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EQUIVALENCE CLASSES OF MIXED INVARIANT SUBSPACES OVER THE BIDISK

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ABSTRACT. A closed subspace N of the Hardy space H^2 over the bidisk is said to be mixed invariant under T_z and T_w^* if $T_zN \subset N$ and $T_w^*N \subset N$. In this paper, we study unitary, similar and quasi-similar module maps for mixed invariant subspaces. We give some characterization of these maps. All unitary module maps are multiplication operators of unimodular functions. Under the condition $\dim(N \ominus zN) = 1$, we can describe similar and quasi-similar module maps by outer functions.

1. Introduction

Let D^2 be the bidisk and Γ^2 be the distinguished boundary of D^2. We use z, w as variables over Γ^2. Let $L^2 = L^2(\Gamma^2)$ and $H^2 = H^2(\Gamma^2)$ be the usual Lebesgue and Hardy spaces over Γ^2. We denote by $H^2(z)$ and $H^2(w)$ the z and w variable Hardy spaces, respectively. For $\varphi \in L^\infty(\Gamma^2)$, we define the Toeplitz operator T_φ on H^2 by $T_\varphi f = P_{H^2}(\varphi f)$, where P_{H^2} is the orthgonal projection from L^2 onto H^2.

A closed subspace M of H^2 is called invariant if $T_zM \subset M$ and $T_wM \subset M$. In [10, 11], K. H. Izuchi and the first author studied M satisfying $\operatorname{rank}(R_zR_w^* - R_w^*R_z) = 1$, where $R_z = T_z|_M$ and $R_w = T_w|_M$. It is still open to describe all M satisfying the above condition. Let $L = H^2 \ominus M$. Then $T_z^*L \subset L$ and $T_w^*L \subset L$. The space L is called backward shift invariant. In [12], K. H. Izuchi and the first author showed that the form of L can be described under the condition $\operatorname{rank}(S_zS_w^* - S_w^*S_z) = 1$, where $S_z = P_LT_z|_L$, $S_w = P_LT_w|_L$. From such a thing, the authors feel that some problems on L are easier than same type problems on M. To overcome this thing, in [13], K. H. Izuchi and the authors introduced the concept of "mixed invariant" for closed subspace on H^2.

A closed subspace N of H^2 with $N \neq \{0\}$ and $N \neq H^2$ is called mixed invariant under T_z and T_w^* if $T_zN \subset N$ and $T_w^*N \subset N$. We define the operators V_z and V_w on N by $V_zf = T_zf$ and $V_wf = P_NT_wf$. In [13], K. H. Izuchi and the authors described the form of mixed invariant subspaces N under the condition $V_zV_w = V_wV_z$. This

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is a similar result for invariant and backward shift invariant subspaces. Moreover, we showed that a wandering subspace $N \ominus V_z N$ has a deep connection with the de Branges-Rovnyak spaces studied by Sarason [15]. See [13] in detail.

It is well known result due to Beurling that for every invariant subspace M of the Hardy space over the unit circle, $M = \varphi H^2(\Gamma)$ for an inner function φ. But it is easy to see that Beurling-type characterization is not possible for invariant subspaces of $H^2(\Gamma^2)$ [14]. Hence this directs one's attention to investigate equivalence classes of invariant subspaces of $H^2(\Gamma^2)$, naturally. See [1, 3, 4, 5, 6, 9] for the related subjects. In [1], Agrawal, Clark and Douglas introduced the concept of unitary equivalence of invariant subspaces. They showed that two invariant subspaces of finite codimension are unitarily equivalent if and only if they are equal. In [9], the first author gave a complete characterization of pairs of invariant subspaces I and J of $H^2(\Gamma^2)$ such that $I = \varphi J$ for an inner function φ. This is a generalization of Agrawal, Clark and Douglas's results. In [5, 6], Guo studied unitary equivalence from a module theoretic viewpoint.

