega(\rho)$. By virtue of the admissible characteristics of Sub-Riemannian connections and $g_{\kappa\omega}g^{\iota\omega} = \delta^\iota_\kappa$, we get

$$e_\lambda(g^{\kappa\omega}) = -g^{\kappa\iota}\Gamma^{\omega}_{\lambda\iota} - g^{\iota\omega}\Gamma^{\kappa}_{\lambda\iota} \tag{3.1.5}$$

Thus we conclude

$$\begin{aligned}
& e_\lambda(g^{\kappa\omega}e_\omega(\rho)) + \Gamma^{\kappa}_{\lambda\omega}g^{\iota\omega}e_\iota(\rho) \\
&= e_\lambda(g^{\kappa\omega}e_\omega(\rho)) - e_\omega(\rho)e_\lambda(g^{\kappa\omega}) - g^{\kappa\omega}\Gamma^{\iota}_{\lambda\omega}e_\iota(\rho) \\
&= g^{\kappa\omega}(e_\lambda \circ e_\omega(\rho) - \Gamma^{\iota}_{\lambda\omega}e_\iota(\rho)).
\end{aligned}$$

That is to say

$$\nabla_{e_\lambda}(g^{\kappa\omega}e_\omega(\rho)) = g^{\kappa\omega}\nabla_{e_\lambda}(e_\omega(\rho)) \tag{3.1.6}$$

For convenience, we let

$$\rho_{\lambda\nu} = \nabla_{e_\lambda} e_\nu(\rho) - e_\lambda(\rho)e_\nu(\rho) + \frac{1}{2} g_{\lambda\nu} g^{\iota\omega} e_\omega(\rho) e_\iota(\rho),$$

From (3.1.6), we know

$$\rho^\kappa_\lambda \hat{=} g^{\kappa\omega}\rho_{\lambda\omega} = \nabla_{e_\lambda}(g^{\kappa\omega}e_\omega(\rho)) - g^{\kappa\omega}e_\omega(\rho)e_\lambda(\rho) + \frac{1}{2}\delta^\kappa_\lambda g^{\iota\omega} e_\omega(\rho)e_\iota(\rho),$$

$$\rho_{\lambda\nu} - \rho_{\nu\lambda} = M^\alpha_{\lambda\nu} e_\alpha(\rho),$$

Thus

$$\overset{*}{K}{}^{\kappa}_{\nu\lambda\mu} = K^\kappa{}_{\nu\lambda\mu} + \delta^\kappa_\mu \rho_{\lambda\nu} - \delta^\kappa_\lambda \rho_{\mu\nu} + g_{\lambda\nu}\rho^\kappa_\mu - g_{\mu\nu}\rho^\kappa_\lambda + \delta^\kappa_\nu M^\alpha_{\lambda\mu} e_\alpha(\rho) \qquad (3.1.7)$$

Considering the retraction with κ and ν, we have

$$\overset{*}{K}{}^{\epsilon}{}_{\epsilon\lambda\mu} = K^\epsilon{}_{\epsilon\lambda\mu} + kM^\alpha_{\lambda\mu} e_\alpha(\rho).$$

so

$$M^\alpha_{\lambda\mu} e_\alpha(\rho) = \frac{1}{k}(\overset{*}{K}{}^{\epsilon}{}_{\epsilon\lambda\mu} - K^\epsilon{}_{\epsilon\lambda\mu}) \qquad (3.1.8)$$

Put

$$S^\kappa{}_{\nu\lambda\mu} = K^\kappa{}_{\nu\lambda\mu} - \frac{1}{k}\delta^\kappa_\nu K^\epsilon{}_{\epsilon\lambda\mu},$$

then

$$\overset{*}{S}{}^{\kappa}{}_{\nu\lambda\mu} = S^\kappa{}_{\nu\lambda\mu} + \delta^\kappa_\mu \rho_{\lambda\nu} - \delta^\kappa_\lambda \rho_{\mu\nu} + g_{\lambda\nu}\rho^\kappa_\mu - g_{\mu\nu}\rho^\kappa_\lambda.$$

