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Groups, Lie algebras and Gauss decompositions for one dimensional tilings

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Abstract

For one dimensional tilings, we will define associated groups and Lie algebras. Then, we will prove that the groups have Gauss decompositions as well as that the Lie algebras also have additive Gauss decompositions.

Key words: tiling, group, Lie algebra, Gauss decomposition

0. Introduction. In this paper, we will give a certain Lie theoretical approach to tilings, and establish some basic decompositions. What is a tiling ? A tiling is roughly a decomposition or a filling of a given space using suitable pieces. Recently it was found that lots of mathematical areas are deeply related to tilings and associated topics. However, not so many algebraic approaches have been given. This article shows that it is possible to study tilings using Lie theory. Mathematically it is very nice to have certain pure algebraic objects (invariants) arising from tilings.

Tiling Structure	$\longrightarrow$ $\longleftarrow$	Mathematics Pure Algebra

Here we will discuss a tiling of the real line $\mathbb{R}$. Then, we will construct tiling monoids (Section 2), tiling bialgebras (Section 3), tiling Lie algebras (Section 4) and tiling groups (Section 5). Then we will establish Gauss decompositions for our tiling groups (Sections 6,7,8). Also we will reach certain additive Gauss decompositions for tiling Lie algebras (Section 9). Those decompositions are fundamental in algebra, which can be very helpful to study many kinds of invariants for mathematical objects.

Next, we will make a rough review of Gauss decompositions. Let us consider the following system of linear equations:

$$\begin{cases} ax + by = s \\ cx + dy = t \end{cases}$$

Generically we can solve it using the so-called Gauss eliminaton. This is corresponding to the following group theoretical decomposition over the field $\mathbb{C}$ of complex numbers (cf. [12]):

$$GL_2(\mathbb{C}) = \begin{pmatrix} 1 & \mathbb{C} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \mathbb{C} & 1 \end{pmatrix} \begin{pmatrix} \mathbb{C}^\times & 0 \\ 0 & \mathbb{C}^\times \end{pmatrix} \begin{pmatrix} 1 & \mathbb{C} \\ 0 & 1 \end{pmatrix}$$

Such a decomposition is sometimes called a Gauss decomposition (in the sense of Group Theory). There is also a Lie algebra version of this kind of decompositions, namely additive Gauss decompositions. For example, if we take a Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ with a standard basis e, h, f satisfying the relations $[h, e] = 2e$, $[h, f] = -2f$, $[e, f] = h$, then the universal enveloping algebra $U(\mathfrak{sl}_2(\mathbb{C}))$ of $\mathfrak{sl}_2(\mathbb{C})$ has the following description (cf. [4]):

$$U(\mathfrak{sl}_2(\mathbb{C})) = \sum_{i,j,k\geq 0} \mathbb{C}e^i f^j e^k.$$

The usual PBW theorem says $U(\mathfrak{sl}_2(\mathbb{C})) = \sum_{i,j,k\geq 0} \mathbb{C}e^i h^j f^k$, but we can erase "$h$" in this new decomposition, which gives some nice application to representation theory. More precisely, to obtain "locally finiteness" of an infinite dimensional module, we have to show that h is a "diagonal" operator, which seems to be rather complicated. However, this new decomposition says directly and visually that it is enough to confirm for e and f to be "locally nilpotent" operators.

1. Tilings. Let $\mathbb{R}$ be the real line. A tile in $\mathbb{R}$ is a connected closed bounded subset of $\mathbb{R}$, namely a closed interval $[a, b]$ whose interior is nonempty. A tiling $\mathcal{T}$ of $\mathbb{R}$ is an infinite set of tiles which covers $\mathbb{R}$ overlapping, at most, at their boundaries. In this note, we identify a tiling of $\mathbb{R}$ with a bi-infinite sequence of letters, equivalently saying, a bi-infinite word of letters. Let $W = W(\mathcal{T})$ be the set of all finite subwords in $\mathcal{T}$. If $w = X_1 \cdots X_r \in W$, then $l(w) = r$ is called the length of w. Let $W_r = W_r(\mathcal{T})$ be the set of all finite subwords with length r. Put $\Omega = \Omega(\mathcal{T}) = W_1$, the set of all letters appearing in $\mathcal{T}$. For convenience, we assume that Ω is finite.