In this paper, we study unitary, similar, and quasi-similar module maps for mixed invariant subspaces. For mixed invariant subspaces N_1 and N_2 of H^2 under T_z and T_w^*, we write $V_z^{(j)} = V_z$ and $V_w^{(j)} = V_w$ on N_j. Note that $V_z^{(j)} = T_z|_{N_j}$ and $V_w^{(j)*} = T_w^*|_{N_j}$. A bounded linear map $T : N_1 \to N_2$ is called a module map with respect to (V_z, V_w^*), (V_z, V_w), (V_z^*, V_w), and (V_z^*, V_w^*) if

$$TV_z^{(1)} = V_z^{(2)}T \quad \text{and} \quad TV_w^{(1)*} = V_w^{(2)*}T,$$

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respectively. We say that N_1 and N_2 are unitarily equivalent (similar) if there is a unitary (invertible) module map $T : N_1 \to N_2$ for each respective type. We also say that N_1 and N_2 are quasi-similar if there are one to one module maps $T_1 : N_1 \to N_2$ and $T_2 : N_2 \to N_1$ with dense range for each respective type. For a fixed N_1, we denote by $orb_{(u,V_z,V_w^*)}(N_1)$, $orb_{(s,V_z,V_w^*)}(N_1)$, and $orb_{(qs,V_z,V_w^*)}(N_1)$ the family of mixed invariant subspaces N which are unitarily equivalent, similar, and quasi-similar to N_1 with respect to (V_z, V_w^*), respectively. We may consider other types of orbits of N_1. We have a characterization of unitary equivalence by unimodular functions. In Corollary 2.2, we shall prove that the followings are equivalent;

(i) $T : N_1 \to N_2$ is a unitary module map with respect to (V_z, V_w^*).
(ii) $T : N_1 \to N_2$ is a unitary module map with respect to (V_z, V_w).
(iii) $T : N_1 \to N_2$ is a unitary module map with respect to (V_z^*, V_w^*).
(iv) $T : N_1 \to N_2$ is a unitary module map with respect to (V_z^*, V_w).
(v) There is a unimodular function $\psi(z)$ satisfying $Th = \psi(z)h$ for $h \in N_1$.

Under the conditon $\dim(N \ominus zN) = 1$, we can describe similar and quasi-similar module maps by outer functions.

2. Theorems

First, we prove the following theorem. The idea of the proof comes from Douglas and Foias [4].

Theorem 2.1. *Let N_1 and N_2 be mixed invariant subspaces of H^2 under T_z and T_w^*. Let $T : N_1 \to N_2$ be a unitary map. Then the following conditions are equivalent.*

(i) *$T : N_1 \to N_2$ is a unitary module map with respect to (V_z, V_w^*).*
(ii) *$T : N_1 \to N_2$ is a unitary module map with respect to (V_z, V_w).*
(iii) *There is a unimodular function $\psi(z)$ satisfying $Th = \psi(z)h$ for every $h \in N_1$.*

Proof. (i) (or (ii)) $\Rightarrow$ (iii): Suppose that $T : N_1 \to N_2$ is a unitary module map with respect to (V_z, V_w^*) (or (V_z, V_w)). Let $\widetilde{N}_j$ be the closed linear span of $\{T_w^n N_j : n \geq 0\} = \{w^n N_j : n \geq 0\}$. Then $\widetilde{N}_j$ is a mixed invariant subspace under T_z and T_w^*, and $T_w \widetilde{N}_j \subset \widetilde{N}_j$. By [13, Corollary 2.5], there are inner functions $q_1(z)$ and $q_2(z)$ satisfying

$$\widetilde{N}_1 = q_1(z)H^2 \quad \text{and} \quad \widetilde{N}_2 = q_2(z)H^2. \tag{2.1}$$

