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WEYL TYPE THEOREMS FOR A CERTAIN CLASS OF OPERATORS

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ABSTRACT. Let A be a bounded linear operator acting on infinite dimensional separable Hilbert space H. Let $H_0(A)$ denote the quasi-nilpotent part

$$H_0(A) = \{x \in H : \lim_{n\to\infty} ||A^n x||^{\frac{1}{n}} = 0\}$$

of an operator A, and let $H(q)$ denote the class of $A \in B(H)$ for which $H_0(A - \lambda I) = \ker(A - \lambda I)^q$ for all complex numbers λ and some integer $q \geq 1$. In this paper we prove that if A is an algebraically class $H(q)$ operator, then generalized Weyl's theorem holds for A. We also show that if A is an algebraically class $H(q)$ operator, then $f(A)$ satisfies genralized Weyl's theorem for every analytic function f in an open neighborhood of $\sigma(A)$. More generally we prove that generalized a-Weyl's theorem holds for A and $f(A)$, where A is algebraically class $H(q)$ operator. By this we generalize some recent results in the literature.

1. INTRODUCTION

Let $B(H)$ and $K(H)$ denote, respectively, the algebra of bounded linear operators and the ideal of compact operators acting on infinite dimensional separable Hilbert space H. If $A \in B(H)$ we shall write $N(A)$ and $R(T)$ for the null space and the range of A, respectively. Also, let $\alpha(A) := dimN(A)$, $\beta(A) := dimN(A^*)$, and let $\sigma(A)$, $\sigma_a(A)$ and $\pi_0(A)$ denote the spectrum, approximate point spectrum and point spectrum of A, respectively.

An operator $A \in B(H)$ is called Fredholm if it has closed range, finite dimensional null space, and its range has finite co-dimension. The index of a Fredholm operator is given by

$$I(A) := \alpha(A) - \beta(A).$$

A Fredholm operator A is called Weyl if it is of index zero, and Browder if its ascent and descent are finite, equivalently ([23], Theorem 7.9.3) if A is Fredholm and $A - \lambda$ is invertible for sufficiently small $|\lambda| > 0$, $\lambda \in \mathbb{C}$. The essential spectrum $\sigma_e(A)$, the Weyl spectrum $\sigma_w(A)$ and the Browder spectrum $\sigma_b(A)$ of A are defined by [22, 23]

$$\sigma_e(A) = \{\lambda \in \mathbb{C} :\ A - \lambda \text{ is not Fredholm}\},$$
$$\sigma_w(A) = \{\lambda \in \mathbb{C} :\ A - \lambda \text{ is not Weyl}\},$$

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$$\sigma_b(A) = \{\lambda \in \mathbb{C} : A - \lambda \text{ is not Browder}\},$$

respectively. Evidently

$$\sigma_e(A) \subseteq \sigma_w(A) \subseteq \sigma_b(A) = \sigma_e(A) \cup acc\sigma(A),$$

where we write $accK$ for the accumulation points of $K \subseteq \mathbb{C}$. If we write $isoK = K \setminus accK$, then we let

$$\pi_{00}(A) := \{\lambda \in iso\sigma A : 0 < \alpha(A - \lambda) < \infty\},$$
$$p_{00}(A) := \sigma(A) \setminus \sigma_b(A).$$

Definition 1.1. *We say that Weyl's theorem holds for A if*

$$\sigma(A) \setminus \sigma_w(A) = \pi_{00}(A).$$

Definition 1.2. *We say that the generalized Weyl's theorem holds for A provided*

$$\sigma(A) \setminus \sigma_{Bw}(A) = E(A),$$

where $E(A)$ and $\sigma_{Bw}(A)$ denote the isolated point of the spectrum which are eigenvalues (no restriction on multiplicity) and the set of all complex numbers λ for which $A - \lambda I$ is not B-Weyl, respectively.