2. Tiling monoids. For $w = X_1 \cdots X_r \in W$, we choose two positions (i, j) with $1 \leq i, j \leq r$ and attach the labels 1 and 2 at X_i and X_j as $\overset{1}{X_i}$ and $\overset{2}{X_j}$ respectively. We note that each of $i < j$, $i = j$ and $i > j$ is allowed. If $i = j$, then we denote by $\overset{12}{X_i}$ to show that X_i has two labels 1 and 2 simultaneously. We call

$$X_1 \cdots \overset{1}{X_i} \cdots \overset{2}{X_j} \cdots X_r$$

a doubly pointed word obtained from $w \in W$. Then $D = D(\mathcal{T})$ denotes the set of all doubly pointed words obtained from W. Let $M = M(\mathcal{T}) = D \cup \{\mathbf{z}, \boldsymbol{\varepsilon}\}$, where $\mathbf{z}$ and $\boldsymbol{\varepsilon}$ are just independent abstract symbols. Now we will introduce a binary operation on M. Let

$$\begin{aligned} \mathbf{x} &= X_1 \cdots \overset{1}{X_i} \cdots \overset{2}{X_j} \cdots X_r \\ \mathbf{y} &= Y_1 \cdots \overset{1}{Y_k} \cdots \overset{2}{Y_\ell} \cdots Y_s \end{aligned}$$

be two elements of D. Put $a = \min\{j, k\}$, $b = \min\{r - j, s - k\}$, $m = \max\{j, k\} - \min\{j, k\}$, $n = \max\{r - j, s - k\} - \min\{r - j, s - k\}$, and set

$$q = a + b = \frac{(r + s) - (m + n)}{2}.$$

If

$$(*)\quad \begin{cases} X_{j-a+1} & = & Y_{k-a+1} \ , \\ \vdots & \vdots & \vdots \\ X_j & = & Y_k \ , \\ \vdots & \vdots & \vdots \\ X_{j+b} & = & Y_{k+b} \ , \end{cases}$$

then we define a new word

$$Z_1 \cdots Z_m Z_{m+1} \cdots Z_{m+q} Z_{m+q+1} \cdots Z_{m+q+n} \ ,$$

where

$$\begin{cases} Z_p & (1 \le p \le m) & = \begin{cases} X_p & \text{if } j > k \\ Y_p & \text{if } j < k \end{cases} \\ Z_{m+p} & (1 \le p \le q) & = X_{j-a+p} \quad (= Y_{k-a+p}) \\ Z_{m+q+p} & (1 \le p \le n) & = \begin{cases} X_{j+b+p} & \text{if } r-j > s-k \\ Y_{k+b+p} & \text{if } r-j < s-k \ . \end{cases} \end{cases}$$

Put

$$i' = \begin{cases} i & \text{if } j \ge k \\ m+i & \text{if } j < k \ , \end{cases} \qquad j' = \begin{cases} m+\ell & \text{if } j > k \\ \ell & \text{if } j \le k \ , \end{cases} \qquad r' = m+q+n.$$

If $(*)$ holds and the new word $Z_1 \cdots Z_{r'}$ belongs to W, then we define

$$\mathbf{xy} = Z_1 \cdots \overset{1}{Z_{i'}} \cdots \overset{2}{Z_{j'}} \cdots Z_{r'} \in D,$$

otherwise we define $\mathbf{xy} = \mathbf{z}$. Also we define $\mathbf{mz} = \mathbf{zm} = \mathbf{z}$ as well as $\mathbf{m}\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}\mathbf{m} = \mathbf{m}$ for all $\mathbf{m} \in M$. Then, the set M becomes a monoid with the above operation. We call M the tiling monoid of a given tiling $\mathcal{T}$. In another sense, M can also be regarded as an inverse monoid with zero (cf. [9]). Then, some of our basic properties can be reduced to those of an inverse monoid (cf. [6],[9]).

It might be better for the readers to see several examples of our product here. If

$$\mathbf{x} = \overset{1}{X}Y\overset{2}{X} \ , \quad \mathbf{y} = \overset{2}{X}Y\overset{1}{X} \ , \quad \mathbf{v} = XY\overset{1}{X}\overset{2}{X}Y \ \in D \ , \quad \text{and if} \quad XYXXYX \in W \ ,$$

then we have

$$\begin{aligned} \mathbf{xy} &= \overset{12}{X}YX \ , & \mathbf{yx} &= XY\overset{12}{X} \ , \\ \mathbf{xv} &= \overset{1}{X}YX\overset{2}{X}Y \ , & \mathbf{vx} &= XY\overset{1}{X}XY\overset{2}{X} \ , \\ \mathbf{yv} &= \mathbf{z} \ , & \mathbf{vy} &= \mathbf{z} \ . \end{aligned}$$

Furthermore, we find $\mathbf{xyx} = \mathbf{x}$ and $\mathbf{yxy} = \mathbf{y}$ in this case.