Considering the retraction again with κ and λ, then we arrive at

$$\overset{*}{S}{}^{\epsilon}{}_{\nu\epsilon\mu} = S^\epsilon{}_{\nu\epsilon\mu} + (k-2)\rho_{\mu\nu} - g_{\mu\nu}\rho^\epsilon_\epsilon \qquad (3.1.9)$$

takeing the action with $g_{\mu\nu}$ on (3.1.9), one has

$$e^{2\rho}\,\overset{*}{S} = S - 2(k-1)\rho^\epsilon_\epsilon,$$

where $S = g^{\nu\mu}S^\epsilon{}_{\nu\epsilon\mu}$. Thus we obtain

$$\rho^\epsilon_\epsilon = \frac{1}{2(k-1)}(S - e^{2\rho})\,\overset{*}{S} \qquad (3.1.10)$$

Substituting (3.1.10) into (3.1.9), one gets

$$\overset{*}{S}{}^{\epsilon}{}_{\nu\epsilon\mu} = S^\epsilon{}_{\nu\epsilon\mu} - (k-2)\rho_{\nu\mu} - \frac{1}{2(k-1)} g_{\nu\mu}(S - e^{2\rho}\,\overset{*}{S}),$$

or

$$\overset{*}{S}{}^{\epsilon}{}_{\nu\epsilon\mu} - \frac{\overset{*}{S}}{2(k-1)}\,\overset{*}{g}_{\nu\mu} = S^\epsilon{}_{\nu\epsilon\mu} - \frac{S}{2(k-1)} g_{\nu\mu} - (k-2)\rho_{\nu\mu}.$$

Put

$$L_{\nu\mu} = S_{\nu\mu} - \frac{S}{2(k-1)} g_{\nu\mu},$$

then

$$\rho_{\nu\mu} = \frac{1}{k-2}(L_{\nu\mu} - \overset{*}{L}_{\nu\mu}) \tag{3.1.11}$$

$$\rho_{\nu}^{\kappa} = g^{\mu\kappa}\rho_{\nu\mu} = \frac{1}{k-2}(L_{\nu}^{\kappa} - e^{2\rho}\,\overset{*}{L}{}^{\kappa}_{\nu}) \tag{3.1.12}$$

Finally, substituting (3.1.11), (3.1.12) and (3.1.8) into (3.1.7), we know

$$\begin{aligned}\overset{*}{K}{}^{\kappa}{}_{\nu\lambda\mu} &= K^{\kappa}{}_{\nu\lambda\mu} + \frac{1}{k-2}[\delta_{\mu}^{\kappa}(L_{\lambda\nu} - \overset{*}{L}_{\lambda\nu}) - \delta_{\lambda}^{\kappa}(L_{\mu\nu} - \overset{*}{L}_{\mu\nu})\\ &+ g_{\lambda\nu}(L_{\mu}^{\kappa} - e^{2\rho}L_{\lambda\nu}\,\overset{*}{L}{}^{\kappa}_{\mu}) - g_{\mu\nu}(L_{\lambda}^{\kappa} - e^{2\rho}L_{\lambda\nu}\,\overset{*}{L}{}^{\kappa}_{\lambda})] + \frac{1}{k}\delta_{\nu}^{\kappa}(\overset{*}{K}{}^{\epsilon}{}_{\epsilon\lambda\mu} - K^{\epsilon}{}_{\epsilon\lambda\mu}).\end{aligned}$$