Let X be a Banach space. An operator $A \in B(X)$ is called B-Fredholm by Berkani [3] if there exists $n \in \mathbb{N}$ for which A^n is closed and the restriction of A on it

$$A_n : A^n(X) \to A^n(X)$$

is Fredholm in the usual sense, and B-Weyl if in addition A_n has index zero. Note that, if the generalized Weyl's theorem holds for A, then so does Weyl's theorem [3]. We say that Browder's theorem holds for A if

$$\sigma(A) \setminus \sigma_w(A) = p_{00}(A).$$

For a $A \in B(H)$, let $H_0(A)$ denote the quasi-nilpotent part

$$H_0(A) = \{x \in H : \lim_{n \to \infty} ||A^n x||^{\frac{1}{n}} = 0\}$$

of the operator A, and let $H(q)$ denote the class of $A \in B(H)$ for which $H_0(A - \lambda I) = \ker(A - \lambda I)^q$ for all complex numbers λ and some integer $q \geq 1$. The class $H(q)$ is large, it contains, amongst others, the classes consisting of generalized scalar, hyponormal, p-hyponormal $(0 < p < 1)$ and M-hyponormal operators on a Hilbert space (see [2, 14, 29]. An operator A is called class $H(q)$ if it belongs to the class $H(q)$. An operator A is called algebraically class $H(q)$, simply alg-$H(q)$, if $p(A)$ is class $H(q)$ for some non-constant polynomial p.

In [44], H. Weyl proved that weyl's theorem holds for hermitian operators. Weyl's theorem has been extended from hermitian operators to hyponormal and Toeplitz operators [11], and to several classes of operators including semi-normal operators ([7, 8]). Recently W.Y.Lee [31] showed that Weyl's theorem holds for algebraically hyponormal operators. R.Curto and Y.M.Han [13] have extended Lee's results to algebraically paranormal operator $A \in B(H)$, where H is a separable Hilbert space.

In [17] the authors showed that Weyl's theorem holds for algebraically p-hyponormal operators. In [33] the authors showed that Weyl's theorem holds for algebraically (p,k)-quasihyponormal or paranormal operator $A \in B(H)$, where H is a general Hilbert space. Berkani [3] showed that if A is a hyponormal operator, then A satisfies generalized Weyl's theorem $\sigma_{Bw}(A) = \sigma(A) \setminus E(A)$, and the B-Weyl spectrum $\sigma_{Bw}(A)$ of A satisfies the spectral mapping theorem.

B.Duggal *et al* [18] showed that Weyl's theorem holds for $f(A)$, where f is an analytic function on an open neighborhhod of $\sigma(A)$ in the case where A is an algebraically class $H(q)$ operator. In this paper we prove that if A is algebraically class $H(q)$ operator, then generalized Weyl's theorem holds for A. We also show that if A is algebraically class $H(q)$ operator, then genralized Weyl's theorem holds for $f(A)$, where f is an analytic function in an open neighborhood of $\sigma(A)$. More generally we prove that Generalized a-Weyl's theorem holds for A and $f(A)$, where A is algebraically class $H(q)$ operator. Other related results are also given.

2. Main results

Lemma 2.1. [18] *Let A be a class $H(q)$ operator and $\lambda \in \mathbb{C}$. If $\sigma(A) = \{\lambda\}$, then $A = \lambda$.*

Lemma 2.2. *Let A be a quasinilpotent algebraically class $H(q)$ operator. Then A is nilpotent.*

Proof. Assume that $p(A)$ is a class $H(q)$ operator for some nonconstant polynomial p. Since $\sigma(p(A)) = p(\sigma(A))$, the operator $p(A) - p(0)$ is quasinilpotent. Thus Lemma 2.1 would imply that

$$cA^m(A-\lambda_1)...(A-\lambda_n) \equiv p(A) - p(0) = 0,$$

where $m \geq 1$. Since $A - \lambda_i$ is invertible for every $\lambda \neq 0$, we must have $A^m = 0$. $\square$

In [18] the authors proved that if A is an algebraically class $H(q)$ operator, then A is isoloid by using some properties of a Kato type operator. In the following lemma we will prove the same result by using a simple techniques as Curto [13] has used for algebraically paranormal operators.