3. Tiling bialgebras. Let $A = \mathbb{C}[M] = \oplus_{\mathbf{m}\in M} \mathbb{C}\mathbf{m}$ be the monoid algebra of M over $\mathbb{C}$. Then $\mathbb{C}\mathbf{z}$ is a two-sided ideal of A. We set $B = B(\mathcal{T}) = A/\mathbb{C}\mathbf{z}$. Then, B is sometimes called the tiling bialgebra (cf. [1],[10]), of $\mathcal{T}$. Here, we also consider B as a Lie algebra with the standard bracket $[x, y] = xy - yx$. We use the notation 1 for ε mod $\mathbb{C}\mathbf{z}$ and the same symbols $\mathbf{x} \in D$ for their images modulo $\mathbb{C}\mathbf{z}$ respectively. For a subset $V \subset W$, we define $E(V)$ to be the subset of D consisting of all doubly pointed words obtained from V with the pointed positions of type $(i, i+1)$ for all $i \geq 1$, and $F(V)$ the subset of D consisting of all doubly pointed words obtained from V with the pointed positions of type $(i+1, i)$ for all $i \geq 1$.

4. Tiling Lie algebras. Here we will make our Lie theoretical approach to tilings. For a given tiling $\mathcal{T}$, we want to make the following substitution:

$$\sigma : X \mapsto XX'X'' \quad (\forall\, X \in \Omega),$$

where the letters X' and X'' are totally new symbols. That is, the given bi-infinite sequence

$$\cdots XYZ \cdots$$

is changed into

$$\cdots XX'X''YY'Y''ZZ'Z'' \cdots$$

by σ, and a finite subword

$$w = X_1X_2 \cdots X_r \in W$$

is changed into

$$\sigma(w) = X_1X_1'X_1''X_2X_2'X_2'' \cdots X_rX_r'X_r''.$$

Hence, the substitution σ creates a new tiling $\mathcal{T}^*$ from $\mathcal{T}$. By the definition, $|\Omega(\mathcal{T}^*)| = 3 \times |\Omega(\mathcal{T})|$. That is, $\Omega(\mathcal{T}^*) = \{\ X, X', X'' \mid X \in \Omega(\mathcal{T})\ \}$ without any redundancy. Put $V^* = \sigma(W(\mathcal{T})) = \{\ \sigma(w) \mid w \in W(\mathcal{T})\ \} \subset W(\mathcal{T}^*)$.

Now we define L to be the Lie subalgebra of $B(\mathcal{T}^*)$ generated by e and f for all $e \in E$ and $f \in F$, where $E = E(V^*)$ and $F = F(V^*)$. We call L the tiling Lie algebra associated with an original tiling $\mathcal{T}$. Now we define three Lie subalgebras of L as follows:

$$\begin{aligned} L_+ &= \langle\ e \mid e \in E\ \rangle, \\ L_0 &= \langle\ h \mid h = [e, f],\ e \in E,\ f \in F\ \rangle, \\ L_- &= \langle\ f \mid f \in F\ \rangle. \end{aligned}$$

Then, we have $L = L_- \oplus L_0 \oplus L_+$ (triangular decomposition cf. [11]). If $e \in E$ is obtained from $v \in V^*$ with the pointed positions of type $(i, i+1)$, then we denote by $\hat{e}$ the element of F which is obtained from the same v with the opposite pointed positions of type $(i+1, i)$. If $f = \hat{e}$, then we also define $\hat{f} = e$. Note that $e\hat{e}e = e$ and $f\hat{f}f = f$ in $B(\mathcal{T}^*)$. We denote by $\mathcal{U}(\mathfrak{a})$ the universal enveloping algebra of a Lie algebra $\mathfrak{a}$. Then we can obtain the following result.

Proposition 1. Notation is as above. Then, we have $\mathcal{U}(L) = \mathcal{U}(L_\pm)\mathcal{U}(L_\mp)\mathcal{U}(L_\pm)$ (additive Gauss decomposition).

The proof of this proposition will be given later (cf. Section 9).

Examples. (1) Let $\mathcal{T}$ be a trivial tiling, that is,

$$\mathcal{T} \ : \ \cdots XXXXX \cdots .$$

This original tiling never produces $\mathfrak{sl}_2$. To obtain $\mathfrak{sl}_2$, our method says that we need a new tiling $\mathcal{T}^*$, that is,

$$\mathcal{T}^* \ : \ \cdots XX'X''XX'X''XX'X''XX'X''XX'X'' \cdots .$$

Then,

$$V^* = \{ \ XX'X'', \ XX'X''XX'X'', \ XX'X''XX'X''XX'X'', \ \cdots \}$$

keeps all local information which $\mathcal{T}$ has.
(2) Let $\mathcal{T}$ be a Fibonacci-type tiling, that is for instance,

$$\mathcal{T} \ : \ \cdots XYXXYXYX \cdots .$$

Then,

$$\mathcal{T}^* \ : \ \cdots XX'X''YY'Y''XX'X''XX'X''YY'Y''XX'X''YY'Y''XX'X'' \cdots ,$$