Denote by

$$C^{\kappa}{}_{\nu\lambda\mu} = K^{\kappa}{}_{\nu\lambda\mu} + \frac{1}{k-2}[\delta_{\mu}^{\kappa}L_{\lambda\nu} - \delta_{\lambda}^{\kappa}L_{\mu\nu} + g_{\lambda\nu}L_{\mu}^{\kappa} - g_{\mu\nu}L_{\lambda}^{\kappa}] - \frac{1}{k}\delta_{\nu}^{\kappa}K^{\epsilon}{}_{\epsilon\lambda\mu} \tag{3.1.13}$$

The formula (3.1.13) is equivalent to the following

$$\begin{aligned}C^{\kappa}{}_{\nu\lambda\mu} &= K^{\kappa}{}_{\nu\lambda\mu} + \frac{1}{k-2}(\delta_{\mu}^{\kappa}K^{\epsilon}{}_{\lambda\epsilon\nu} - \delta_{\lambda}^{\kappa}K^{\epsilon}{}_{\mu\epsilon\nu} + g_{\lambda\nu}K^{\epsilon}{}_{\mu\epsilon}{}^{\kappa} - g_{\mu\nu}K^{\epsilon}{}_{\lambda\epsilon}{}^{\kappa})\\ &- \frac{S}{(k-1)(k-2)}(g_{\mu\nu}\delta_{\lambda}^{\kappa} - g_{\lambda\nu}\delta_{\mu}^{\kappa})\\ &+ \frac{1}{k(k-2)}(\delta_{\lambda}^{\kappa}K^{\epsilon}{}_{\epsilon\mu\nu} - \delta_{\mu}^{\kappa}K^{\epsilon}{}_{\epsilon\lambda\nu} + g_{\mu\nu}K^{\epsilon}{}_{\epsilon\lambda}{}^{\kappa} - g_{\lambda\nu}K^{\epsilon}{}_{\epsilon\mu}{}^{\kappa})\end{aligned} \tag{3.1.14}$$

Then

$$\overset{*}{C}{}^{\kappa}{}_{\nu\lambda\mu} = C^{\kappa}{}_{\nu\lambda\mu} \tag{3.1.15}$$

Similar to the case of Riemannian manifolds, we also call $\hat{C} = (C^{\kappa}{}_{\nu\lambda\mu})$ defined by (3.1.14) the *Weyl* conformally curvature tensors. This implies the following

Theorem 3.1.1 The Weyl conformally curvature tensors are invariants under the sub-conformal transformation.

Remark 3.1.1

(1) If Q is integrable, then (3.1.14) is exactly the analogue of Riemannian manifolds.

(2) For a flat metric g, it is obvious that $C^{\kappa}{}_{\nu\lambda\mu} = 0$. If $\overset{*}{g}$ is conformal to a flat metric g, then $\overset{*}{C}{}^{\kappa}{}_{\nu\lambda\mu} = 0$.

Proposition 3.1.1 The Conformal curvature tensors $\hat{C} = (C^{\kappa}{}_{\nu\lambda\mu}), C_{\kappa\nu\lambda\mu} = g_{\kappa\omega}C^{\omega}{}_{\nu\lambda\mu}$ satisfy

1) $C^{\kappa}{}_{\nu\lambda\mu} = -C^{\kappa}{}_{\nu\mu\lambda}$;

2) $C^{\kappa}{}_{\nu\lambda\mu} + C^{\kappa}{}_{\lambda\mu\nu} + C^{\kappa}{}_{\mu\nu\lambda} = \frac{2}{k}(\delta^{\kappa}_{\mu}K^{\epsilon}{}_{\epsilon\lambda\nu} + \delta^{\kappa}_{\lambda}K^{\epsilon}{}_{\epsilon\nu\mu} + \delta^{\kappa}_{\nu}K^{\epsilon}{}_{\epsilon\mu\lambda})$;

3) $C_{\kappa\nu\lambda\mu} = -C_{\kappa\nu\mu\lambda}$;

4) $C^{\epsilon}{}_{\epsilon\lambda\mu} = 0$;