Lemma 2.3. *Let A be an algebraically class $H(q)$ operator. Then A is isoloid.*

Proof. Let $\lambda \in iso\sigma(A)$ and let

$$P := \frac{1}{2\pi i}\int_{\partial D}(\mu - T)^{-1}d\mu$$

be the associated Riesz idempotent, where D is a closed disk centered at λ which contains no other points of $\sigma(A)$. We can then represent A as the direct sum

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \text{ where } \sigma(\mathrm{A}_1) = \{\lambda\} \text{ and } \sigma(\mathrm{A}_2) = \sigma(\mathrm{A}) \setminus \{\lambda\}.$$

Since A is a class $H(q)$ operator, $p(A)$ is a class $H(q)$ operator for some nonconstant polynomial p. Since $\sigma(A_1) = \lambda$, we must have

$$\sigma(p(A_1)) = p(\sigma(A_1)) = \{p(\lambda)\}.$$

Therefore $p(A_1) - p(\lambda)$ is quasinilpotent. Since $p(A_1)$ is a class $H(q)$ operator, it follows from lemma 2.1 that $p(A_1) - p(\lambda) = 0$. Put $q(z) := p(z) - p(\lambda)$. Then $q(A_1) = 0$, so A_1 is algebraically class $H(q)$ operator . Since $A_1 - \lambda$ is quasinilpotent and algebraically class $H(q)$ operator, it follows from Lemma 2.2 that $A_1 - \lambda$ is nilpotent. Therefore $\lambda \in \pi_0(A_1)$, and hence $\lambda \in \pi_0(A)$. This shows that A is isoloid. □

Recall that Duggal *et al* [18] have extended Weyl's theorem to algebraically class $H(q)$ operators. It is known [3] that Weyl's theorem don't imply Generalized Weyl's theorem. In the following theorem we will extend generalized Weyl's theorem to algebraically class $H(q)$ operators. We start by the following lemma

Lemma 2.4. [18] *Let $A \in B(H)$ be algebraically class $H(q)$ operator. Then A has SVEP, i.e., the single valued extension property.*

It is known that SVEP is stable under the functional calculus, i.e., if $A \in B(H)$ has SVEP, then so does $f(A)$ for each $f(A)$ analytic on an open neighborhood of $\sigma(A)$. The following lemma is immediate.

Lemma 2.5. *Let $A \in B(H)$ be algebraically class $H(q)$ operator. Then $f(A)$ has SVEP for each analytic function f on a neighborhood of $\sigma(A)$.*

Theorem 2.1. *Let A be an algebraically class $H(q)$ operator. Then generalized Weyl's theorem holds for A.*

Proof. Assume that $\lambda \in \sigma(A) \setminus \sigma_{Bw}(A)$. Then $A - \lambda I$ is B-Weyl and not invertible. We claim that $\lambda \in \partial\sigma(A)$. Assume to the contrary that λ is an interior point of $\sigma(A)$. Then there exists a neigborhood U of λ such that $dim(A - \mu) > 0$ for all $\mu \in U$. It follows from ([19], Theorem 10) that A does not have SVEP. On the other hand, since $p(A)$ is a class $H(q)$ operator for nonconstant polynomial p, it follows from Lemma 2.4 that $p(A)$ has SVEP. Hence by ([29], Theorem 3.3.9), A has SVEP, a contradiction. Therefore $\lambda \in \partial\sigma(A)$. Conversely, assume that $\lambda \in E(A)$, then λ is isolated in $\sigma(A)$. From ([27], Theorem 7.1) we have $X = M \oplus N$, where M, N are closed subspaces of X, $U = (A - \lambda I)|_M$ is an invertible operator and $V = (A - \lambda I)|_N$ is a quasinilpotent operator. Since A is algebraically class $H(q)$ operator, V is also algebraically class $H(q)$ operator, from Lemma 2.2 V is nilpotent. Therefore $A - \lambda I$ is Drazin invertible ([39], Proposition 6) and ([28], Corollary 2.2). By ([5], Lemma 4.1) $A - \lambda I$ is a B-Fredholm operator of index 0. Thus $\lambda \in \sigma(A) \setminus \sigma_{Bw}(A)$. □