and

$$V^* = \{ \ XX'X'', \ YY'Y'', \ XX'X''XX'X'', \ XX'X''YY'Y'', \ YY'Y''XX'X'', \ \cdots \}.$$

In this case, without using our substitution σ, we can locally obtain $\mathfrak{sl}_2$ from $\mathcal{T}$. In fact, we see

$$\left\langle \overset{1}{X}\overset{2}{X}, \ \overset{2}{X}\overset{1}{X} \right\rangle \simeq \left\langle \overset{1}{X}\overset{2}{Y}, \ \overset{2}{X}\overset{1}{Y} \right\rangle \simeq \left\langle \overset{1}{Y}\overset{2}{X}, \ \overset{2}{Y}\overset{1}{X} \right\rangle \simeq \mathfrak{sl}(2, \mathbb{C}) ,$$

since there are no subwords XXX and YY in $\mathcal{T}$. However, we need the substitution σ and the associated new tiling $\mathcal{T}^*$ to obtain a nice property as a global structure like a Gauss decomposition.

5. Tiling groups. For each $t \in \mathbb{C}$ and $\xi \in E \cup F$, we put $x_\xi(t) = 1 + t\xi \in B(\mathcal{T}^*)^\times$, where $B(\mathcal{T}^*)^\times$ is the multiplicative group of all units in $B(\mathcal{T}^*)$. Let G be the subgroup of $B(\mathcal{T}^*)^\times$ generated by $x_\xi(t)$ for all $\xi \in E \cup F$ and $t \in \mathbb{C}$. We call G the tiling group associated with an original tiling $\mathcal{T}$. For each $\xi \in E \cup F$ and $u \in \mathbb{C}^\times$, we set

$$\begin{aligned} w_\xi(u) &= x_\xi(u) x_{\hat{\xi}}(-u^{-1}) x_\xi(u) \\ &= 1 - \xi\hat{\xi} - \hat{\xi}\xi + u\xi - u^{-1}\hat{\xi}, \\ h_\xi(u) &= w_\xi(u) w_\xi(-1) \\ &= 1 + (u-1)\xi\hat{\xi} + (u^{-1} - 1)\hat{\xi}\xi. \end{aligned}$$

Then, we define three subgroups of G as follows:

$$\begin{array}{rcl} G_+ & = & \langle\, x_e(t) \mid e \in E,\ t \in \mathbb{C}\,\rangle, \\ G_0 & = & \langle\, h_\xi(u) \mid \xi \in E \cup F,\ u \in \mathbb{C}^\times\,\rangle, \\ G_- & = & \langle\, x_f(t) \mid f \in F,\ t \in \mathbb{C}\,\rangle. \end{array}$$

Then, we can establish the following result.

Proposition 2. Notation is as above. Then we have $G = G_\pm G_\mp G_0 G_\pm$ (Gauss decomposition).

The proof of this proposition will be given later (cf. Section 8). For $\alpha \in W_2(\mathcal{T}^*)$ with $\alpha = XY$ and $\xi \in E \cup F$, we say $\xi \vdash \alpha$ if and only if

$$\xi = Z_1 Z_2 \cdots \overset{i}{X}\overset{j}{Y} \cdots Z_r$$

with $\{i, j\} = \{1, 2\}$: namely $(i, j) = (1, 2)$ or $(i, j) = (2, 1)$. Let

$$\begin{array}{rcl} U_{\alpha,+} & = & \langle\, x_\xi(t) \mid t \in \mathbb{C},\ \xi \in E,\ \xi \vdash \alpha\,\rangle, \\ U_{\alpha,-} & = & \langle\, x_\xi(t) \mid t \in \mathbb{C},\ \xi \in F,\ \xi \vdash \alpha\,\rangle, \\ T_\alpha & = & \langle\, h_\xi(u) \mid u \in \mathbb{C}^\times,\ \xi \in E \cup F,\ \xi \vdash \alpha\,\rangle, \\ G_\alpha & = & \langle\, U_{\alpha,\pm}\,\rangle \end{array}$$

for each $\alpha \in W_2(\mathcal{T}^*)$.

6. Some relations. We will find several relations in our group G. For $\xi \in E \cup F$ and $s, t \in \mathbb{C}$, we have

(R1) $\quad x_\xi(s) x_\xi(t) = x_\xi(s + t)$,

which can be obtained by direct computation:

$$(1 + s\xi)(1 + t\xi) = 1 + s\xi + t\xi = 1 + (s + t)\xi$$

with $\xi^2 = 0$. In G, the commutator $[g_1, g_2]$ means $g_1 g_2 g_1^{-1} g_2^{-1}$ for $g_1, g_2 \in G$ as usual. Let $[H, H] = \langle\, [g_1, g_2] \mid g_1, g_2 \in H\,\rangle$ be the derived subgroup of a subgroup, H, of G. For $\alpha \in W_2(\mathcal{T}^*)$, we have