5) $C^{\epsilon}{}_{\nu\epsilon\mu} = \frac{k-2}{k}K^{\epsilon}{}_{\epsilon\nu\mu}; C^{\epsilon}{}_{\nu\epsilon\mu} = -C^{\epsilon}{}_{\mu\epsilon\nu}$

Proof. It is not hard to derive that the properties above are tenable. We omit it here. □

Definition 3.1.2 Let $\phi : M^n \to M^n$ be a diffeomorphic mapping, and $\{M, Q, g\}$ and $\{M, Q, \widetilde{g}\}$ be two $k-$dimensional Sub-Riemannian manifolds. If the mapping ϕ satisfies $\phi_*(Q) = Q, g = e^{2\rho}\Phi(\widetilde{g})$, we say that ϕ is a Sub-conformal transformation. In this setting we say that Sub-Riemannian manifolds $\{M, Q, g\}$ and Sub-riemannian manifolds $\{M, Q, \widetilde{g}\}$ are the Sub-Riemannian isometric homoeomorphic.

Theorem 3.1.2 Conformal curvature tensors are invariants under the Sub-conformal transformations over Sub-Riemannian manifolds $M^n(n > 3)$ with k $(k \geq 3)$ dimensional smooth distribution.

Definition 3.1.3 For all t, $\{\phi_t\}$ are sub-conformal transformations, then $\{\phi_t\}$ is called the sub-conformal transformative group. Let X be the horizontal vector fields induced by $\{\phi_t\}$. If $\{\phi_t\}$ is the sub-conformal transformative group, then there exists a scalar function ρ_t such that

$$\Phi_t(g_{\bar{p}}) = \exp(2\rho_t)g_p, \ \ \bar{p} = \phi_t(p).$$

Since ϕ_0 is an identical map, i.e., $\rho_0 = 0$. Thus, one arrives at

$$[\frac{\partial}{\partial t}\Phi_t(g_{\bar{p}})]_{t=0} = 2[\frac{\partial \rho_t}{\partial t}\exp(2\rho_t)]_{t=0} \cdot g_p$$

Let $\frac{\partial \rho_t}{\partial t}|_{t=0} = \rho$, then we have

$$\mathcal{L}_X g = 2\rho g$$

Conversely, if there exists a scalar function ρ such that $\mathcal{L}_X g = 2\rho g$.

Put $\bar{p} = \phi_s(p)$, considering the deformation of the left side of the following formula

$$[\frac{\partial}{\partial t}\Phi_t(g_{\phi_t(\bar{p})})]_{t=0} = 2\rho(\bar{p})g_{\bar{p}}$$

and getting the following

$$[\frac{\partial}{\partial t}\Phi_t(g_{\phi_t(\bar{p})})]_{t=0} = [\frac{\partial}{\partial t}\Phi_t(g_{\phi_{t+s}(p)})]_{t=0} = \Phi_s^{-1}[\frac{\partial}{\partial l}\Phi_l(g_{\phi l(p)})]_{l=s}$$

Thus, by a direct computation, we have

$$\Phi_s(g_{\bar{p}}) = \exp(2\rho_s)g_p$$

This implies that there holds the following

Theorem 3.1.3 The transformative group $\{\phi_t\}$ is the sub-conformal transformative group if and only if there exists a scalar function ρ such that the vector fields X induced by $\{\phi_t\}$ satisfies

$$\mathcal{L}_X g = 2\rho g \tag{3.1.16}$$

Similarly, we also call the vector fields $X = X^\lambda e_\lambda$ satisfying (3.1.16) the sub-conformal Killing vector fields.

We now consider the geometric characteristics for conformal curvature tensor $C^\kappa{}_{\nu\lambda\mu}$. Let $C^\kappa{}_{\nu\lambda\mu} = 0$, by a direct computation, one has $K^\kappa{}_{\kappa\lambda\mu} = 0$ for $k \neq 2$. This says that the distribution Q is involutive. In other words, we obtain the following

Theorem 3.1.4 The Sub-Riemannian manifold $\{M, Q, g\}$ is conformal flat if and only if the distribution Q is involutive.