For $F = \sum T_w^n h_n, h_n \in N_1$, we define $\widetilde{T}F = \sum T_w^n T h_n$. Since $TT_w^* = T_w^* T$ (or $TV_w^{(1)} = V_w^{(2)} T$) on N_1 and $T : N_1 \to N_2$ is unitary, we have

$$\begin{aligned}
\|\widetilde{T}F\|^2 &= \sum_{n,k} \left\langle T_w^n T h_n, T_w^k T h_k \right\rangle \\
&= \sum_{n \geq k} \left\langle T h_n, T_w^{*(n-k)} T h_k \right\rangle + \sum_{n<k} \left\langle T_w^{*(k-n)} T h_n, T h_k \right\rangle \\
\Big(\text{or} &= \sum_{n \geq k} \left\langle V_w^{(2)n-k} T h_n, T h_k \right\rangle + \sum_{n<k} \left\langle T h_n, V_w^{(2)k-n} T h_k \right\rangle \Big) \\
&= \sum_{n \geq k} \left\langle T h_n, T T_w^{*(n-k)} h_k \right\rangle + \sum_{n<k} \left\langle T T_w^{*(k-n)} h_n, T h_k \right\rangle \\
\Big(\text{or} &= \sum_{n \geq k} \left\langle T V_w^{(1)n-k} h_n, T h_k \right\rangle + \sum_{n<k} \left\langle T h_n, T V_w^{(1)k-n} h_k \right\rangle \Big) \\
&= \sum_{n \geq k} \left\langle h_n, T_w^{*(n-k)} h_k \right\rangle + \sum_{n<k} \left\langle T_w^{*(k-n)} h_n, h_k \right\rangle \\
\Big(\text{or} &= \sum_{n \geq k} \left\langle V_w^{(1)n-k} h_n, h_k \right\rangle + \sum_{n<k} \left\langle h_n, V_w^{(1)k-n} h_k \right\rangle \Big)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{n\ge k}\langle T_w^n h_n, T_w^k h_k\rangle + \sum_{n<k}\langle T_w^n h_n, T_w^k h_k\rangle \\
&= \|\sum T_w^n h_n\|^2 \quad = \quad \|F\|^2.
\end{aligned}
$$

Hence $\widetilde{T} : \widetilde{N}_1 \to \widetilde{N}_2$ is well defined and a unitary map.

We shall prove that

$$
\widetilde{T}T_w = T_w\widetilde{T} \quad \text{and} \quad \widetilde{T}T_z = T_z\widetilde{T} \quad \text{on } \widetilde{N}_1. \tag{2.2}
$$

Since $TT_z = T_zT$ on N_1, we have

$$
\widetilde{T}T_zF = \widetilde{T}\big(\sum T_w^n T_z h_n\big) = \sum T_w^n TT_z h_n = T_z\widetilde{T}F.
$$

We also have

$$
\widetilde{T}T_wF = \widetilde{T}\big(\sum T_w^{n+1} h_n\big) = \sum T_w^{n+1} Th_n = T_w\widetilde{T}F.
$$

Thus we get (2.2).

By (2.1), we can define the operator $\widetilde{\widetilde{T}}$ on H^2 by

$$
\widetilde{\widetilde{T}} : H^2 = \overline{q_1(z)}\widetilde{N}_1 \ni \overline{q_1(z)}F \to \overline{q_2(z)}\widetilde{T}F \in H^2.
$$

Since $\widetilde{T} : \widetilde{N}_1 \to \widetilde{N}_2$ is unitary, $\widetilde{\widetilde{T}} : H^2 \to H^2$ is unitary. By (2.2), it is easy to see that $\widetilde{\widetilde{T}}T_z = T_z\widetilde{\widetilde{T}}$ and $\widetilde{\widetilde{T}}T_w = T_w\widetilde{\widetilde{T}}$ on H^2. Hence we get $\widetilde{\widetilde{T}} = cI$ for some $c \in \mathbb{C}$ with $|c| = 1$. Thus we get $\overline{q_2(z)}\widetilde{T}F = c\overline{q_1(z)}F$ for $F \in \widetilde{N}_1$. Therefore $\widetilde{T}F = c\overline{q_1(z)}q_2(z)F$ for every $F \in \widetilde{N}_1$. Since $\widetilde{T}|_{N_1} = T$, $Th = c\overline{q_1(z)}q_2(z)h$ for every $h \in N_1$. Thus we get (iii).