(R2) $\quad [U_{\alpha,+}, U_{\alpha,+}] = [U_{\alpha,-}, U_{\alpha,-}] = 1$,

which is confirmed by the reason that there is no pattern

$$\cdots XX \cdots$$

in $W(\mathcal{T}^*)$ and by the fact that the relation $\xi\eta = 0$ holds for all

$$\xi, \eta \in E_\alpha = \{\, \zeta \in E \mid \zeta \vdash \alpha \,\}$$

or

$$\xi, \eta \in F_\alpha = \{ \zeta \in F \mid \zeta \vdash \alpha \}.$$

One can also obtain a general commutation formula for $U_{\alpha,\pm}$ and $U_{\beta,\pm}$, but we do not use it explicitly here. For each $\xi \in E \cup F$, we have

(R3) $\langle x_\xi(t),\ x_{\hat{\xi}}(t) \mid t \in \mathbb{C} \rangle \simeq SL(2, \mathbb{C})$,

which is actually given by the correspondence:

$$x_\xi(t) \longleftrightarrow \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \qquad x_{\hat{\xi}}(t) \longleftrightarrow \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}.$$

For $\xi, \eta \in E \cup F$ and for $u, v \in \mathbb{C}^\times$, we have

(R4) $h_\xi(u)h_\xi(v) = h_\xi(uv)$,
(R5) $h_\xi(u)h_\eta(v) = h_\eta(v)h_\xi(u)$,
(R6) $h_\xi(u) = h_{\hat{\xi}}(u^{-1})$,

which are given by the direct computation. For $\xi \in E \cup F$ and $u \in \mathbb{C}^\times$, and for $\alpha \in W_2(\mathcal{T}^*)$, we have

(R7) $h_\xi(u)U_{\alpha,\pm}h_\xi(u)^{-1} = U_{\alpha,\pm}$.

This is obtained from the observation that for each $\eta \in E \cup F$ with $\eta \vdash \alpha$ we can find some $u_i \in \mathbb{C}^\times$ and some $\eta_i \vdash \alpha$ such that

$$h_\xi(u)\eta h_\xi(u)^{-1} = \sum_{i=1}^{k} u_i \eta_i,$$

and the relation

$$h_\xi(u)x_\eta(t)h_\xi(u)^{-1} = x_{\eta_1}(u_1 t) \cdots x_{\eta_k}(u_k t)$$

obtained from the remark after (R2).

7. Several Lemmas. We need three lemmas to show Proposition 2.

Lemma 3. Let $\alpha \in W_2(\mathcal{T}^*)$. Then, we have $G_\alpha = U_{\alpha,\pm}U_{\alpha,\mp}T_\alpha U_{\alpha,\pm}$.

Proof of Lemma 3. Let $g \in G$, and write

$$g = x_{\xi_1}(t_1)x_{\xi_2}(t_2) \cdots x_{\xi_k}(t_k)$$

with $\xi_i \in E \cup F$ and $t_i \in \mathbb{C}$ for $i = 1, 2, \ldots, k$. For each $g \in G$, we fix one expression in this way. Then, we put

$$L(g) = \langle \xi_i,\ \hat{\xi}_i \mid 1 \leq i \leq k \rangle$$

as a Lie subalgebra of L, and define $E(g) = E \cap L(g)$ and $F(g) = F \cap L(g)$. Let $G(g)$ be the subgroup of G generated by $x_\xi(t)$ for all $\xi \in E(g) \cup F(g)$ and $t \in \mathbb{C}$. Then we have that $L(g)$ is isomorphic to the direct sum of finite copies of $\mathfrak{sl}(2, \mathbb{C})$, and that $G(g)$ is isomorphic to the direct product, $\mathfrak{G}$, of finite copies of $SL(2, \mathbb{C})$. Therefore, setting

$$\begin{aligned} U_+(g) &= \langle\, x_\xi(t) \mid \xi \in E(g),\ t \in \mathbb{C}\,\rangle, \\ T(g) &= \langle\, h_\xi(u) \mid \xi \in E(g) \cup F(g),\ u \in \mathbb{C}^\times\,\rangle, \\ U_-(g) &= \langle\, x_\xi(t) \mid \xi \in F(g),\ t \in \mathbb{C}\,\rangle, \end{aligned}$$

we obtain $G(g) = U_+(g)U_-(g)T(g)U_+(g)$. In fact, there is a Gauss decomposition $\mathfrak{G} = \mathfrak{U}_\pm \mathfrak{U}_\mp \mathfrak{T} \mathfrak{U}_\pm$, where $\mathfrak{U}_+$ (resp. $\mathfrak{U}_-$) is the upper (resp. lower) triangular unipotent part of $\mathfrak{G}$ and $\mathfrak{T}$ is the diagonal part of $\mathfrak{G}$, and we see that $\mathfrak{U}_\pm$ and $\mathfrak{T}$ are corresponding to $U_\pm(g)$ and $T(g)$ respectively. Therefore, we see

$$g \in G(g) \subset U_{\alpha,+}U_{\alpha,-}T_\alpha U_{\alpha,+},$$

which implies $G_\alpha = U_{\alpha,+}U_{\alpha,-}T_\alpha U_{\alpha,+}$. Similarly we can obtain $G_\alpha = U_{\alpha,-}U_{\alpha,+}T_\alpha U_{\alpha,-}$. Q.E.D

