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LEFT UNITARILY INVARIANT NORMS ON MATRICES

MASUMI DOMON, TAKASHI SANO, AND TOMOHIDE TOBA

ABSTRACT. In operator theory, unitarily invariant norms and related inequalities have been studied intensively. In this article, weaker invariance; left unitary invariance and related results are observed.

1. INTRODUCTION

Unitarily invariant norms and their inequalities have been studied well in operator theory. See [1] for instance. And weaker invariances of norms have been observed; weakly unitarily invariance, unitary similarity invariance, and so on. C.K. Li and N.K. Tsing gave such invariant norms a general point of view: they consider G-actions on matrices and G-invariant norms and their properties ([6, 7, 8]). See also [9].

In this article, we would like to study the following norm μ on the set of all complex n-square matrices:

$$\mu(UA) = \mu(A)$$

for all matrices A and all unitary U. This is equivalent to

$$\mu(|A|) = \mu(A)$$

for all matrices A. This property is called unitary row equivalence invariance in [7, pp. 436], and we call this property left unitary invariance. By studying left unitarily invariant norms, we think that we can recognize the theory of unitarily invariant norms from another point of view: we present properties of left unitarily invariant norms; characterizations of left unitary invariance, submultiplicativity, essential subclasses, non-existence of a finite essential subclass, and a characterization of left unitarily invariant norms given by inner products.

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2. LEFT UNITARILY INVARIANT NORMS ON MATRICES

Let $M_n(\mathbb{C})$ be the set of all complex n-square matrices. A norm μ on $M_n(\mathbb{C})$ is said to be unitarily invariant (abbreviated to ui) if

$$\mu(UAV) = \mu(A)$$

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for all $A \in M_n(\mathbb{C})$ and all unitary $U, V \in M_n(\mathbb{C})$. In this article, a norm μ on $M_n(\mathbb{C})$ is said to be left unitarily invariant (abbreviated to lui) if

$$\mu(UA) = \mu(A)$$

for all $A \in M_n(\mathbb{C})$ and all unitary $U \in M_n(\mathbb{C})$.

Left unitary invariance is characterized as follows:

Theorem 2.1. *Let μ be a norm on $M_n(\mathbb{C})$. Then the following are equivalent:*
(i) *μ is left unitarily invariant.*
(ii) *$\mu(|A|) = \mu(A)$ for all $A \in M_n(\mathbb{C})$.*
(iii) *$\mu(AB) \leqq \|A\|\mu(B)$ for all $A, B \in M_n(\mathbb{C})$.*
Here $\|\cdot\|$ is the operator norm on $M_n(\mathbb{C})$ and $|A| = \sqrt{A^*A}$.

Proof. (i) $\Leftrightarrow$ (ii) and (iii) $\Rightarrow$ (i) are easy to see.
(i) $\Rightarrow$ (iii): we may and do assume that $\|A\| = 1$. Since a contraction is a convex combination of unitary matrices, A is of the form: $A = \sum_{j=1}^{N} \lambda_j U_j$ for $\lambda_j (\geqq 0)$ and unitary U_j $(j = 1, 2, \dots, N)$ with $\sum_{j=1}^{N} \lambda_j = 1$. It follows that

$$\begin{aligned} \mu(AB) &= \mu((\sum_{j=1}^{N} \lambda_j U_j)B) \\ &\leqq \sum_{j=1}^{N} \lambda_j \mu(U_j B) = \sum_{j=1}^{N} \lambda_j \mu(B) = \mu(B). \end{aligned}$$

Hence, the proof is complete. □

Example. Let μ be a left unitarily invariant norm on $M_n(\mathbb{C})$ and let $C \in M_n(\mathbb{C})$ be invertible. Then we have the left unitarily invariant norm μ_C defined by

$$\mu_C(A) := \mu(AC) \quad (A \in M_n(\mathbb{C})).$$

Proposition 2.2. *Let μ be a unitarily invariant norm on $M_n(\mathbb{C})$ and let $C \in M_n(\mathbb{C})$ be invertible. Then the left unitarily invariant norm μ_C defined above is unitarily invariant if and only if C is a scalar multiple of a unitary matrix.*