(iii) $\Rightarrow$ (i) and (ii): Suppose that $Th = \psi(z)h$ for $h \in N_1$, where $\psi(z)$ is a unimodular function. It is trivial that $TV_z^{(1)} = V_z^{(2)}T$. We have

$$
TV_w^{(1)*}h = \psi(z)T_w^*h = T_w^*(\psi(z)h) = V_w^{(2)*}Th.
$$

Hence $TV_w^{(1)*} = V_w^{(2)*}T$.

We write $wh = h_1 \oplus g_1 \in N_1 \oplus (H^2 \ominus N_1)$. Since $\psi(z)N_1 = N_2 \subset H^2$, $\psi(z)g_1 \in H^2$. Since $g_1 \perp N_1$, we have $\psi(z)g_1 \perp \psi(z)N_1 = N_2$. Thus

$$
\psi(z)wh = \psi(z)h_1 \oplus \psi(z)g_1 \in N_2 \oplus (H^2 \ominus N_2).
$$

Hence $P_{N_2}(\psi(z)wh) = \psi(z)h_1$ and

$$
TV_w^{(1)}h = Th_1 = \psi(z)h_1 = P_{N_2}(\psi(z)wh) = V_w^{(2)}Th.
$$

Therefore we get $TV_w^{(1)} = V_w^{(2)}T$. □

Corollary 2.2. *Let N_1 and N_2 be mixed invariant subspaces of H^2 under T_z and T_w^*. Let $T : N_1 \to N_2$ be a unitary map. Then the following conditions are equivalent.*

(i) *$T : N_1 \to N_2$ is a unitary module map with respect to (V_z, V_w^*).*

(ii) *$T : N_1 \to N_2$ is a unitary module map with respect to (V_z, V_w).*

(iii) *$T : N_1 \to N_2$ is a unitary module map with respect to (V_z^*, V_w^*).*

(iv) $T : N_1 \to N_2$ *is a unitary module map with respect to* (V_z^*, V_w).

(v) *There is a unimodular function* $\psi(z)$ *satisfying* $Th = \psi(z)h$ *for* $h \in N_1$.

Proof. Conditions (iii) and (iv) are equivalent to that $T^* : N_2 \to N_1$ are unitary module maps with respect to (V_z, V_w) and (V_z, V_w^*), respectively. By Theorem 2.1, (iii) and (iv) are equivalent, and also they are equivalent to that $T^*h = \varphi(z)h, h \in N_2$, for a unimodular function $\varphi(z)$. Hence $Th_1 = \overline{\varphi(z)}h_1$ for every $h_1 \in N_1$. □

Corollary 2.3. *Let* N_1 *be a mixed invariant subspace of* H^2 *under* T_z *and* T_w^*. *Then*

$$orb_{(u,V_z,V_w^*)}(N_1) = orb_{(u,V_z,V_w)}(N_1) = orb_{(u,V_z^*,V_w^*)}(N_1) = orb_{(u,V_z^*,V_w)}(N_1)$$

and this family consists of mixed invariant subspaces N *of* H^2 *such that* $N = \psi(z)N_1$ *for some unimodular function* $\psi(z)$.

In the above argument, the condition of unitarity of the module map T is important. It seems difficult to describe similar-orbits of N_1 generally, so we study for a special case of N_1 with $dim\,(N_1 \ominus zN_1) = 1$, which is studied in [13].