For each $\alpha \in W_2(\mathcal{T}^*)$, we define

$$\begin{aligned} U'_{\alpha,\pm} &= \langle\, xU_{\beta,\pm}x^{-1} \mid x \in U_{\alpha,\pm},\ \beta \in W_2(\mathcal{T}^*),\ \beta \neq \alpha\,\rangle, \\ T'_\alpha &= \langle\, T_\beta \mid \beta \in W_2(\mathcal{T}^*),\ \beta \neq \alpha\,\rangle. \end{aligned}$$

Then we obtain the following two lemmas.

Lemma 4. Let $\alpha, \beta \in W_2(\mathcal{T}^*)$. Then we have:
(1) $G_\pm = U_{\alpha,\pm}U'_{\alpha,\pm} = U'_{\alpha,\pm}U_{\alpha,\pm}$,
(2) $G_0 = T_\alpha T'_\alpha = T'_\alpha T_\alpha$,
(3) $T_\alpha U_{\beta,\pm} = U_{\beta,\pm}T_\alpha$.

Proof of Lemma 4. (1) follows from the definition of $U'_{\alpha,\pm}$. (2) follows from (R5). (3) follows from (R7). Q.E.D.

Lemma 5. Let $\alpha \in W_2(\mathcal{T}^*)$. Then, we have $U_{\alpha,\mp}U'_{\alpha,\pm} = U'_{\alpha,\pm}U_{\alpha,\mp}$.

Proof of Lemma 5. Let $\xi \in E$ with $\xi \vdash \beta$, where

$$\beta \in W_2(\mathcal{T}^*),\ \beta \neq \alpha,$$

and let

$$\begin{aligned} x &= x_{\eta_1}(t_1)\cdots x_{\eta_k}(t_k), \\ y &= x_{\hat{\eta}}(s), \end{aligned}$$

where

$$\eta_i, \eta \in E, \quad \eta_i \vdash \alpha,\ \eta \vdash \alpha, \quad t_i, s \in \mathbb{C}.$$

Then, we see

$$yx\xi x^{-1}y^{-1} = \xi + \sum_{i=1}^{k} t_i[\eta_i, \xi] + \sum_{i=1}^{k} st_i[\hat{\eta}, [\eta_i, \xi]],$$

and all the components at the right hand side mutually commute. Hence, we obtain

$$yxx_\xi(t')x^{-1}y^{-1} = (xx_\xi(t')x^{-1})x_{\xi_1}(st_1t') \cdots x_{\xi_k}(st_kt'),$$

where $\xi_i = [\hat{\eta}, [\eta_i, \xi]]$ for $1 \le i \le k$. We note that $x_{\xi_i}(st_it')$ belongs to $U'_{\alpha,+}$ if $\xi_i \neq 0$, and that $x_{\xi_i}(st_it')$ must be omitted if $\xi_i = 0$. Anyway we reached

$$yxx_\xi(t')x^{-1}y^{-1} \in U'_{\alpha,+},$$

which implies $U_{\alpha,-}U'_{\alpha,+} = U'_{\alpha,+}U_{\alpha,-}$. Similarly we can prove $U_{\alpha,+}U'_{\alpha,-} = U'_{\alpha,-}U_{\alpha,+}$. Q.E.D.

8. Proof of Proposition 2. Put $\mathfrak{X} = G_+G_-G_0G_+$. Let $\xi \in E \cup F$ and $t \in \mathbb{C}$. Then, there is $\alpha \in W_2(\mathcal{T}^*)$ such that $\xi \vdash \alpha$. If $\xi \in E$, then $x_\xi(t)\mathfrak{X} = \mathfrak{X}$. If $\xi \in F$, then we have, in the same way as in [12],

