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COMMON INVARIANT SUBSPACES OF TWO DOUBLY COMMUTING OPERATORS ON $\ell^2 \otimes \mathbb{C}^2$

TAKAHIKO NAKAZI AND MICHIO SETO

ABSTRACT. In this paper, we study common invariant subspaces of $\mathbb{T}$ and $\mathbb{S}$ on $\ell^2 \otimes \mathbb{C}^2$ where $\mathbb{T} = T \otimes I_{\mathbb{C}^2}$ and $\mathbb{S} = I_{\ell^2} \otimes S$. We describe such invariant subspaces using T and S.

1. Introduction

Let $H = H_1 \otimes H_2$ be a Hilbert space where H_j is a Hilbert space for $j = 1, 2$. Let T_j be a bounded linear operator on H_j and I_j an identity operator on H_j. We will write

$$\mathbb{T}_1 = T_1 \otimes I_2 \quad \text{and} \quad \mathbb{T}_2 = I_1 \otimes T_2.$$

For $X = \mathbb{T}_1, \mathbb{T}_2, \mathbb{T}_1^*$ or $\mathbb{T}_2^*$, $\mathrm{Lat}X$ denotes the set of all invariant subspaces of X in H. In this paper, we are interested in $\mathrm{Lat}\mathbb{T}_1 \cap \mathrm{Lat}\mathbb{T}_2$ and $\mathrm{Lat}\mathbb{T}_1^* \cap \mathrm{Lat}\mathbb{T}_2^*$.

For M in $\mathrm{Lat}\mathbb{T}_1 \cap \mathrm{Lat}\mathbb{T}_2$ put

$$V_j = \mathbb{T}_j \mid M \quad (j = 1, 2).$$

For N in $\mathrm{Lat}\mathbb{T}_1^* \cap \mathrm{Lat}\mathbb{T}_2^*$, put

$$S_j^* = \mathbb{T}_j^* \mid N \quad (j = 1, 2).$$

For a closed subspace K in H, P_K denotes the orthogonal projection from H onto K. When $H = M \oplus N$, put

$$A = P_M \mathbb{T}_2 P_N \quad \text{and} \quad B = P_N \mathbb{T}_1^* P_M$$

then

$$\mathbb{T}_2 = \begin{bmatrix} V_2 & A \\ 0 & S_2 \end{bmatrix} \quad \text{and} \quad \mathbb{T}_1^* = \begin{bmatrix} V_1^* & 0 \\ B & S_1^* \end{bmatrix}.$$

Hence

$$\mathbb{T}_2 \mathbb{T}_1^* = \begin{bmatrix} V_2 V_1^* + AB & A S_1^* \\ S_2 B & S_2 S_1^* \end{bmatrix}$$

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and

$$\mathbb{T}_1^*\mathbb{T}_2 = \begin{bmatrix} V_1^*V_2 & V_1^*A \\ BV_2 & S_1^*S_2 + BA \end{bmatrix}.$$

Since $\mathbb{T}_2\mathbb{T}_1^* = \mathbb{T}_1^*\mathbb{T}_2$,

$$AB \mid M = V_1^*V_2 - V_2V_1^*$$

and

$$BA \mid M = S_2S_1^* - S_1^*S_2.$$

Thus $V_1^*V_2 = V_2V_1^*$ if and only if $AB = 0$, and $S_2S_1^* = S_1^*S_2$ if and only if $BA = 0$. If $A = 0$ then $V_1^*V_2 = V_2V_1^*$ and $S_2S_1^* = S_1^*S_2$.

H^2 denotes the usual Hardy space on the unit circle in $\mathbb{C}$ and q is called inner when q is a unimodular function in H^2. Such a problem has been studied in the following cases.

(1) $H_1 = H_2 = H^2$ and $T_1 = T_2$ are a usual shift on H^2 ([1],[2],[5],[6]).

(2) $H_1 = H_2 = H^2$ and $T_1 = T_2$ are a backward shift ([4]).

(3) $H_1 = H^2$ and $H_2 = \mathbb{C}^2$, and T_1 is the shift on H^2 and T_2 is the truncated shift on $\mathbb{C}^2$ ([3]).

Even if in very special examples, our problem is still very difficult. Our motivation is to make clear the causes by considering most special case. Hence we will not dare to generalize our results. In this paper, we assume that $\dim H_2 = 2$, that is, $H_2 = \mathbb{C}^2$. $\{e_1, e_2\}$ denotes the standard basis for $\mathbb{C}^2$, that is, $e_1 =^t [1, 0]$ and $e_2 =^t [0, 1]$. We will write $P_K = P_1$ for $K = H_1 \otimes [e_1]$ and $P_K = P_2$ for $K = H_1 \otimes [e_2]$. If T_2 is a bounded linear operator on $\mathbb{C}^2$ then we may assume that T_2 is a triangular matrix under the standard basis. In order to study $\mathrm{Lat}\mathbb{T}_1 \cap \mathrm{Lat}\mathbb{T}_2$ it is enough to consider when

$$T_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{or} \quad T_2 = \begin{bmatrix} 1 & x \\ 0 & 0 \end{bmatrix} \quad \text{for} \quad x \neq 0.$$

Then $\mathbb{T}_2^2 = 0$ or $\mathbb{T}_2^2 = \mathbb{T}_2$. In this paper, for arbitrary $\mathbb{T}_1$ we study $\mathrm{Lat}\mathbb{T}_1 \cap \mathrm{Lat}\mathbb{T}_2$ when $\mathbb{T}_2^2 = 0$ or $\mathbb{T}_2^2 = \mathbb{T}_2$. We determine $M \in \mathrm{Lat}\mathbb{T}_1 \cap \mathrm{Lat}\mathbb{T}_2$ when $A = 0$. Moreover, when $\mathbb{T}_1$ does not have orthogonal invariant subspaces, we show that $AB = 0$ if and only if $\mathrm{Lat}\mathbb{T}_1 \cap \mathrm{Lat}\mathbb{T}_2 = \mathrm{Lat}T_1 \otimes \mathrm{Lat}T_2$.