3.2 Projection Transformations

In this section, we assume that there exists a basis satisfying $[e_\lambda, e_\mu] \in \widetilde{Q}$, $\lambda, \mu = 1, \cdots, k$ over distribution Q, then the connection coefficients of non-holonomic connections are symmetric with respect to subscripts, i.e., $\Gamma^\nu_{\lambda\mu} = \Gamma^\nu_{\mu\lambda}$.

Definition 3.2.1 Assume that there exist two classes of non-holonomic connections $\Gamma, \overset{*}{\Gamma}$ with symmetry with respect to subscripts on $\{M, Q\}$. If the paths corresponding to Γ coincides always with that corresponding to $\overset{*}{\Gamma}$, then we say that $\overset{*}{\Gamma}$ is projective correspondence to Γ.

Let curve $\gamma : \dot{\gamma}(t) = r^\lambda(t)e_\lambda$ be a path associated with Γ, then

$$r^\lambda(t)e_\lambda(r^\nu) + \Gamma^\nu_{\lambda\mu}r^\lambda r^\mu = \alpha(t) \tag{3.2.1}$$

where $\alpha(t)$ is a function on γ, t is a arbitrary parameter. Since γ is also a path associated with $\overset{*}{\Gamma}$, so we know

$$r^\lambda(t)e_\lambda(r^\nu) + \Gamma^\nu_{\lambda\mu}r^\lambda r^\mu = \beta(t) \tag{3.2.2}$$

Define a tensor $L^{\nu}_{\lambda\mu} = \overset{*}{\Gamma}{}^{\nu}_{\lambda\mu} - \Gamma^{\nu}_{\lambda\mu}$, then

$$L^{\nu}_{\lambda\mu} r^{\lambda} r^{\mu} = (\beta - \alpha) r^{\nu}, \quad r^{\kappa} L^{\nu}_{\lambda\mu} r^{\lambda} r^{\mu} = (\beta - \alpha) r^{\nu} r^{\kappa}$$

Since the right hand of this equality is symmetric with respect to κ and ν, so one can interchange κ, ν and consider the subtraction between these equalities, then

$$(\delta^{\kappa}_{\omega} L^{\nu}_{\lambda\mu} - \delta^{\nu}_{\omega} L^{\kappa}_{\lambda\mu}) r^{\omega} r^{\lambda} r^{\mu} = 0 \tag{3.2.3}$$

Considering the symmetric part of (3.2.3), it is not hard to see that

$$\delta^{\kappa}_{\omega} L^{\nu}_{\lambda\mu} - \delta^{\nu}_{\omega} L^{\kappa}_{\lambda\mu} + \delta^{\kappa}_{\lambda} L^{\nu}_{\omega\mu} - \delta^{\nu}_{\lambda} L^{\kappa}_{\omega\mu} + \delta^{\kappa}_{\mu} L^{\nu}_{\omega\lambda} - \delta^{\nu}_{\mu} L^{\kappa}_{\omega\lambda} = 0.$$

Let $\kappa = \omega$, and find the summation for this equality above, we have

$$L^{\nu}_{\lambda\mu} = \delta^{\nu}_{\lambda} \varphi_{\mu} + \delta^{\nu}_{\mu} \varphi_{\lambda},$$

where $\varphi_{\mu} = \frac{1}{k+1} L^{\epsilon}_{\epsilon\mu}$.