Let Φ be the family of pairs $(a(z), b(z))$ in $H^\infty(z)$ satisfying $|a(z)| < 1$ a.e. on Γ and $|a(z)|^2 + |b(z)|^2 = 1$ a.e. on Γ. For $(a(z), b(z)) \in \Phi$, we write

$$N = N_{(a,b)} = GH^2(z), \quad \text{where} \quad G = \frac{b(z)}{1 - wa(z)}.$$

By [13, Theorems 2.4 and 3.2], N is a mixed invariant subspace of H^2 under T_z and T_w^* with $N \ominus zN = \mathbb{C} \cdot G$, and $a(z)$ is constant if and only if $[V_z, V_w] = 0$. We note that $\|G\| = 1$,

$$\|\xi(z)G\| = \|\xi(z)\| \quad \text{and} \quad \langle \xi(z)G, \eta(z)G \rangle = \langle \xi(z), \eta(z) \rangle, \tag{2.3}$$

$$V_w^*(\xi(z)G) = a(z)\xi(z)G, \tag{2.4}$$

and $V_z^*(\xi(z)G) = (T_z^*\xi(z))G$ for every $\xi(z), \eta(z) \in H^2(z)$. Moreover by [13, Lemma 5.1] we have

$$V_w(\xi(z)G) = (T_a^*\xi(z))G. \tag{2.5}$$

Lemma 2.4. *Let*

$$N_1 = N_{(a_1,b_1)} = G_1H^2(z), \quad G_1 = \frac{b_1(z)}{1 - wa_1(z)}$$

for some $(a_1(z), b_1(z)) \in \Phi$ *and* N_2 *be a mixed invariant subspace of* H^2 *under* T_z *and* T_w^*. *Let* $T : N_1 \to N_2$ *be a one to one bounded linear map with dense range. If* $TV_z^{(1)} = V_z^{(2)}T$, *then*

$$N_2 = N_{(a_2,b_2)} = G_2H^2(z), \quad G_2 = \frac{b_2(z)}{1 - wa_2(z)}$$

for some $(a_2(z), b_2(z)) \in \Phi$ *and there is an outer function* $h(z)$ *in* $H^\infty(z)$ *satisfying* $T(\xi(z)G_1) = h(z)\xi(z)G_2$ $\xi(z) \in H^2(z)$ *and* $T^*(\eta(z)G_2) = (T_h^*\eta(z))G_1$ *for every* $\xi(z), \eta(z) \in H^2(z)$.

Proof. We have $T(zN_1) = zTN_1 \subset zN_2$. Since $\mathbb{C} \cdot TG_1 + zTN_1$ is dense in N_2, $\mathbb{C} \cdot TG_1 + zN_2$ is dense in N_2. Hence $dim\,(N_2 \ominus zN_2) = 1$. When $[V_z^{(2)}, V_w^{(2)}] = 0$, by [13, Theorem 2.4] there exist an inner function $q(z)$ and $c \in D$ satisfying

$$N_2 = G_2 H^2(z), \quad \text{where} \quad G_2 = \frac{\sqrt{1-|c|^2}q(z)}{1-cw}.$$

Here we used condition $dim\,(N_2 \ominus zN_2) = 1$. Write $a_2(z) = c$ and $b_2(z) = \sqrt{1-|c|^2}q(z)$. Then $(a_2(z), b_2(z)) \in \Phi$ and $N_2 = N_{(a_2,b_2)}$.

Suppose that $[V_z^{(2)}, V_w^{(2)}] \neq 0$. By [13, Theorem 3.2], there exists $(a_2(z), b_2(z)) \in \Phi$ such that $a_2(z)$ is nonconstant and

$$N_2 = N_{(a_2,b_2)} = G_2 H^2(z), \quad \text{where} \quad G_2 = \frac{b_2(z)}{1 - wa_2(z)}.$$

Since $TG_1 \in N_2$, there is $h(z) \in H^2(z)$ with $TG_1 = h(z)G_2$. For $\xi(z) \in H^2(z)$, we have $T(\xi(z)G_1) = h(z)\xi(z)G_2$. By (2.3), it is not difficult to see that $h(z)H^2(z)$ is dense in $H^2(z)$, so $h(z)$ is an outer function in $H^\infty(z)$. For $\eta(z) \in H^2(z)$, we have

$$\begin{aligned} \big\langle T^*(\eta(z)G_2), \xi(z)G_1 \big\rangle &= \big\langle \eta(z)G_2, h(z)\xi(z)G_2 \big\rangle \\ &= \big\langle \overline{h(z)}\eta(z)G_2, \xi(z)G_2 \big\rangle \\ &= \big\langle (T_h^*\eta(z))G_2, \xi(z)G_2 \big\rangle \qquad \text{by (2.5).} \end{aligned}$$