In this paper, $[S]$ denotes the closed linear span of a subset S in H. If $\mathbb{T}_2^2 = 0$ then $\mathbb{T}_2H = H_1 \otimes [e_1]\mathrm{Ker}\mathbb{T}_2 = H_1 \otimes [e_1]$ and $\mathrm{Ker}\mathbb{T}_2^* = H_1 \otimes [e_2]$, and if $\mathbb{T}_2^2 = \mathbb{T}_2$ then $\mathbb{T}_2H = H_1 \otimes [e_1]$, $\mathrm{Ker}\mathbb{T}_2 = H_1 \otimes [e_2 - xe_1]$ and $\mathrm{Ker}\mathbb{T}_2^* = H_1 \otimes [e_2]$. In general, if M is in $\mathrm{Lat}\mathbb{T}_1 \cap \mathrm{Lat}\mathbb{T}_2$ then $M = \mathrm{Ker}V_2^* \oplus [V_2M]$. It is clear that if $V_1V_2^* = V_2^*V_1$ then $V_1\mathrm{Ker}V_2^* \subseteq \mathrm{Ker}V_2^*$. This will be used several times in this paper.

The nilpotent case of $\mathbb{T}_2$ is studied in Section 2. The idempotent case of $\mathbb{T}_2$ is studied in Section 3. In Section 4 several concrete examples are given and it is noted that one of them can be applied to some invariant subspaces of the two variable Hardy space.

2. Nilpotent case

In this section, we assume that $T_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, that is, $\mathbb{T}_2^2 = 0$.

Theorem 2.1 *Suppose $M \in \mathrm{Lat}\mathbb{T}_1 \cap \mathrm{Lat}\mathbb{T}_2$, then the following are valid.*

(i) $M = M_2 \oplus [\mathbb{T}_2 M_2]$ *and* $[\mathbb{T}_2 M_2] = K_3 \otimes [e_1]$ *where* $K_3 \in \mathrm{Lat}T_1$.

(ii) $M_2 = M_0 \oplus M_2 \cap \mathrm{Ker}\mathbb{T}_2 \oplus M_2 \cap \mathrm{Ker}\mathbb{T}_2^*$, $M_2 \cap \mathrm{Ker}\mathbb{T}_2 = K_1 \otimes [e_1]$ *and* $M_2 \cap \mathrm{Ker}\mathbb{T}_2^* = K_2 \otimes [e_2]$ *where* $K_2 \in \mathrm{Lat}\mathbb{T}_1$, $K_2 \subseteq K_3$ *and* $K_1 \oplus K_3 \in \mathrm{Lat}T_1$.

(iii) $\dim M_0 = \dim P_1 M_0 = \dim P_2 M_0$.

(iv) $P_1 M_0 \subseteq (H_1 \ominus (K_1 \oplus K_3)) \otimes [e_1]$ *and* $P_2 M_0 \subseteq (K_3 \ominus K_2) \otimes [e_2]$.

(v) M=*[RangeA]*$\oplus M \cap \mathrm{Ker}A^*$ *where* $M \cap \mathrm{Ker}A^* = \{f \otimes e_1 + g \otimes e_2 \in M : f \otimes e_2 \in M\}$ *and* $M \cap \mathrm{Ker}A^* \in \mathrm{Lat}\mathbb{T}_1 \cap \mathrm{Lat}\mathbb{T}_2^*$. *Hence* $M \cap \mathrm{Ker}A^* \supseteq K_2 \otimes [e_2]$.

Proof. (i) Put $M_2 = M \ominus [\mathbb{T}_2 M]$ then $\mathbb{T}_2 M = \mathbb{T}_2 M_2$ because $\mathbb{T}_2^2 M = [0]$. Since $[\mathbb{T}_2 M] = K_3 \otimes [e_1]$ and $\mathbb{T}_1 \mathbb{T}_2 = \mathbb{T}_2 \mathbb{T}_1$, K_3 belongs to $\mathrm{Lat}T_1$.

(ii) Since $\mathrm{Ker}\mathbb{T}_2 = H_1 \otimes [e_1]$ and $\mathrm{Ker}\mathbb{T}_2^* = H_1 \otimes [e_2]$, $M_2 = M_0 \oplus M_2 \cap \mathrm{Ker}\mathbb{T}_2 \oplus M_2 \cap \mathrm{Ker}\mathbb{T}_2^*$, and $M_2 \cap \mathrm{Ker}\mathbb{T}_2 = K_1 \otimes [e_1]$ and $M_2 \cap \mathrm{Ker}\mathbb{T}_2^* = K_2 \otimes [e_2]$. It is easy to see that $K_2 \in \mathrm{Lat}T_1, K_1 \perp K_3$ and $K_2 \subset K_3$. Since $M \cap \mathrm{Ker}\mathbb{T}_2 = (K_1 \oplus K_3) \otimes [e_1]$, $K_1 \oplus K_3 \in \mathrm{Lat}T_1$.

(iii) It is enough to show that if $\{f_\alpha \otimes e_1 + g_\alpha \otimes e_2\}_\alpha$ is a basis in M_0 then $\{f_\alpha \otimes e_1\}_\alpha$ is a basis in $P_1 M_0$ and $\{g_\alpha \otimes e_2\}_\alpha$ is in $P_2 M_0$. If $\{f_\alpha \otimes e_1\}_\alpha$ is not a basis in $P_1 M_0$ then there exists a nonzero $g_\alpha \otimes e_2$ in M_0. For if $g_\alpha = 0$ then $f_\alpha \otimes e_1 \in \mathrm{Ker}\mathbb{T}_2 \cap M_0 = [0]$. This contradiction implies that if $\{f_\alpha \otimes e_1 + g_\alpha \otimes e_2\}_\alpha$ is a basis in M_0 then $\{f_\alpha \otimes e_1\}_\alpha$ is a basis in $P_1 M_0$. Similarly we can show that if $\{f_\alpha \otimes e_1 + g_\alpha \otimes e_2\}_\alpha$ is in M_0 then $\{g_\alpha \otimes e_2\}_\alpha$ is a basis in $P_2 M_0$.