Proof. If μ_C is unitarily invariant, $\mu(AUCV) = \mu(AC)$ for all $A \in M_n(\mathbb{C})$ and all unitary U, V. For instance,

$$\mu(\mathrm{diag}(1, 0, \cdots, 0)U\mathrm{diag}(s_1(C), s_2(C), \cdots, s_n(C))V) \qquad (\sharp)$$

is constant for all unitary U, V, where $s_i(C)$ denotes the i-th singular value of C with $s_1(C) \geqq s_2(C) \geqq \cdots \geqq s_n(C)$. Taking U as a permutation matrix for 1 and i, and V as U^*, we see that the above value ($\sharp$) is $s_i(C)\mu(\mathrm{diag}(1, 0, \cdots, 0))$. Hence, the singular values are identical. The converse is clear and the proof is complete. □

Example. ($G(C)$-radius $r_{G(C)}$ in [8].) For an invertible matrix $C \in M_n(\mathbb{C})$, we have the left unitarily invariant norm f_C on $M_n(\mathbb{C})$ defined by

$$\begin{aligned} f_C(A) &:= \max_{U:\ \text{unitary}} |\mathrm{Tr}(UAC)| \\ &= \max_{S:\ \text{contraction}} |\mathrm{Tr}(SAC)| = \|AC\|_1 \end{aligned}$$

for $A \in M_n(\mathbb{C})$, where $\|\cdot\|_1$ is the trace norm. Equalities follow from the fact that a contraction is a convex combination of unitary matrices and the duality between the trace norm and the operator norm. This f_C is the $G(C^*)$- radius $r_{G(C^*)}$ in [8, pp. 183]; $r_{G(C)}(A) = \max_{U:\ \text{unitary}} |\mathrm{Tr}((UC)^*A)|$.

Remark that we have

$$\max_{U,V:\text{unitary}} |\mathrm{Tr}(UAVC)| = \sum_{i=1}^{n} s_i(A)s_i(C)$$

due to von Neumann ([10, pp. 514]).

Although some of Theorem 2.3 are known in [8, Theorem 3.4], we have the following with a simple proof for our case:

Theorem 2.3. *Let $A, B \in M_n(\mathbb{C})$. Then the following are equivalent:*
(i) $\mu(A) \leqq \mu(B)$ *for all lui norms* μ.
(ii) $\mu(AC) \leqq \mu(BC)$ *for all lui norms* μ *and all invertible* C.
(iii) $\mu(AC) \leqq \mu(BC)$ *for some lui norms* μ *and all invertible* C.
(iv) $\mu(AC) \leqq \mu(BC)$ *for some lui norms* μ *and all matrices* C.
(v) $\mu(AP) \leqq \mu(BP)$ *for some lui norms* μ *and all projections* P.
(vi) $A = VB$ *for some contraction* V.

Proof. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) and (iv) $\Rightarrow$ (v) are clear.
(iii) $\Rightarrow$ (iv): since the set of invertible matrices is dense in $M_n(\mathbb{C})$ and the map $C \mapsto \mu(\cdot\, C)$ is continuous, (iii) implies (iv).
(vi) $\Rightarrow$ (i): this follows from Theorem 2.1.
(v) $\Rightarrow$ (vi): take a unit vector $x \in \mathbb{C}^n$ and denote the orthogonal projection to the subspace $\mathbb{C}x$ by P. By assumption,

$$\mu(|AP|) = \mu(AP) \leqq \mu(BP) = \mu(|BP|).$$

Since the matrices $|AP|, |BP|$ are at most rank-one, $|AP| = \alpha P, |BP| = \beta P$ for $\alpha, \beta \geqq 0$. $\mu(\alpha P) \leqq \mu(\beta P)$ implies $\alpha \leqq \beta$, or

$$PA^*AP = \alpha^2 P \leqq \beta^2 P = PB^*BP.$$

Hence,

$$\|Ax\| \leqq \|Bx\|$$

for all $x \in \mathbb{C}^n$, and (vi) follows. □

Remark that we can replace "lui" with "ui" in Theorem 2.3.