Thus we get $T^*(\eta(z)G_2) = (T_h^*\eta(z))G_2$. □

Theorem 2.5. *Let*

$$N_1 = N_{(a_1,b_1)} = G_1 H^2(z), \quad G_1 = \frac{b_1(z)}{1 - wa_1(z)}$$

for some $(a_1(z), b_1(z)) \in \Phi$ and N_2 be a mixed invariant subspace of H^2 under T_z and T_w^*. Let $T : N_1 \to N_2$ be a one to one bounded linear map with dense range. If T is a module map with respect to (V_z, V_w^*), then there exists $b_2(z) \in H^\infty(z)$ satisfying $(a_1(z), b_2(z)) \in \Phi$ and

$$N_2 = N_{(a_1,b_2)} = G_2 H^2(z), \quad \textit{where} \quad G_2 = \frac{b_2(z)}{1 - wa_1(z)},$$

and there exists an outer function $h(z) \in H^\infty(z)$ satisfying

$$T(\xi(z)G_1) = h(z)\xi(z)G_2 = \frac{h(z)b_2(z)}{b_1(z)}\xi(z)G_1$$

for every $\xi(z) \in H^2(z)$.

Proof. By Lemma 2.4, we have

$$N_2 = N_{(a_2,b_2)} = G_2 H^2(z), \quad G_2 = \frac{b_2(z)}{1 - wa_2(z)}$$

for some $(a_2(z), b_2(z)) \in \Phi$, and there is an outer function $h(z) \in H^\infty(z)$ satisfying $T(\xi(z)G_1) = h(z)\xi(z)G_2$ for every $\xi(z) \in H^2(z)$. By (2.4),

$$TV_w^{(1)*}G_1 = T(a_1(z)G_1) = h(z)a_1(z)G_2.$$

Also we have

$$V_w^{(2)*}TG_1 = V_w^{(2)*}(h(z)G_2) = h(z)a_2(z)G_2.$$

Since $TV_w^{(1)*} = V_w^{(2)*}T$, we get $a_1(z) = a_2(z)$. Hence

$$T(\xi(z)G_1) = h(z)\xi(z)\frac{b_2(z)}{1 - wa_2(z)} = \frac{h(z)b_2(z)}{b_1(z)}\xi(z)G_1.$$

□

Theorem 2.6. *Let*

$$N_1 = N_{(a_1,b_1)} = G_1H^2(z), \quad G_1 = \frac{b_1(z)}{1 - wa_1(z)}$$

for some $(a_1(z), b_1(z)) \in \Phi$ and N_2 be a mixed invariant subspace of H^2 under T_z and T_w^*. Let $T : N_1 \to N_2$ be a one to one bounded linear map with dense range. If T is a module map with respect to (V_z, V_w), then there exists $b_2(z) \in H^\infty(z)$ satisfying $(a_1(z), b_2(z)) \in \Phi$ and

$$N_2 = N_{(a_1,b_2)} = G_2H^2(z), \quad \textit{where} \quad G_2 = \frac{b_2(z)}{1 - wa_1(z)},$$

and there exists an outer function $h(z) \in H^\infty(z)$ satisfying

$$T(\xi(z)G_1) = h(z)\xi(z)G_2 = \frac{h(z)b_2(z)}{b_1(z)}\xi(z)G_1$$

for every $\xi(z) \in H^2(z)$. Moreover if $a_1(z)$ is nonconstant, then $h(z)$ is a nonzero constant function.