(iv) and (v) are clear. □

Corollary 2.1 *Suppose $M \in \mathrm{Lat}\mathbb{T}_1 \cap \mathrm{Lat}\mathbb{T}_2$. The following are valid.*

(i) $M_0 = [0]$ *if and only if* $M = (K_2 \otimes [e_2]) \oplus ((K_1 \oplus K_3) \otimes [e_1])$ *where* $K_2 = K_3$, $K_1 \perp K_3$ *and* $K_2, K_3, K_1 \oplus K_3 \in \mathrm{Lat}T_1$.

(ii) $M_2 = M_0$ *if and only if* $M = M_0 \oplus (K_3 \otimes [e_1])$ *where* $K_3 \in \mathrm{Lat}T_1$. *Then* $P_1 M_0 = K_4 \otimes [e_1]$, $P_2 M_0 = K_3 \otimes [e_2]$, $\dim K_4 = \dim K_3 = \dim M_0$ *and* $K_4 \perp K_3$.

(iii) *In (ii), for* $f \otimes e_1 + g \otimes e_2$ *in* M, *if* $T_1 f = 0$ *then* $T_1 g = 0$ *and if* $T_1 g = 0$ *then* $T_1 f \in K_3$.

(iv) $\mathbb{T}_2 M_2 = [0]$ *if and only if* $M = K_1 \otimes [e_1]$ *where* $K_1 \in \mathrm{Lat}T_1$.

(v) $M \neq M_0$.

Proof. (i) and (iv) are clear.

(ii) If $M = M_0 \oplus (K_3 \otimes [e_1])$ then $P_2 M_0 = K_3 \otimes [e_2]$ and $P_1 M_0 = K_4 \otimes [e_1]$ and $K_3 \perp K_4$. By (iii) of Theorem 2.1, $\dim K_4 = \dim K_3 = \dim M_0$.

(iii) Let $F = f \otimes e_1 + g \otimes e_2$ in $M = M_0 \oplus (K_3 \otimes [e_1])$. If $T_1 f = 0$ then $\mathbb{T}_1 F = T_1 g \otimes e_2 \in M$. By (ii) $T_1 g = 0$. If $T_1 g = 0$ then $\mathbb{T}_1 F = T_1 f \otimes e_1 \in M$. By (ii) $T_1 f \in K_3$.

(v) If $M = M_0$ then $M_0 \supset \mathbb{T}_2 M_2$ and so $\mathbb{T}_2 M_2 = [0]$. (iii) of Theorem 2.1 and the above (iv) imply $M \neq M_0$. □

Corollary 2.2 *Suppose $M \in \mathrm{Lat}\mathbb{T}_1 \cap \mathrm{Lat}\mathbb{T}_2$ then the following are equivalent.*

(i) $A = 0$.

(ii) $M \in \mathrm{Lat}\mathbb{T}_2^*$.

(iii) $M = K \otimes [e_1, e_2]$ *where* $K \in \mathrm{Lat}T_1$.

Proof. (i)⇔(ii) is a result of (v) of Theorem 2.1 because (i) is equivalent to $M \subset \mathrm{Ker}A^*$.

(ii)⇒(iii) If $f \otimes e_1 + g \otimes e_2 \in M$ then $f \otimes e_2$ and $g \otimes e_1$ belong to M. Hence both $f \otimes e_1$ and $g \otimes e_2$ belong to M. Thus $M = K \otimes [e_1, e_2]$.

(iii)⇒(ii) is clear. □

Corollary 2.3 *Suppose $M \in \mathrm{Lat}\mathbb{T}_1 \cap \mathrm{Lat}\mathbb{T}_2$, then the following are equivalent.*

(i) $[RangeA] = M$.

(ii) $\mathrm{Ker}A^* \cap M = [0]$.

(iii) $M_2 \cap \mathrm{Ker}\mathbb{T}_2^* = [0]$.

(iv) $M = M_0 \oplus \{(K_1 \oplus K_3) \otimes [e_1]\}$.

Proof. (i)⇔(ii)⇒(iii) is a result of (v) of Theorem 2.1.

(iii)⇒(iv) is a result of (i) and (ii) of Theorem 2.1.

(iv)⇒(ii) Since $\mathbb{T}_2^* M = \mathbb{T}_2^* M_0 \oplus ((K_1 \oplus K_3) \otimes [e_2])$, $\mathbb{T}_2^* M \cap M = [0]$ and so $\mathrm{Ker}A^* \cap M = [0]$. □

Theorem 2.2 *Suppose $M \in \mathrm{Lat}\mathbb{T}_1 \cap \mathrm{Lat}\mathbb{T}_2$.*

(i) *If $AB = 0$ then $M_2 = M_0 \oplus (K_1 \otimes [e_1]) \oplus (K_2 \otimes [e_2])$ where $K_j \in \mathrm{Lat}T_1$ for $j = 1, 2$.*

(ii) $AB = 0$ *on* $\mathrm{Ker}\mathbb{T}_2^* \cap M$.

(iii) *If* $A = 0$ *then* $M_0 = [0]$.

Proof. We will use the notations in Theorem 2.1.

(i) If $AB = 0$ then $V_2^*V_1 = V_1V_2^*$ and so $V_1M_2 \subset M_2$. $K_2 \in \mathrm{Lat}T_1$ by Theorem 2.1 and $K_1 \in \mathrm{Lat}T_1$ by that $(T_1K_1) \otimes e_1 \subset M_2$.

(ii) $\mathrm{Ker}\mathbb{T}_2^* \cap M = K \otimes [e_2]$ and $V_2^*(K \otimes [e_2]) = 0$. Hence $(V_1V_2^* - V_2^*V_1)(K \otimes [e_2]) = -V_2^*(T_1K \otimes [e_2]) = 0$ because $K \in \mathrm{Lat}T_1$.

(iii) Corollaries 2.1 and 2.2 show (iii). □

Corollary 2.4 *Suppose* T_1 *does not have orthogonal invariant subspaces. When* $M \in \mathrm{Lat}\mathbb{T}_1 \cap \mathrm{Lat}\mathbb{T}_2$, $AB = 0$ *if and only if* $M = K \otimes [e_1]$ *or* $K \otimes [e_1, e_2]$ *for some* $K \in \mathrm{Lat}T_1$.