By Theorem 2.3, we have several essential subclasses for the set of left unitarily invariant norms in the following sense: a subclass $\mathcal{C}$ in lui norms is said to be essential if for $A, B \in M_n(\mathbb{C})$,

$$\mu(A) \leqq \mu(B)$$

for all lui norms μ whenever

$$\mu(A) \leqq \mu(B)$$

for all $\mu \in \mathcal{C}$.

In [8], C. K. Li and N. K. Tsing consider the existence of finite essential subclasses for G-invariant norms. We apply [8, Theorem 4.3] to our case: if there were a finite essential subclass for lui norms, then we would have

$$\mathrm{span}\{UC : \text{unitary } \ U\} = M_n(\mathbb{C})$$

for all $C \neq O \in M_n(\mathbb{C})$. But for any non-invertible matrix $C \neq O$,

$$\mathrm{span}\{UC : \text{unitary } \ U\} = M_n(\mathbb{C}) \cdot C \neq M_n(\mathbb{C}).$$

Hence, we have

Proposition 2.4. *There is no finite essential subclass for left unitarily invariant norms.*

For the trace norm $\|\cdot\|_1$, the following may be considered as an essential subclass in Theorem 2.3:

$$\mathrm{Tr}(|AC|) = \|AC\|_1 \leqq \|BC\|_1 = \mathrm{Tr}(|BC|)$$

for all $C \in M_n(\mathbb{C})$. This is equivalent to

$$|\mathrm{Tr}(AC)| \leqq \mathrm{Tr}(|BC|)$$

for all $C \in M_n(\mathbb{C})$. Here we observe another inequality:

Proposition 2.5. *Let $A, B \in M_n(\mathbb{C})$.*

$$|\mathrm{Tr}(AC)| \leqq |\mathrm{Tr}(BC)|$$

for all $C \in M_n(\mathbb{C})$ if and only if there is a complex number α ($|\alpha| \leqq 1$) such that $A = \alpha B$.

Proof. Let φ, ψ be the linear functionals on $M_n(\mathbb{C})$ defined by

$$\varphi(\cdot) = \mathrm{Tr}(A\ \cdot), \quad \psi(\cdot) = \mathrm{Tr}(B\ \cdot).$$

Then the assumption implies

$$\ker\ \psi \subseteqq \ker\ \varphi.$$

By the well-known fact ([2, Appendix A]), there exists a complex number α such that

$$\varphi = \alpha\ \psi.$$

This means the assertion. □

We study the submultiplicativity of lui norms. See [4, Theorem] or [5, Theorem 5.5.7] for the corresponding result for ui norms.

Theorem 2.6. *Let μ be a lui norm on $M_n(\mathbb{C})$. Then the following are equivalent:*
(i) *μ dominates the operator norm, i.e., $\|A\| \leqq \mu(A)$ for all $A \in M_n(\mathbb{C})$.*
(ii) *μ is submultiplicative, i.e., $\mu(AB) \leqq \mu(A)\mu(B)$ for all $A, B \in M_n(\mathbb{C})$.*
(iii) *μ is spectrally dominant, i.e., $\rho(A) \leqq \mu(A)$ for all $A \in M_n(\mathbb{C})$, where $\rho(A)$ is the spectral radius of A; $\rho(A) := \max\{|\lambda| : \lambda$ is an eigenvalue of $A\}$.*

Proof. (i) $\Rightarrow$(ii): this follows from Theorem 2.1.
(ii)$\Rightarrow$(iii): take an eigenvector $x(\neq 0)$ for an eigenvalue λ of A and put $X = (xx\dots x) \in M_n(\mathbb{C})$. Then we have

$$|\lambda|\mu(X) = \mu(AX) \leqq \mu(A)\mu(X).$$

(See also [3, Theorem 5.6.9].)
(iii) $\Rightarrow$ (i): for $A \in M_n(\mathbb{C})$,

$$\|A\|^2 = \|A^*A\| = \rho(A^*A) \leqq \mu(A^*A) \leqq \|A^*\|\mu(A) = \|A\|\mu(A).$$

□

It is well-known that for each unitarily invariant norm μ, we have

$$\mu(A \circ B) \leqq \|A\|\mu(B).$$

See [4, pp. 95], for instance. But this is not the case for lui norms:
For

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, C = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

we have

$$\|(A \circ B)C\|_1 > \|A\|\|BC\|_1.$$

The following characterizes lui norms given by inner products on matrices. This kind of norms is called elliptical and studied in [7]. Here we have a proof from the viewpoint of the theory of operator algebras.