Proof. By Lemma 2.4,

$$N_2 = N_{(a_2,b_2)} = G_2H^2(z), \quad \text{where} \quad G_2 = \frac{b_2(z)}{1 - wa_2(z)}$$

for some $(a_2(z), b_2(z)) \in \Phi$, and $T(\xi(z)G_1) = h(z)\xi(z)G_2, \xi(z) \in H^2(z)$ for an outer function $h(z) \in H^\infty(z)$. By (2.5), we have $TV_w^{(1)}(\xi(z)G_1) = h(z)(T_{a_1}^*\xi(z))G_2$ and

$$V_w^{(2)}T(\xi(z)G_1) = V_w^{(2)}(h(z)\xi(z)G_2) = \big(T_{a_2}^*(h(z)\xi(z))\big)G_2.$$

Since $TV_w^{(1)} = V_w^{(2)}T$, we have $h(z)T_{a_1}^*\xi(z) = T_{a_2}^*(h(z)\xi(z))$ for every $\xi(z) \in H^2(z)$. Hence $T_hT_{a_1}^* = T_{a_2}^*T_h$ on $H^2(z)$. Therefore $T_hT_{\overline{a_1}} = T_{\overline{a_2}h}$ on $H^2(z)$. By the Brown-Halmos theorem (see [7]), either $\overline{h(z)} \in H^\infty(z)$ or $\overline{a_1(z)} \in H^\infty(z)$, so either $h(z)$ or $a_1(z)$ is constant.

If $h(z) = c$ for some $c \in \mathbb{C}$, since T has dense range, $c \neq 0$ and $T_{c\overline{a_1}} = T_{c\overline{a_2}}$. Hence $a_1(z) = a_2(z)$.

If $a_1(z) = d, d \in \mathbb{C}$, then $T_{\overline{d}h} = T_{\overline{a_2}h}$. Moreover if $d = 0$, then $a_2(z) = 0$ and this is a contradiction. If $d \neq 0$, then $a_2(z) = d$, so $a_1(z) = a_2(z)$. Thus we get the assertion. □

Theorem 2.7. *Let*

$$N_1 = N_{(a_1,b_1)} = G_1 H^2(z), \quad G_1 = \frac{b_1(z)}{1 - wa_1(z)}$$

for some $(a_1(z), b_1(z)) \in \Phi$ and N_2 be a mixed invariant subspace of H^2 under T_z and T_w^*. Let $T : N_1 \to N_2$ be an invertible bounded linear map. If T is a module map with respect to (V_z^*, V_w), then there exists $b_2(z) \in H^\infty(z)$ satisfying $(a_1(z), b_2(z)) \in \Phi$ and

$$N_2 = N_{(a_1,b_2)} = G_2 H^2(z), \quad \textit{where} \quad G_2 = \frac{b_2(z)}{1 - wa_1(z)},$$

and there exists an invertible outer function $h(z) \in H^\infty(z)$ satisfying $T(\xi(z)G_1) = (T_h^*\xi(z))G_2$ for every $\xi(z) \in H^2(z)$.

Proof. Since $TV_z^{(1)*} = V_z^{(2)*}T$, we have $V_z^{(2)*}TG_1 = 0$, so $TG_1 \in N_2 \ominus zN_2$. Suppose that $N_2 \ominus zN_2 \neq \mathbb{C} \cdot TG_1$. Then there exists a nonzero $F_2 \in N_2 \ominus zN_2$ with $F_2 \perp \mathbb{C} \cdot TG_1$. Since T is invertible, there is $F_1 \in N_1$ with $TF_1 = F_2$. Then $TV_z^{(1)*}F_1 = V_z^{(2)*}TF_1 = 0$, so $V_z^{(1)*}F_1 = 0$. Thus we get $F_1 \in N_1 \ominus zN_1$. Since $N_1 \ominus zN_1 = \mathbb{C} \cdot G_1$, we have $F_1 = cG_1$, and $F_2 = TF_1 = cTG_1$. But this is a contradiction. Thus $dim\,(N_2 \ominus zN_2) = 1$. By [13, Theorems 2.4 and 3.2], there exists $(a_2(z), b_2(z)) \in \Phi$ satisfying

$$N_2 = N_{(a_2,b_2)} = G_2 H^2(z), \quad \text{where} \quad G_2 = \frac{b_2(z)}{1 - wa_2(z)}.$$