Proof. Since $AB|M = V_1V_2^* - V_2^*V_1$, it is easy to see the 'if' part and so it is enough to show the 'only if' part. If $AB = 0$ then $V_1V_2^* = V_2^*V_1$ and so $V_1\mathrm{Ker}V_2^* \subseteq \mathrm{Ker}V_2^*$. If $f \otimes e_1 + g \otimes e_2 \in M_2$ then $f \otimes e_1 \perp \mathbb{T}_2M_2$. Since $\mathbb{T}_1(f \otimes e_1 + g \otimes e_2) \in M_2$ and $[\mathbb{T}_2M_2] = K \otimes [e_1]$ for some $K \in \mathrm{Lat}T_1$, $\bigcup_{n=0}^{\infty} T_1^n f$ is orthogonal to K. If $f \neq 0$ then $K = [0]$ by hypothesis on $\mathrm{Lat}T_1$ and so $\mathbb{T}_2M = [0]$. Hence $M = K' \otimes [e_1]$ for some $K' \in \mathrm{Lat}T_1$. If there does not exist f such that $f \neq 0$ whenever $f \otimes e_1 + g \otimes e_2 \in M_2$, then $M_2 = K'' \otimes [e_2]$ for some $K'' \in \mathrm{Lat}T_1$ and so $M = K'' \otimes [e_1, e_2]$. □

3. Idempotent case

In this section, we assume that $T_2 = \begin{bmatrix} 1 & x \\ 0 & 0 \end{bmatrix}$, that is, $\mathbb{T}_2^2 = \mathbb{T}_2$. If $x = 0$ then everything is trivial and so we assume $x \neq 0$.

Theorem 3.1 *Suppose* $M \in \mathrm{Lat}\mathbb{T}_1 \cap \mathrm{Lat}\mathbb{T}_2$. *then the following are valid.*

(i) $M = M_2 \oplus [\mathbb{T}_2M], M_2 = M_2' \oplus \mathrm{Ker}\mathbb{T}_2^* \cap M_2$ *and* $M_2' = M_0 \oplus \mathrm{Ker}\mathbb{T}_2 \cap M_2'$.

(ii) $[\mathbb{T}_2M] = K_3 \otimes [e_1]$, $\mathrm{Ker}\mathbb{T}_2^* \cap M_2 = K_2 \otimes [e_2]$ *and* $\mathrm{Ker}\mathbb{T}_2 \cap M_2' = K_1 \otimes [e_2 - xe_1]$. *Here* $K_2 \subset K_3, K_1 \perp K_3$ *where* $K_3 \in \mathrm{Lat}T_1$ *and* $K_2 \in \mathrm{Lat}T_1$

(iii) $\dim M_0 = \dim P_1M_0 = \dim P_2M_0$

(iv) $P_1M_0 \subseteq (H_1 \ominus K_3) \otimes [e_1]$ *and* $P_2M_0 \subseteq (H_1 \ominus K_2) \otimes [e_2)$

(v) $M = [\mathit{RangA}] \oplus M \cap \mathrm{Ker}A^*$ *where* $M \cap \mathrm{Ker}A^* = \{f \otimes e_1 + g \otimes e_2 \in M : f \otimes (e_1 + \bar{x}e_2) \in M\}$ *and* $M \cap \mathrm{Ker}A^* \in \mathrm{Lat}\mathbb{T}_1 \cap \mathrm{Lat}\mathbb{T}_2^*$. *Moreover* $M \cap \mathrm{Ker}A^* \supset K_2 \otimes [e_2]$.

Proof. (i) is clear.

(ii) The first part is clear. Since $\mathbb{T}_2(K_2 \otimes [e_2]) \subseteq K_3 \otimes [e_1]$, $K_2 \subseteq K_3$. Since $[\mathbb{T}_2 M] \perp \mathrm{Ker}\mathbb{T}_2 \cap M_2', K_1 \perp K_3$ because $x \neq 0$.

(iii) It is enough to show that if $\{f_\alpha \otimes e_1 + g_\alpha \otimes e_2\}_\alpha$ is a basis in M_0 then $\{f_\alpha \otimes e_1\}_\alpha$ is a basis in $P_1 M_0$ and $\{g_\alpha \otimes e_2\}_\alpha$ is a basis in $P_2 M_0$. If $\{f_\alpha \otimes e_1\}_\alpha$ is not a basis in $P_1 M_0$ then there exists a nonzero $g_\alpha \otimes e_2$ in M_0. For if $g_\alpha = 0$ then $f_\alpha \otimes e_1 \in M_0 \cap \mathrm{Ker}\mathbb{T}_2^* = [0]$. This contradiction implies that if $\{f_\alpha \otimes e_1 + g_\alpha \otimes e_2\}_\alpha$ is a basis in M_0 then $\{f_\alpha \otimes e_1\}_\alpha$ is a basis in $P_1 M_0$. If $\{g_\alpha \otimes e_2\}_\alpha$ is not a basis in $P_2 M_0$ then there exists a nonzero $f_\alpha \otimes e_1$ in M_0 and so $f_\alpha \in K_3$. By the definitions of M_0 and $K_3 \otimes [e_1], f_\alpha \perp K_3$. This implies $f_\alpha = 0$.

(iv) and (v) are clear. □

Corollary 3.1 *Suppose $M \in \mathrm{Lat}\mathbb{T}_1 \cap \mathrm{Lat}\mathbb{T}_2$. Then the following are valid.*

(i) $M_0 = [0]$ *if and only if* $M = (K_1 \otimes [e_2 - xe_1]) \oplus (K_2 \otimes [e_2]) \oplus (K_3 \otimes [e_1])$ *where* $K_j \in \mathrm{Lat}T_1$ $(j = 2, 3)$, $K_2 = K_3$ *and* $K_1 \perp K_3$. *Hence if* $M_0 = [0]$ *then* $T_1 M_2 \subseteq M_2$.