Conversely, if there exist the covariant vector fields φ_{μ} such that for non-holonomic connections over $\{M, Q\}$ there holds

$$\overset{*}{\Gamma}{}^{\nu}_{\lambda\mu} = \Gamma^{\nu}_{\lambda\mu} + \delta^{\nu}_{\mu} \varphi_{\lambda} + \delta^{\nu}_{\lambda} \varphi_{\mu}$$

Then we obtain that (3.2.1) and (3.2.2) are satisfied simultaneously. Thus we have

Theorem 3.2.1 The non-holonomic connections Γ and $\overset{*}{\Gamma}$ on $\{M, Q\}$ are projective correspondence if and only if there exists a horizontal vector field φ_{μ} such that

$$\overset{*}{\Gamma}{}^{\nu}_{\lambda\mu} = \Gamma^{\nu}_{\lambda\mu} + \delta^{\nu}_{\mu} \varphi_{\lambda} + \delta^{\nu}_{\lambda} \varphi_{\mu} \tag{3.2.4}$$

Since $[e_{\lambda}, e_{\mu}] \in \widetilde{Q}, \lambda, \mu = 1, \cdots, k$, so $\Omega^{\nu}_{\lambda\mu} = 0$. Considering the following formula

$$\overset{*}{K}{}^{\kappa}_{\nu\lambda\mu} = e_{\lambda}(\overset{*}{\Gamma}{}^{\kappa}_{\mu\nu}) - e_{\mu}(\overset{*}{\Gamma}{}^{\kappa}_{\lambda\nu}) + \overset{*}{\Gamma}{}^{\kappa}_{\lambda\omega} \overset{*}{\Gamma}{}^{\omega}_{\mu\nu} - \overset{*}{\Gamma}{}^{\kappa}_{\mu\omega} \overset{*}{\Gamma}{}^{\omega}_{\lambda\nu} - M^{\alpha}_{\lambda\mu} \Lambda^{\kappa}_{\alpha\nu}.$$

Substituting (3.2.4) into this formula we get

$$\begin{aligned} \overset{*}{\Gamma}{}^{\nu}_{\lambda\mu} &= K^{\kappa}{}_{\nu\lambda\mu} + \delta^{\kappa}_{\mu}(e_{\lambda}(\varphi_{\nu}) - \Gamma^{\omega}_{\lambda\nu}\varphi_{\omega} - \varphi_{\lambda}\varphi_{\nu}) \\ &\quad - \delta^{\kappa}_{\lambda}(e_{\mu}(\varphi_{\nu}) - \Gamma^{\omega}_{\mu\nu}\varphi_{\omega} - \varphi_{\mu}\varphi_{\nu}) + \delta^{\kappa}_{\nu}(e_{\lambda}(\varphi_{\mu}) - e_{\mu}(\varphi_{\lambda})) \end{aligned} \tag{3.2.5}$$

Definition 3.2.2 Let $\{M, Q\}$ be a Sub-riemannian space with bundle metrics $g, \overset{*}{g}$, and denote by Sub-Riemannian connections $\Gamma, \overset{*}{\Gamma}$ determined by $g, \overset{*}{g}$, respectively. If there exists a horizontal vector field φ_{b} such that (3.2.4) is tenable, then we say that g is projective correspondence to $\overset{*}{g}$.

Let $\varphi_{\lambda\nu} = \nabla_{e_\lambda}\varphi_\nu - \varphi_\lambda\varphi_\nu$, by using (2.7), then

$$\Gamma^{\epsilon}_{\lambda\epsilon} = \frac{1}{2} g^{\epsilon\kappa} e_\lambda(g_{\epsilon\kappa}) = \frac{1}{2} e_\lambda(\log g),$$

where $g = \det(g_{\lambda\mu})$

By virtue of (3.2.4), there holds

$$\frac{1}{2} e_\lambda(\log \overset{*}{g}) = \frac{1}{2} e_\lambda(\log g) + (k+1)\varphi_\lambda.$$

Thus

$$\varphi_\lambda = \frac{1}{2(k+1)} e_\lambda(\log \overset{*}{g} - \log g)$$

are the gradient vector fields. So we have

$$\nabla_{e_\lambda}\varphi_\nu = \nabla_{e_\nu}\varphi_\lambda, \quad \varphi_{\lambda\nu} = \varphi_{\nu\lambda} \tag{3.2.6}$$