Theorem 2.7. ([7, Theorem 4.4]) *Let μ be a norm on $M_n(\mathbb{C})$ that is given by an inner product $\langle \cdot, \cdot \rangle_0$ on $M_n(\mathbb{C})$;*

$$\mu(A) := \sqrt{\langle A, A \rangle_0} \quad (A \in M_n(\mathbb{C})).$$

If μ is left unitarily invariant, then we have a positive definite matrix $C \in M_n(\mathbb{C})$ such that

$$\mu(A) = \|AC^{\frac{1}{2}}\|_2 = \sqrt{\mathrm{Tr}(|A|^2 C)} \quad (A \in M_n(\mathbb{C})).$$

Proof. We recall basic facts of the standard representation of $M_n(\mathbb{C})$: we identify $M_n(\mathbb{C})$ with $\mathbb{C}^{n^2}$ as a Hilbert space via the inner product $\langle \cdot, \cdot \rangle$ induced by the trace Tr. For $A \in M_n(\mathbb{C})$, we denote the corresponding vector in $\mathbb{C}^{n^2}$ by $\eta(A)$ and we have the inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{C}^{n^2}$ defined by

$$\langle \eta(A), \eta(B) \rangle = \mathrm{Tr}(B^*A) \quad (A, B \in M_n(\mathbb{C})).$$

By the left multiplication, we have the $*$- representation π of $M_n(\mathbb{C})$ on $\mathbb{C}^{n^2}$; for $A \in M_n(\mathbb{C})$,

$$\pi(A)\eta(C) := \eta(AC) \quad (C \in M_n(\mathbb{C})).$$

And we define the conjugation J on $\mathbb{C}^{n^2}$ by $J\eta(C) := \eta(C^*)$ $(C \in M_n(\mathbb{C}))$ and the commutant of $\pi(M_n(\mathbb{C}))$ by

$$\pi(M_n(\mathbb{C}))' := \{X \in B(\mathbb{C}^{n^2}) | X\pi(A) = \pi(A)X \ (A \in M_n(\mathbb{C}))\}.$$

It is easy to see that $J\pi(A)J\eta(B) = \eta(BA^*)$ and it is known that

$$\pi(M_n(\mathbb{C}))' = J\pi(M_n(\mathbb{C}))J.$$

The inner product $\langle\cdot,\cdot\rangle_0$ in the assertion induces one on $\mathbb{C}^{n^2}$;

$$\langle\eta(A),\eta(B)\rangle_0^{\sim} := \langle A, B\rangle_0 \quad (A, B \in M_n(\mathbb{C})).$$

This is represented by a positive definite $X \in B(\mathbb{C}^{n^2})$ such as

$$\langle\eta(A),\eta(B)\rangle_0^{\sim} = \langle X\eta(A),\eta(B)\rangle.$$

Then the left unitary invariance of μ yields

$$\langle X\eta(A),\eta(A)\rangle = \mu(A) = \mu(UA) = \langle\pi(U)^*X\pi(U)\eta(A),\eta(A)\rangle$$

for all $A \in M_n(\mathbb{C})$ and all unitary $U \in M_n(\mathbb{C})$. Therefore, $X = \pi(U)^*X\pi(U)$ for all unitary U and unitary matrices generate $M_n(\mathbb{C})$ so that

$$X \in \pi(M_n(\mathbb{C}))'.$$

The preceding argument implies

$$X\eta(A) = \eta(AC)$$

for a positive definite $C \in M_n(\mathbb{C})$ and the proof is complete. □

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Masumi Domon
Department of Mathematical Sciences
Yamagata University
Yamagata, 990-8560, Japan

Takashi Sano
Department of Mathematical Sciences
Yamagata University
Yamagata, 990-8560, Japan

Tomohide Toba
Department of Mathematical Sciences
Yamagata University
Yamagata, 990-8560, Japan

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