(ii) $M_2 = M_0$ *if and only if* $M = M_0 \oplus (K_3 \otimes [e_1])$ *where* $K_3 \in \mathrm{Lat}T_1$. *Then* $P_1 M_0 = K_5 \otimes [e_1]$, $P_2 M_0 = K_4 \otimes [e_2]$, $\dim K_5 = \dim K_4 = \dim M_0$, $K_5 \perp K_3$ *and* $K_4 + xK_5 = K_3$.

(iii) *In (ii), if* $M_0 \neq [0]$ *then* $K_4 \not\subset K_3$ *and* $K_5 \not\subset K_3$.

(iv) $\mathbb{T}_2 M_2 = [0]$ *if and only if* $M = (K_1 \otimes [e_2 - xe_1]) \oplus (K_3 \otimes [e_1])$ *where* $K_1, K_3 \in \mathrm{Lat}T_1$ *and* $K_1 \perp K_3$.

(v) $\mathbb{T}_2 M = [0]$ *if and only if* $M = K_1 \otimes [e_2 - xe_1]$ *for* $K_1 \in \mathrm{Lat}T_1$.

Proof. It is clear except (iii) and (iv). (iii) Suppose $K_4 \subset K_3$. Then $K_4 \perp K_5$ and if $F \in K_4$ then $F = f + xg$ for some $f \in K_4$ and $g \in K_5$ by (ii). Hence $F - f = xg \in K_4 \cap K_5 = [0]$. Since $x \neq 0$, $g = 0$ and by (iii) of Theorem 3.1 $f = F = 0$ and so $M_0 = [0]$. This contradiction implies $K_4 \not\subset K_3$.

(iv) If $f \in K_1$ then

$$T_1 f \otimes (e_2 - xe_1) = f_1 \otimes (e_2 - xe_1) + f_2 \otimes e_1$$

where $f_1 \in K_1$ and $f_2 \in K_3$. Hence $T_1 f = f_1$ and $xT_1 f = xf_1 - f_2$. Therefore $f_2 = 0$ and $T_1 f = f_1 \in K_1$. □

Corollary 3.2 *Suppose $M \in \mathrm{Lat}\mathbb{T}_1 \cap \mathrm{Lat}\mathbb{T}_2$. Then $A = 0$ if and only if $M = K \otimes [e_1, e_2]$ where $K \in \mathrm{Lat}T_1$.*

Proof. By (v) of Theorem 3.1, $A = 0$ if and only if $\mathbb{T}_2^* M \subset M$. Hence the 'if' part is clear. We will show the 'only if' part. Since $\mathbb{T}_2^* M \subset M$, $M = \mathrm{Ker}\mathbb{T}_2^* \cap M \oplus [\mathbb{T}_2 M] = (K_2 \otimes [e_2]) \oplus (K_3 \otimes [e_1])$ where $K_2 \subset K_3$ and $K_j \in \mathrm{Lat}T_1$ for $j = 1, 2$. Since $\mathbb{T}_2^* M \subset M$, $K_2 \supset K_3$ and so $K_2 = K_3$. □

Theorem 3.2 *Suppose* $M \in \mathrm{Lat}\mathbb{T}_1 \cap \mathrm{Lat}\mathbb{T}_2$.

(i) *If* $AB = 0$ *then* $T_1 P_1 M_0 \subseteq P_1 M_0 + K_1 \otimes [e_1]$ *and* $T_1 P_2 M_0 \subseteq P_2 M_0 + K_2 \otimes [e_2]$ *where* M_0, K_1 *and* K_2 *are defined in Theorem 3.1.*

(ii) $AB = 0$ *on* $\mathrm{Ker}\mathbb{T}_2^* \cap M$

(iii) *If* $M = K \otimes [e_1]$, *or* $M = K \otimes [e_1, e_2]$ *for some* $K \in \mathrm{Lat}T_1$ *then* $AB = 0$.

(iv) *If* $A = 0$ *then* $M_0 = [0]$.

Proof. (i) If $AB = 0$ then $V_1 V_2^* = V_2^* V_1$ and so $V_1 M_2 \subseteq M_2$. Since $M_2 = M_0 \oplus (K_2 \otimes [e_2]) \oplus (K_1 \otimes [e_2 - x e_1])$ by Theorem 3.1, $T_1 P_1 M_0 \subseteq P_1 M_0 + K_1 \otimes [e_1]$ and $T_1 P_2 M_0 \subseteq P_2 M_0 + K_2 \otimes [e_2]$

(ii) $V_1 V_2^* (\mathrm{Ker}\mathbb{T}_2^* \cap M) = V_1 P_M (\mathbb{T}_2^* (\mathrm{Ker}\mathbb{T}_2^* \cap M)) = [0]$. Since $\mathrm{Ker}\mathbb{T}_2^* \cap M \subset H_1 \otimes [e_2], V_2^* V_1 (\mathrm{Ker}\mathbb{T}_2^* \cap M) = [0]$ by Theorem 3.1. Since $AB \mid M = V_1 V_2^* - V_2^* V_1, AB = 0$ on $\mathrm{Ker}\mathbb{T}_2^* \cap M$.

(iii) By the proof of (ii) $AB = 0$ on $K \otimes [e_2]$. Hence we will prove $AB = 0$ on $K \otimes [e_1]$. If $f \otimes e_1 \in M$ then

$$V_1 V_2^* (f \otimes e_1) = V_1 (f \otimes e_1 + \bar{x} f \otimes e_2) = T_1 f \otimes e_1 + \bar{x} T_1 f \otimes e_2$$

and

$$V_2^* V_1 (f \otimes e_1) = V_2^* (T_1 f \otimes e_1) = T_1 f \otimes e_1 + \bar{x} T_1 f \otimes e_2.$$

Hence $AB = 0$ on $K \otimes [e_1]$.

(iv) Corollaries 3.1 and 3.2 show (iv). □

Corollary 3.3 *Suppose* T_1 *does not have orthogonal invariant subspaces and* $\mathbb{T}_2 M \neq [0]$. *When* $M \in \mathrm{Lat}\mathbb{T}_1 \cap \mathrm{Lat}\mathbb{T}_2$, $AB = 0$ *if and only if* $M = K \otimes [e_1]$ *or* $M = K \otimes [e_1, e_2]$ *for some* $K \in \mathrm{Lat}T_1$.