We have $V_z^{(1)}T^* = T^*V_z^{(2)}$ and $V_w^{(1)*}T^* = T^*V_w^{(2)*}$. By Theorem 2.5, we have $a_1(z) = a_2(z)$ and there is an outer function $h(z) \in H^\infty(z)$ satisfying

$$T^*(\eta(z)G_2) = \frac{h(z)b_1(z)}{b_2(z)}\eta(z)G_2$$

for every $\eta(z) \in H^2(z)$. Note that $|b_1(z)| = |b_2(z)|$ a.e. on Γ. For $\xi(z) \in H^2(z)$, we have

$$\begin{aligned}
\big\langle T(\xi(z)G_1), \eta(z)G_2 \big\rangle &= \big\langle \xi(z)G_1, T^*(\eta(z)G_2) \big\rangle \\
&= \Big\langle \xi(z)G_1, \frac{h(z)b_1(z)}{b_2(z)}\eta(z)G_2 \Big\rangle \\
&= \Big\langle \overline{h(z)}\xi(z)\frac{b_2(z)}{b_1(z)}G_1, \eta(z)G_2 \Big\rangle \\
&= \big\langle \overline{h(z)}\xi(z)G_2, \eta(z)G_2 \big\rangle \\
&= \big\langle (T_h^*\xi(z))G_2, \eta(z)G_2 \big\rangle.
\end{aligned}$$

Thus we get $T(\xi(z)G_1) = (T_h^*\xi(z))G_2$. Since T is invertible, T_h^* is invertible on $H^2(z)$. By [7, p. 140], $h(z)$ is invertible in $H^\infty(z)$. □

Theorem 2.8. *Let*

$$N_1 = N_{(a_1,b_1)} = G_1 H^2(z), \quad G_1 = \frac{b_1(z)}{1 - wa_1(z)}$$

for some $(a_1(z), b_1(z)) \in \Phi$ and N_2 be a mixed invariant subspace of H^2 under T_z and T_w^*. Let $T : N_1 \to N_2$ be an invertible bounded linear map. If T is a module map with respect to (V_z^*, V_w^*), then there exists $b_2(z) \in H^\infty(z)$ satisfying $(a_1(z), b_2(z)) \in \Phi$ and

$$N_2 = N_{(a_1,b_2)} = G_2 H^2(z), \quad \textit{where} \quad G_2 = \frac{b_2(z)}{1 - wa_1(z)},$$

and there exists an invertible outer function $h(z) \in H^\infty(z)$ satisfying $T(\xi(z)G_1) = (T_h^*\xi(z))G_2$ for every $\xi(z) \in H^2(z)$. Moreover if $a_1(z)$ is nonconstant, $h(z)$ is a nonzero constant function.

Proof. As the first paragraph of the proof of Theorem 2.7,

$$N_2 = N_{(a_2,b_2)} = G_2 H^2(z), \quad \text{where} \quad G_2 = \frac{b_2(z)}{1 - wa_2(z)}.$$

By the assumption, $T^* : N_2 \to N_1$ is an invertible bounded module map with respect to (V_z, V_w). Then by Theorem 2.6, $a_1(z) = a_2(z)$ and there is an outer function $h(z) \in H^\infty(z)$ satisfying

$$T^*(\eta(z)G_2) = \frac{h(z)b_1(z)}{b_2(z)}\eta(z)G_2$$

for every $\eta(z) \in H^2(z)$. By the second paragraph of the proof of Theorem 2.7, we have $T(\xi(z)G_1) = (T_h^*\xi(z))G_2$ for every $\xi(z) \in H^2(z)$. □

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