As consequences of the previous theorem, we obtain

Corollary 2.1. [18] *Let A be an algebraically class $H(q)$ operator. Then Weyl's theorem holds for A.*

Corollary 2.2. [3] *Let A be an algebraically hyponormal operator. Then generalized Weyl's theorem holds for A.*

Corollary 2.3. [45] *Let A be a p-hyponormal operator. Then generalized Weyl's theorem holds for A.*

Corollary 2.4. [45] *Let A be M-hyponormal. Then generalized Weyl's theorem holds for A.*

Corollary 2.5. [17] *Let A be an algebraically p-hyponormal operator. Then Weyl's theorem holds for A.*

Corollary 2.6. *Let A be an algebraically M-hyponormal operator. Then generalized Weyl's theorem holds for A.*

Corollary 2.7. *Let A be an algebraically totally paranormal operator. Then generalized Weyl's theorem holds for A.*

Theorem 2.2. *Let A be an algebraically class $H(q)$ operator. Then generalized Weyl's theorem holds for $f(A)$ for every analytic function f in a neighborhood of $\sigma(A)$.*

Proof. Since A is isoloid by Lemma 2.3, has the SVEP and satifies generalized Weyl's theorem, it follows from ([46], Theorem 2.2) that $f(A)$ satisfies generalized Weyl's theorem. □

As a consequence of the previous theorem, we obtain

Corollary 2.8. [18] *Let A be an algebraically class $H(q)$ operator. Then Weyl's theorem holds for $f(A)$ for every analytic function f in a neighborhood of $\sigma(A)$.*

Corollary 2.9. [5] *Let A be an algebraically hyponormal operator. Then generalized Weyl's theorem holds for $f(A)$ for every analytic function f in a neighborhood of $\sigma(A)$.*

Corollary 2.10. [45] *Let A be an algebraically p-hyponormal operator. Then generalized Weyl's theorem holds for $f(A)$ for every analytic function f in a neighborhood of $\sigma(A)$.*

Corollary 2.11. [45] *Let A be an algebraically M-hyponorma operator. Then generalized Weyl's theorem holds for $f(A)$ for every analytic function f in a neighborhood of $\sigma(A)$.*

Corollary 2.12. [46] *Let A be an algebraically paranormal operator. Then generalized Weyl's theorem holds for $f(A)$ for every analytic function f in a neighborhood of $\sigma(A)$.*

The essential approximate point spectrum $\sigma_{ea}(A)$ is defined by

$$\sigma_{ea}(A) = \cap\{\sigma_a(A+K) : K \text{ is a compact operator }\}$$

where $\sigma_a(T)$ is the approximate point spectrum of T. By definition

$$\sigma_{ab}(A) = \cap\{\sigma_a(A+K) : TK = KT \text{ and } K \in K(H)\},$$

We consider the set

$$\Phi_+^-(H) = \{A \in B(H) : T \text{ is left semi-Fredholm and } \operatorname{ind} A \leq 0\}.$$

V. Rakočević [35] proved that

$$\sigma_{ea}(A) = \{\lambda \in \mathbb{C} : A - \lambda \notin \Phi_+^-(H)\}$$

and the inclusion $\sigma_{ea}(f(A)) \subset f(\sigma_{ea}(A))$ holds for all functions $f(z)$ which are analytic on some open neighborhood of $\sigma(T)$ with no restriction on A. The next theorem shows the spectral mapping theorem on the essential approximate point spectrum of algebraically class $H(q)$ operators.