$$\begin{aligned} x_\xi(t)\mathfrak{X} &\in U_{\alpha,-}\mathfrak{X} \\ &= U_{\alpha,-}(G_+G_-G_0G_+) \\ &= U_{\alpha,-}(U'_{\alpha,+}U_{\alpha,+})(U'_{\alpha,-}U_{\alpha,-})(T_\alpha T'_\alpha)(U_{\alpha,+}U'_{\alpha,+}) \\ &= U'_{\alpha,+}U_{\alpha,-}U'_{\alpha,-}U_{\alpha,+}U_{\alpha,-}T_\alpha U_{\alpha,+}T'_\alpha U'_{\alpha,+} \\ &= U'_{\alpha,+}U'_{\alpha,-}(U_{\alpha,-}U_{\alpha,+}U_{\alpha,-}T_\alpha U_{\alpha,+})T'_\alpha U'_{\alpha,+} \\ &= U'_{\alpha,+}U'_{\alpha,-}(U_{\alpha,+}U_{\alpha,-}T_\alpha U_{\alpha,+})T'_\alpha U'_{\alpha,+} \\ &= U'_{\alpha,+}U_{\alpha,+}U'_{\alpha,-}U_{\alpha,-}T_\alpha T'_\alpha U_{\alpha,+}U'_{\alpha,+} \\ &= G_+G_-G_0G_+ = \mathfrak{X}. \end{aligned}$$

Therefore, $G\mathfrak{X} = \mathfrak{X}$, which shows $G = \mathfrak{X}$. Similarly we can establish $G = G_-G_+G_0G_-$. We have finished to prove Proposition 2.

9. Proof of Proposition 1. We set $\Pi = W_2(\mathcal{T}^*)$, and $Q = \oplus_{\alpha\in\Pi}\mathbb{Z}\alpha$. We define

$$\begin{aligned} \deg(\xi) &= \alpha \quad \text{if} \quad \xi \in E,\ \xi \vdash \alpha, \\ \deg(\xi) &= -\alpha \quad \text{if} \quad \xi \in F,\ \xi \vdash \alpha. \end{aligned}$$

Then, denoting by L_μ the subspace of L consisting of all elements of degree μ, we have $L = \oplus_{\mu\in Q}L_\mu$. Let $\Delta = \{\ \mu \in Q \mid L_\mu \neq 0\ \}$, the grading set of L. Put

$$\begin{aligned} \Delta_+ &= \Delta \cap (\textstyle\sum_{\alpha\in\Pi} \mathbb{Z}_{\ge 0}\alpha \setminus \{0\}), \\ \Delta_- &= \Delta \cap (\textstyle\sum_{\alpha\in\Pi} \mathbb{Z}_{\le 0}\alpha \setminus \{0\}). \end{aligned}$$

For each $\alpha \in \Pi$, we set

$$\begin{aligned} \Delta'_\pm(\alpha) &= \Delta_\pm \setminus \{\pm\alpha\}, \\ L'_{\pm\alpha} &= \oplus_{\mu\in\Delta'_\pm(\alpha)}L_\mu, \\ S_\alpha &= \langle\ L_{\pm\alpha}\ \rangle. \end{aligned}$$

We note that S_α normalizes $L'_{\pm\alpha}$, and we also see

$$\begin{aligned} \mathcal{U}(L_\pm) &= \mathcal{U}(L_{\pm\alpha})\mathcal{U}(L'_{\pm\alpha}) = \mathcal{U}(L'_{\pm\alpha})\mathcal{U}(L_{\pm\alpha}), \\ \mathcal{U}(L_{\pm\alpha})\mathcal{U}(L'_{\mp\alpha}) &= \mathcal{U}(L'_{\mp\alpha})\mathcal{U}(L_{\pm\alpha}). \end{aligned}$$

First, we will establish $\mathcal{U}(S_\alpha) = \mathcal{U}(L_\alpha)\mathcal{U}(L_{-\alpha})\mathcal{U}(L_\alpha)$. Let $x \in \mathcal{U}(S_\alpha)$. We can suppose, for our purpose, that x is of the form:

$$x = \xi_1\xi_2\cdots\xi_k,$$

where $\xi_i \in E \cup F$ with $\xi_i \vdash \alpha$ for $1 \leq i \leq k$. We fix this expression $x = \xi_1\cdots\xi_k$. Let $L(x)$ be the Lie subalgebra of L generated by ξ_i and $\hat{\xi}_i$ for all $1 \leq i \leq k$. Then, $L(x)$ is isomorphic to the direct sum, $\mathfrak{L}$, of finite copies of $\mathfrak{sl}(2, \mathbb{C})$. Therefore, we have $\mathcal{U}(L(x)) = \mathcal{U}(L_+(x))\mathcal{U}(L_-(x))\mathcal{U}(L_+(x))$ with $L_\pm(x) = L(x) \cap L_\pm$. In fact, there is an additive Gauss decomposition $\mathcal{U}(\mathfrak{L}) = \mathcal{U}(\mathfrak{L}_+)\mathcal{U}(\mathfrak{L}_-)\mathcal{U}(\mathfrak{L}_+)$, where $\mathfrak{L}_+$ (resp. $\mathfrak{L}_-$) is the upper (resp. lower) triangular nilpotent part of $\mathfrak{L}$, and we see that $\mathfrak{L}_+$ and $\mathfrak{L}_-$ are corresponding to $L_+(x)$ and $L_-(x)$ respectively (cf. [4]). Hence,