Proof. By (iii) of Theorem 3.2, it is enough to show the 'only if' part. If $AB = 0$ then $\mathbb{T}_1 M_2 \subseteq M_2$. Suppose $[\mathbb{T}_2 M] = K_3 \otimes [e_1]$. If $f \otimes e_1 + g \otimes e_2 \in M_2$ then $T_1 f \otimes e_1 + T_1 g \otimes e_2 \in M_2$ and so $T_1 f \perp K_3$. If there exists a nonzero f such that $f \otimes e_1 + g \otimes e_2 \in M_2$ then there exists $K_3' \in \mathrm{Lat}T_1$ such that $K_3' \perp K_3$ as in the proof of Theorem 3.1. The hypothesis on T_1 implies that $K_3 = [0]$. Hence it contradicts $\mathbb{T}_2 M \neq [0]$. Hence there does not exist any nonzero f such that $f \otimes e_1 + g \otimes e_2 \in M_2$, that is, $M_2 \subseteq H_1 \otimes [e_2]$ and so $M_2 = \mathrm{Ker}\mathbb{T}_2^* \cap M$ then $M = (K_2 \otimes [e_2]) \oplus (K_3 \otimes [e_1])$

and $K_2 \subseteq K_3$. If $K_2 = [0]$ then $M = K_3 \otimes [e_1]$. If $K_2 \neq [0]$ we will show $K_2 = K_3$. If $f \in K_3 \ominus K_2$ is nonzero then

$$V_2^* V_1(f \otimes e_1) = V_2^*(T_1 f \otimes e_1) = T_1 f \otimes e_1 + P_M(\bar{x} T_1 f \otimes e_2)$$

and

$$V_1 V_2^*(f \otimes e_1) = T_1 f \otimes e_1 + T_1 P_M(\bar{x} f \otimes e_2).$$

Since $V_2^* V_1 = V_1 V_2^*$, $P_M(T_1 f \otimes e_2) = T_1 P_M(f \otimes e_2)$. Since $f \otimes e_2 \perp K_2 \otimes e_2$, $f \otimes e_2 \perp M$ and so $P_M(f \otimes e_2) = 0$. Hence $T_1 f \otimes e_1 \in M$ and $T_1 f \otimes e_2 \perp M = (K_2 \otimes [e_2]) \oplus (K_3 \otimes [e_1])$. Therefore $T_1 f \in K_3 \ominus K_2$. This contradicts the hypothesis on T_1. □

4. Examples

In this section we give several concrete examples for the theorems in Sections 2 and 3.

Example 4.1 Suppose $H_1 = \mathbb{C}^n = [f_1, \ldots, f_n, 0]$ where $\{f_j\}_{j=1}^n$ is a standard basis and $T_1 f_j = f_{j+1}$ for $1 \leq j \leq n$ where $f_{n+1} = 0$. Suppose $\mathbb{T}_2^2 = 0$. If $M \in \mathrm{Lat}\mathbb{T}_1 \cap \mathrm{Lat}\mathbb{T}_2$ then by Theorem 2.1 $\mathbb{T}_2 M = [f_t, \ldots, f_{n+1}] \otimes [e_1]$, $M_2 \cap \mathrm{Ker}\mathbb{T}_2^* = [f_s, \ldots, f_{n+1}] \otimes [e_2]$ with $s \geq t$, $M_2 \cap \mathrm{Ker}\mathbb{T}_2 = [f_\ell, \ldots, f_{t-1}] \otimes [e_1]$, and

$$M_0 \subseteq ([f_1, \ldots, f_{\ell-1}] \otimes [e_1]) \oplus ([f_t, \ldots, f_{s-1}] \otimes [e_2]).$$

If $M_2 = M_0$ then $M_2 \cap \mathrm{Ker}\mathbb{T}_2^* = M_2 \cap \mathrm{Ker}\mathbb{T}_2 = [0]$, and so $P_1 M_0 \subseteq [f_1, \ldots, f_{t-1}] \otimes [e_1]$ and $P_2 M_0 = [f_t, \ldots, f_{n+1}] \otimes [e_2]$. Hence $t-1 \geq n-t+1$ and so $2t \geq n+2$. $M_0 = [0]$ if and only if $s = t$, that is, $M = ([f_\ell, \ldots, f_{n+1}] \otimes [e_1]) \oplus ([f_s, \ldots, f_{n+1}] \otimes [e_2])$ where $\ell \leq s$. By Corollary 2.3, $M = [\mathrm{Ran}\ A]$ if and only if $M = M_0 \oplus [f_\ell, \ldots, f_{n+1}] \otimes [e_1]$. By Corollary 2.2, $A = 0$ if and only if $M = [f_\ell, \ldots, f_{n+1}] \otimes [e_1, e_2]$. By Corollary 3.2, $AB = 0$ if and only if $M = [f_s, \ldots, f_{n+1}] \otimes [e_1]$ or $M = [f_s, \ldots, f_{n+1}] \otimes [e_1, e_2]$.

We consider when $n = 2$. We assume $M \neq [0]$. If $\mathbb{T}_2 M_2 = [f_1, f_2] \otimes [e_1]$ then $M = H$. Suppose $\mathbb{T}_2 M_2 = [f_2] \otimes [e_1]$. If $M_0 = [0]$ then $M = [f_2] \otimes [e_2]$ or $M = [f_2] \otimes [e_1, e_2]$. If $M_0 \neq [0]$ then $M_0 = [f_2 \otimes (\alpha_1 e_1 + \alpha_2 e_2)]$ where $\alpha_1 \neq 0$ and $\alpha_2 \neq 0$, and so $M = \{[f_2 \otimes (\alpha_1 e_1 + \alpha_2 e_2)]\} \oplus ([f_2] \otimes [e_1])$.