Lemma 2.6. *Let $A \in B(H)$ and $\lambda \in \mathbb{C}$. If $A - \lambda$ is semi-Fredholm and it has finite ascent, then* ind $(A - \lambda) \leq 0$.

Proof. If $A - \lambda$ has finite descent, then ind $(A - \lambda) = 0$ by Theorem V 6.2 of [41]. If $A - \lambda$ does not have finite descent, then

$$n \operatorname{ind}(A-\lambda) = \dim N(A-\lambda)^n - \dim R((T-\lambda)^n)^\perp \to -\infty.$$

Hence ind $(A - \lambda) < 0$.

□

Corollary 2.13. *Let $A \in B(H)$ be algebraically class $H(q)$ operator. If $A - \lambda$ is semi-Fredholm for some $\lambda \in \mathbb{C}$, then* ind $(A - \lambda) \leq 0$.

Theorem 2.3. *Let $A \in B(H)$ be algebraically class $H(q)$ operator. Then*

$$\sigma_{ea}(f(A)) = f(\sigma_{ea}(A))$$

for every functions $f(z)$ which is analytic on some open neighborhood G of $\sigma(A)$.

Proof. It suffices to show that $f(\sigma_{ea}(A)) \subseteq \sigma_{ea}(f(A))$. We may assume that f is nonconstant. Let $\lambda \notin \sigma_{ea}(f(A))$ and

$$f(z) - \lambda = g(z)\Pi_{j=1}^n (z - \lambda_j)$$

where $\lambda_j \in G$ and $g(z) \neq 0$ for all $z \in G$. Then

$$f(A) - \lambda = g(A)\Pi_{j=1}^n (A - \lambda_j).$$

Since $\lambda \notin \sigma_{ea}(f(A))$ and all operators on the right side of above equality commute, each $(A - \lambda_j)$ is left semi-Fredholm and ind $(A - \lambda_j) \leq 0$ by the previous corollary. Thus $\lambda_j \notin \sigma_{ea}(A)$ and $\lambda \notin f(\sigma_{ea}(A))$. □

We say that a-Browder's theorem holds for A if $\sigma_{ea}(A) = \sigma_{ab}(A)$. It is well known

$$a\text{-Browder's theorem} \Rightarrow \text{Browder's theorem}.$$

In general [6] Weyl's theorem does not hold for operators having SVEP only, but a-Browder's theorem holds for operator having SVEP only as we will show in Theorem 2.4.

Theorem 2.4. *Assume $A \in B(H)$ has SVEP. Then a-Browder's theorem holds for A.*

Proof. It is well known that $\sigma_{ea}(A) \subseteq \sigma_{ab}(A)$. Conversely, assume that $\lambda \in \sigma_a(A) \setminus \sigma_{ea}(A)$. Then $A - \lambda \in \Phi_+^-(H)$ and $A - \lambda$ is not bounded below. Since A has SEVP and $A - \lambda \in \Phi_+^-(H)$, [2, Theorem 2.6] implies that $A - \lambda$ has finite acsent. Hence [36, Theorem 2.1] would imply that $\lambda \in \sigma_a(A) \setminus \sigma_{ab}(A)$. This implies that a-Browder theorem holds for A. □

Corollary 2.14. *Let $A \in B(H)$ be algebraically class $H(q)$ operator. Then a-Browder's theorem holds for $f(A)$ for every analytic function on a neighborhood of $\sigma(A)$.*

Proof. By applying Theorem 2.3 we get

$$\sigma_{ab}(f(A)) = f(\sigma_{ab}(A)) = f(\sigma_{ea}(A)) = \sigma_{ea}(f(A)).$$

Therefore a-Browder's theorem holds for $f(A)$. □

Let SBF_+ be the class of all upper semi-Fredholm operators, SBF_+^- the class of $A \in SBF_+$ such that $\text{ind}(A) \leq 0$, and let

$$\sigma_{SBF_+^-}(A) = \{\lambda \in \mathbb{C} : A - \lambda I \text{ is not in } SBF_+^-\}$$

be called the semi-B-essential approximate point spectrum.