$$\begin{aligned} x &\in \mathcal{U}(L_+(x))\mathcal{U}(L_-(x))\mathcal{U}(L_+(x)) \\ &\subset \mathcal{U}(L_\alpha)\mathcal{U}(L_{-\alpha})\mathcal{U}(L_\alpha), \end{aligned}$$

which shows

$$\mathcal{U}(S_\alpha) = \mathcal{U}(L_\alpha)\mathcal{U}(L_{-\alpha})\mathcal{U}(L_\alpha).$$

Now we put $\mathfrak{Y} = \mathcal{U}(L_+)\mathcal{U}(L_-)\mathcal{U}(L_+)$. Then, for every $\alpha \in \Pi$, we obtain

$$\begin{aligned} \mathcal{U}(L_{-\alpha})\mathfrak{Y} &= \mathcal{U}(L_{-\alpha})(\mathcal{U}(L_+)\mathcal{U}(L_-)\mathcal{U}(L_+)) \\ &= \mathcal{U}(L_{-\alpha})(\mathcal{U}(L'_\alpha)\mathcal{U}(L_\alpha))(\mathcal{U}(L'_{-\alpha})\mathcal{U}(L_{-\alpha}))(\mathcal{U}(L_\alpha)\mathcal{U}(L'_\alpha)) \\ &= \mathcal{U}(L'_\alpha)\mathcal{U}(L_{-\alpha})\mathcal{U}(L'_{-\alpha})\mathcal{U}(L_\alpha)\mathcal{U}(L_{-\alpha})\mathcal{U}(L_\alpha)\mathcal{U}(L'_\alpha) \\ &= \mathcal{U}(L'_\alpha)\mathcal{U}(L'_{-\alpha})(\mathcal{U}(L_{-\alpha})\mathcal{U}(L_\alpha)\mathcal{U}(L_{-\alpha})\mathcal{U}(L_\alpha))\mathcal{U}(L'_\alpha) \\ &= \mathcal{U}(L'_\alpha)\mathcal{U}(L'_{-\alpha})\mathcal{U}(L_\alpha)\mathcal{U}(L_{-\alpha})\mathcal{U}(L_\alpha)\mathcal{U}(L'_\alpha) \\ &= \mathcal{U}(L'_\alpha)\mathcal{U}(L_\alpha)\mathcal{U}(L'_{-\alpha})\mathcal{U}(L_{-\alpha})\mathcal{U}(L_\alpha)\mathcal{U}(L'_\alpha) \\ &= \mathcal{U}(L_+)\mathcal{U}(L_-)\mathcal{U}(L_+) = \mathfrak{Y}. \end{aligned}$$

Hence, $\mathcal{U}(L)\mathfrak{Y} = \mathfrak{Y}$, which implies $\mathcal{U}(L) = \mathfrak{Y}$. Similarly we can establish $\mathcal{U}(L) = \mathcal{U}(L_-)\mathcal{U}(L_+)\mathcal{U}(L_-)$. We have just finished to prove Proposition 1.

10. Remarks. (1) There is a number theoretical way to construct interesting tilings and study them (cf. [2], [3]). One may be interested in pure mathematical approaches to tilings and aperiodic orders (cf. [6], [7], [8], [9]), which often induce several algebraic structures. For example in [10], we already found that a couple of one-dimensional tilings $\mathcal{T}$ and $\mathcal{T}'$ are locally nondistinguishable (or locally indistinguishable) if and only if the corresponding bialgebras with triangular decompositions are isomorphic. Here we obtained groups and Lie algebras associated with one dimensional tilings.

(2) Gauss decompositions are important to study group structures. For example, if $G = U_+U_-U_0U_+$ is a Gauss decomposition, then modulo conjugacy an element of G can be expressed as an element of $U_-U_0U_+$. In many cases, we have a unique expression of elements in $U_-U_0U_+$. This is very helpful to study group invariants.
(3) Quite recently a new development was given in extended affine Lie theory. Namely, Bruhat decompositions and Gauss decompositions were established in some groups defined by extended affine Lie algebras with nullity 2 (cf. [13]). We hope that this helps to generalize our method here to higher dimensional tilings, at least 2 dimensional tilings. The main idea arises from K. Saito's marking at his extended affine root systems related to singularity theory.
(4) We used and fixed our special substitution σ for our purpose. However, one can use other substitutions for further developments keeping Gauss decompositions. In this paper, we did not make any comparison among tilings, substitutions and Gauss decompositions.

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