Example 4.2 Suppose $\mathbb{T}_2^2 = \mathbb{T}_2$ in Example 4.1. If $M \in \mathrm{Lat}\mathbb{T}_1 \cap \mathrm{Lat}\mathbb{T}_2$ then by Theorem 3.1 $\mathbb{T}_2 M = [f_t, \ldots, f_{n+1}] \otimes [e_1]$, $\mathrm{Ker}\mathbb{T}_2^* \cap M_2 = [f_s, \ldots, f_{n+1}] \otimes [e_2] (s \geq t)$ and $\mathrm{Ker}\mathbb{T}_2 \cap M_2' \subseteq [f_m, \ldots, f_{t-1}] \otimes [e_2 - x e_1]$. Hence $M_2' = M_0 \oplus [f_m, \ldots, f_{t-1}] \otimes [e_2 - x e_1]$ and

$$M_2' \subseteq ([f_1, \ldots, f_{t-1}] \otimes [e_1]) \oplus ([f_1, \ldots, f_{s-1}] \otimes [e_2]).$$

Therefore

$$\begin{aligned} &\dim M_0 = \dim P_1 M_0 = \dim P_2 M_0 \leq \dim M_2' \\ &\leq \min(t-1, s-1). \end{aligned}$$

$M_0 = [0]$ if and only if $M = ([f_m, \dots, f_{t-1}] \otimes [e_2 - xe_1]) \oplus ([f_t, \dots, f_{n+1}] \otimes [e_2]) \oplus ([f_t, \dots, f_{n+1}] \otimes [e_1])$. By Corollary 3.2 $A = 0$ if and only if $M = [f_t, \dots, f_{n+1}] \otimes [e_1, e_2]$. By Corollary 3.3 when $\mathbb{T}_2 M \neq [0]$, $AB = 0$ if and only if $M = [f_t, \dots, f_{n+1}] \otimes [e_1], [f_t, \dots, f_{n+1}] \otimes [e_1, e_2]$.

We consider when $n = 2$. If $\mathbb{T}_2 M = [f_1, f_2] \otimes [e_1]$ then $M_2 = [0]$, $M_2 = [f_2] \otimes [e_2]$ or $M_2 = [f_1, f_2] \otimes [e_2]$. If $\mathbb{T}_2 M = [f_2] \otimes [e_1]$ then $\mathrm{Ker}\mathbb{T}_2^* \cap M_2 = [0]$ or $[f_2] \otimes [e_2]$. If $\mathrm{Ker}\mathbb{T}_2^* \cap M_2 = [f_2] \otimes [e_2]$ then $M_2' \subseteq [f_1] \otimes [e_1, e_2]$. If $f_1 \otimes (\alpha_1 e_1 + \alpha_2 f_2) \in M_2'$ then $\alpha_1 + x\alpha_2 = 0$ because $\mathbb{T}_2 M_2' \subseteq \mathbb{T}_2 M$. Therefore $M_2' = \mathrm{Ker}\mathbb{T}_2 \cap M$ and so $M = ([f_2] \otimes [e_1, e_2]) \oplus [f_1 \otimes (e_2 - xe_1)]$. If $\mathrm{Ker}\mathbb{T}_2^* \cap M_2 = [0]$ then $M = M_2' \oplus ([f_2] \otimes [e_1])$. Suppose $\alpha_1 f_1 \otimes e_1 + g \otimes e_2 \in M_2'$. Since $\mathbb{T}_1 M_2' \subset M$, $T_1 g \otimes e_2$ belongs to M and so $T_1 g \otimes e_2 \in \mathrm{Ker}\mathbb{T}_2^* \cap M_2$. Hence $T_1 g = 0$ and so $g = \alpha_2 f_2$. Since $\mathbb{T}_2 M_2' \subset [f_2] \otimes [e_1]$, $\alpha_1 = 0$. Therefore $M_2' = [f_2] \otimes [e_2]$ and so $M = [f_2] \otimes [e_1, e_2]$. $\mathbb{T}_2 M = [0]$ if and only if $M = [f_1, f_2] \otimes [e_2 - xe_1]$ or $[f_2] \otimes [e_2 - xe_1]$.

Example 4.3 Suppose $\{f_j\}_{j=1}^\infty$ is a standard orthogonal basis in $H_1 = \ell^2$ and T_1 is a unicellular weighted shift on $\{f_j\}_{j=1}^\infty$ and $f_\infty = 0$. Suppose $\mathbb{T}_2^2 = 0$. If $M \in \mathrm{Lat}T_1 \cap \mathrm{Lat}T_2$ then by Theorem 2.1 $[\mathbb{T}_2 M_2] = [f_s, f_{s+1}, \dots] \otimes [e_1]$, $\mathrm{Ker}\mathbb{T}_2^* \cap M_2 = [f_t, f_{t+1}, \dots] \otimes [e_2]$ for $t \geq s$ and $\mathrm{Ker}\mathbb{T}_2 \cap M_2 = [f_\ell, \dots, f_{s-1}] \otimes [e_1]$. Moreover $M_0 \subseteq ([f_1, \dots, f_{\ell-1}] \otimes [e_1]) \oplus ([f_s, \dots, f_{t-1}] \otimes [e_2])$ and $\dim M_0 \leq \min(\ell - 1, t - s)$. If $M_2 = M_0$ then $\dim M_0 = \infty$ because $P_2 M_0 = [f_s, f_{s+1}, \dots] \otimes [e_2]$. On the other hand, $\dim P_1 M_0 < \infty$ because $P_1 M_0 \subseteq [f_1, \dots, f_{s-1}] \otimes [e_1]$. This contradiction shows that $M_2 \neq M_0$ and $\mathrm{Ker}\mathbb{T}_2^* \cap M_2 \neq [0]$. $M_0 = [0]$ if and only if $M = ([f_\ell, f_{\ell+1}, \dots] \otimes [e_1]) \oplus ([f_t, f_{t+1}, \dots] \otimes [e_2])$ where $\ell \leq t$. $A = 0$ if and only if $M = [f_s, f_{s+1}, \dots] \otimes [e_1, e_2]$. $AB = 0$ if and only if $M = [f_s, f_{s+1}, \dots] \otimes [e_1]$ or $M = [f_s, f_{s+1}, \dots] \otimes [e_1, e_2]$.