Definition 2.1. *We say that A obeys generalized a-Weyl's theorem if*

$$\sigma_{SBF_+^-}(A) = \sigma_{ap}(A) \setminus E^a(A),$$

where $E^a(A)$ is the set of all eigenvalues of A which are isolated in $\sigma_{ap}(A)$.

Definition 2.2. *An operator $A \in B(H)$ is said to be obeys a-weyl's theorem if*

$$\sigma_{ap}(A) \setminus \sigma_{SF_+^-}(A) = E_0^a(A),$$

where E_0^a is the set of all isolated points of $\sigma_{ap}(A)$ which are eigenvalues of finite multiplicity and $\sigma_{SF_+^-}(A)$ is the set of all $\lambda \in \mathbb{C}$ for which $A - \lambda I$ is not an upper semi-Fredholm operators with ind$(A - \lambda I) \leq 0$.

Recall [6] that

Generalized a-Weyl's theorem $\Rightarrow$ Generalized Weyl's theorem $\Rightarrow$ Weyl's theorem

$\Rightarrow$ Browder's theorem.

$$\text{Generalized } a\text{-Weyl's theorem} \Rightarrow a\text{-Weyl's theorem} \Rightarrow \text{Weyl's theorem} \Rightarrow \text{Browder's theorem}.$$

$$\text{Generalized } a\text{-Weyl's theorem} \Rightarrow a\text{-Weyl's theorem} \Rightarrow a\text{-Browder's theorem} \Rightarrow \text{Browder's theorem}.$$

The converse of the previous implications are false (see [6, Examples 3.12]).

Theorem 2.5. *Let A^* be algebraically class $H(q)$ operator. Then generalized a-Weyl's theorem holds for A.*

Proof. We have to prove that $\sigma_{ap}(A) \setminus \sigma_{SBF_+^-}(A) = E^a(A)$. For this, assume that $\lambda \in \sigma_{ap}(A) \setminus \sigma_{SBF_+^-}(A)$. Then $A - \lambda I$ is an upper semi-B- Fredholm operator and $\text{ind}(A - \lambda I) \leq 0$. Hence for n large enough, $A - (\lambda + \frac{1}{n})I$ is an upper semi Fredholm operator and $\text{ind}(A - (\lambda + \frac{1}{n})I) = ind(A - \lambda I)$ [6]. Therefore $\text{ind}(A - (\lambda + \frac{1}{n})I) \leq 0$. Since A^* has SVEP, [4] implies that $ind(A - (\lambda + \frac{1}{n})I) \geq 0$. Thus $\text{ind}(A - (\lambda + \frac{1}{n})I) = 0$. It follows that $\text{ind}(A - \lambda I) = 0$. This implies that $A - \lambda I$ is a B-Fredholm operator of index zero. Since A^* has SVEP, we have $\sigma(A) = \sigma_{ap}(A)$ and we have $\lambda \in \sigma(A) \setminus \sigma_{BW}(A)$. Then it follows from Theorem 2.1 that $\lambda \in E(A)$. Hence $\lambda \in E^a(A)$. Conversely, let $\lambda \in E^a(A)$. Then λ is an isolated point of $\sigma_{ap}(A) = \sigma_a(A)$. Therefore $\overline{\lambda}$ is an isolated point of $\sigma(A^*)$. Let P be the spectral projection

$$P = \int_{\partial B_0} (\lambda_0 I - A^*)^{-1} d\lambda_0,$$

where B_0 is an open disk centred at $\overline{\lambda}$ which contains no other points of $\sigma(A^*)$. Then A^* can be represented as the direct sum

$$A^* = A_1 \oplus A_2, \text{where } \sigma(A_1) = \{\overline{\lambda}\} \text{ and } \sigma(A_2) = \sigma(A^* \setminus \{\overline{\lambda}\}).$$

Then $\overline{\lambda}I - A_2$ is invertible. We have to consider two cases.