By using (3.2.5) and (3.2.6), then

$$\overset{*}{K}{}^{\kappa}{}_{\nu\lambda\mu} = K^{\kappa}{}_{\nu\lambda\mu} + \delta^{\kappa}_{\mu}\varphi_{\lambda\nu} - \delta^{\kappa}_{\lambda}\varphi_{\mu\nu} \tag{3.2.7}$$

Considering the retraction with indexes κ, λ on (3.2.7), then we get

$$\overset{*}{K}{}^{\epsilon}{}_{\nu\epsilon\mu} = K^{\epsilon}{}_{\nu\epsilon\mu} - (k-1)\varphi_{\mu\nu}.$$

Thus

$$\varphi_{\mu\nu} = -\frac{1}{k-1}(\overset{*}{K}{}^{\epsilon}{}_{\nu\epsilon\mu} - K^{\epsilon}{}_{\nu\epsilon\mu}).$$

Substituting this equality into (3.2.7), we get

$$\overset{*}{W}{}^{\kappa}{}_{\nu\lambda\mu} = W^{\kappa}{}_{\nu\lambda\mu} \tag{3.2.8}$$

where $\hat{W} = (W^{\kappa}{}_{\nu\lambda\mu})$ is defined as

$$W^{\kappa}{}_{\nu\lambda\mu} = K^{\kappa}{}_{\nu\lambda\mu} + \frac{1}{k-1}(\delta^{\kappa}_{\mu} K^{\epsilon}{}_{\nu\epsilon\lambda} - \delta^{\kappa}_{\lambda} K^{\epsilon}{}_{\nu\epsilon\mu}) \tag{3.2.9}$$

We call formula (3.2.9) the Weyl sub-projective curvature tensor of Sub-Riemannian spaces. According to (3.2.8), then

Theorem 3.2.2 The Weyl sub-projection curvature tensors are invariants under the projective transformations.

From Theorem 3.2.2, it is not hard to derive that Proposition 3.2.1 below is tenable.

Proposition 3.2.1 For the Weyl sub-projective curvature tensors $W^{\kappa}{}_{\nu\lambda\mu}$, there holds the following

1) $W^{\kappa}{}_{\nu\lambda\mu} = -W^{\kappa}{}_{\nu\mu\lambda}$;

2) $W^{\kappa}{}_{\nu\lambda\mu} + W^{\kappa}{}_{\lambda\mu\nu} + W^{\kappa}{}_{\mu\nu\lambda} = \delta^{\kappa}_{\mu} K^{\omega}{}_{\omega\nu\lambda} + \delta^{\kappa}_{\lambda} K^{\omega}{}_{\omega\mu\nu} + \delta^{\kappa}_{\nu} K^{\omega}{}_{\omega\lambda\mu}$;

3) $W^{\omega}{}_{\omega\lambda\mu} = \frac{k-2}{k-1} K^{\omega}{}_{\omega\mu\lambda}$.

On the other hand, by virtue of (3.2.9), one can prove easily that Theorem 3.2.3 below is tenable.

Theorem 3.2.3 The Weyl sub-projective curvature tensor of Sub-Riemannian space $\{M, Q, g\}$ with dimension 2 is identically vanishing.

Definition 3.2.3 Assume that ϕ_t are the projective transformations for all t, then the transformative group $\{\phi_t\}$ is the projective transformative group.

By a similar argument just as Section 3.1 and the statements [20], we have

Theorem 3.2.4 The transform group $\{\phi_t\}$ is the projective transformative group if and only if there exist vector field φ_λ such that there holds

$$\mathcal{L}_X \Gamma^{\lambda}_{\mu\nu} = \varphi_\mu \delta^{\lambda}_{\nu} + \varphi_\nu \delta^{\lambda}_{\mu}$$

where X is the vector field induced by $\{\phi_t\}$.

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