Case where $\lambda = 0$. Assume that $\lambda = 0$. Then $\sigma(A_1) = \{0\}$. Since A_1 is algebraically class $H(q)$ operator, it follows that $A_1 = 0$ by Lemma 2.1. Therefore $\overline{\lambda}I - A^* = 0 \oplus \overline{\lambda} - A_2$.

Case where $\lambda \neq 0$. Since A_1 is invertible algebraically class $H(q)$ operator, it follows that A_1^{-1} is algebraically class $H(q)$ operator. Then $||A_1|| = |\lambda|$ and $||A_1^{-1}|| = \frac{1}{\lambda}$. Therefore for any $x \in R(P)$, we have

$$||x|| \leq ||A_1^{-1}||||A_1 x|| = \frac{1}{|\lambda|}||A_1 x|| \leq \frac{1}{|\lambda|}|\lambda||x| = ||x||.$$

Hence $\frac{1}{\lambda}A_1$ is unitary. Therefore A_1 is normal and $\overline{\lambda}I - A_1$ is also normal. Since $\overline{\lambda} - A_1$ is quasinilpotent and the only normal quasinilpotent operator is zero, it follows that $\overline{\lambda} - A^* = 0 \oplus \overline{\lambda}I - A_2$. Now since $\overline{\lambda}I - A_2$ is invertible, it is known that $\overline{\lambda}I - A^*$ has finite acsent and descent. Therefore $\lambda I - A$ has finite ascent and descent. This implies that $\lambda \in \sigma_a(A) \setminus \sigma_{SBF_+^-}(A)$. Which completes the proof. □

Let

$$A_2(H) = \left\{A \in B(H) : ind(A - \lambda I)ind(A - \mu I) \geq 0, \text{ for all } \lambda, \mu \in \mathbb{C} \setminus \sigma_{SF_+}(A)\right\}.$$

An operator $A \in B(H)$ is said to be approximate-isoloid (abbrev. a-isoloid) if every isolated point of $\sigma_a(A)$ is an eigenvalue of A and isoloid if every isolated point of $\sigma(A)$ is an eigenvalue of A. Clearly, if A is a-isoloid then it is isoloid. However, the converse is not true.

Lemma 2.7. *Let A be algebraically class $H(q)$ operator. Then A is a-isoloid.*

Proof. Since A^* is algebraically class $H(q)$ operator, Theorem 2.5 would imply that a-Weyl's theorem holds for A and $\sigma(A) = \sigma_a(A)$. If we assume that $\lambda \in iso\sigma_a(A) = iso\sigma(A)$, then $\overline{\lambda} \in iso\sigma(A^*)$. Since A^* is algebraically class $H(q)$ operator, we have A^* is isoloid by Lemma 2.3. Then $N(\overline{\lambda}I - A^*) \neq \{0\}$. Since $N(\overline{\lambda} - A^*) \subseteq N(\lambda I - A)$ [18], we have $N(\lambda I - A) \neq 0$. Thus A is a-isoloid. □

Lemma 2.8. *Let A be algebraically class $H(q)$ operator. Then $A \in A_2(H)$.*

Proof. Let $\lambda \in \mathbb{C} \setminus \sigma_{SF_+}(A)$. Since $N(\overline{\lambda} - A^*) \subseteq N(\lambda I - A)$, we have $\text{ind}(A - \lambda I) \geq 0$. Which implies that $A \in A_2(H)$. □

Theorem 2.6. *Let A be algebraically class $H(q)$ operator. Then $f(A)$ obeys generalized a-Weyl's theorem for every analytic function f on a neigborhood of $\sigma(A)$.*

Proof. Since A is a-isoloid, $A \in A_2(H)$ and A obey's generalized a-Weyl's theorem, [10, Theorem 2.2] implies that $f(A)$ obeys generalized a-Weyl's theorem. □

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