If $s = 2$ then $\dim M_0 \leq 1$. If $M_0 = [0]$ then $M = H$ or $M = [f_2, f_3, \dots] \otimes [e_1, e_2]$. If $M_0 \neq [0]$ then $M_0 = [\alpha(f_1 \otimes e_1) + \beta(f_2 \otimes e_2)]$ and $M_2 = ([f_2, f_3, \dots] \otimes [e_1]) \oplus ([f_3, f_4, \dots] \otimes [e_1]) \oplus M_0$.

Example 4.4 Suppose $\mathbb{T}_2^2 = \mathbb{T}_2$ in Example 4.3. If $M \in \mathrm{Lat}T_1 \cap \mathrm{Lat}T_2$ then by Theorem 3.1 $[\mathbb{T}_2 M] = [f_s, f_{s+1}, \dots] \otimes [e_1]$, $M_2 \cap \mathrm{Ker}\mathbb{T}_2^* = [f_t, f_{t+1}, \dots] \otimes [e_2]$ for $t \geq s$, $M_2' \cap \mathrm{Ker}\mathbb{T}_2 \subseteq [f_\ell, \dots, f_{s-1}] \otimes [e_2 - xe_1]$ and

$$M_0 \subseteq ([f_1, \dots, f_{s-1}] \otimes [e_1]) \oplus ([f_1, \dots, f_{t-1}] \otimes [e_2]).$$

$M_0 = [0]$ if and only if $M = (K_1 \otimes [e_2 - xe_1]) \oplus ([f_s, f_{s+1}, \dots] \otimes [e_1]) \oplus ([f_s, f_{s+1}, \dots] \otimes [e_2])$ where $t \geq s$ and $K_1 \subseteq [f_\ell, \dots, f_{s-1}]$. $A = 0$ if and only if $M = [f_s, f_{s+1}, \dots] \otimes [e_1, e_2]$. $AB = 0$ if and only if $M = [f_s, f_{s+1}, \dots] \otimes [e_1]$ or $M = [f_s, f_{s+1}, \dots] \otimes [e_1, e_2]$.

If $s = 2$ then $\dim M_0 \leq 1$. If $M_0 = [0]$ then $M = ([f_2, f_3, \dots] \otimes [e_1]) \oplus ([f_2, f_3, \dots] \otimes [e_2]) \oplus ([f_1] \otimes [e_2 - xe_1])$. If $M_0 \neq [0]$ then $M_0 = [f_1 \otimes (\alpha_1 e_1 + \alpha_2 e_2)]$ where $\alpha_1 \neq 0$ and $\alpha_2 \neq 0$ and so $\mathbb{T}_2 M_0 = [f_1 \otimes e_1]$. This is a contradiction because $[\mathbb{T}_2 M] = [f_2, f_3, \dots] \otimes [e_1]$. Thus if $s = 2$ then $M_0 = [0]$.

Example 4.5 Suppose $H_1 = H^2$ and T_1 is a unilateral shift on H^2. Suppose $\mathbb{T}_2^2 = 0$. If $M \in \mathrm{Lat}\mathbb{T}_1 \cap \mathrm{Lat}\mathbb{T}_2$, $\mathbb{T}_2 M \neq [0]$, $\mathrm{Ker}\mathbb{T}_2 \cap M \neq [0]$ and $\mathrm{Ker}\mathbb{T}_2^* \cap M \neq [0]$ then by

Theorem 2.1 and Beurling theorem $[\mathbb{T}_2 M_2] = q_1 H^2 \otimes [e_1]$, $\mathrm{Ker}\mathbb{T}_2 \cap M_2 = (q_3 H^2 \ominus q_1 H^2) \otimes [e_1]$ and $\mathrm{Ker}\mathbb{T}_2^* \cap M_2 = q_2 H^2 \otimes [e_2]$ where $q_2 H^2 \subset q_1 H^2$ and q_j is inner for $j = 1, 2, 3$. Hence

$$M_0 \subseteq \{(H^2 \ominus q_3 H^2) \otimes [e_1]\} \oplus \{(q_1 H^2 \ominus q_2 H^2) \otimes [e_2]\}$$

and

$$\dim M_0 = \dim P_1 M_0 = \dim P_2 M_0 \le \min(\deg q_3, \deg q_2 \bar{q}_1).$$

$M_0 = [0]$ if and only if $M = \{q_2 H^2 \otimes [e_2]\} \oplus \{q_3 H^2 \otimes [e_1]\}$ where $\bar{q}_2 q_3 \in H^2$. $A = 0$ if and only if $M = qH^2 \otimes [e_1, e_2]$. $AB = 0$ if and only if $M = qH^2 \otimes [e_1]$ or $M = qH^2 \otimes [e_1, e_2]$. Here q is inner. If $M_2 = M_0$ then q_1 is not a finite Blaschke product. In fact, if q_1 is a finite Blaschke product then $\dim M_0 \le \deg q_1$ and $[\mathbb{T}_2 M_0] = q_1 H^2 \otimes [e_1]$. This contradiction shows that q_1 is not a finite Blaschke product.

If $q_1 = z$ then $q_3 = 1$ or $q_3 = z$. If $q_3 = 1$ then $M_0 = [0]$. If $q_3 = z$ then $M_2 = M_0 \oplus (q_2 H^2 \otimes [e_2])$. If $M_0 = [0]$ then $q_2 = z$. If $M_0 \ne [0]$ then $M_0 = [\alpha(1 \otimes e_1) + \beta(zf \otimes e_2)]$ where $zf \in zH^2 \ominus q_2 H^2$. Hence $\bar{z} q_2$ is a single Blaschke product.

Example 4.6 Suppose $H_1 = H^2$ and T_1 is a unilateral shift on H^2. Suppose $\mathbb{T}_2^2 = \mathbb{T}_2$. By Theorem 3.1, if $\mathbb{T}_2 M \ne [0]$ and $\mathrm{Ker}\mathbb{T}_2^* \cap M_2 \ne [0]$ then $[\mathbb{T}_2 M] = q_1 H^2 \otimes [e_1]$, $\mathrm{Ker}\mathbb{T}_2^* \cap M_2 = q_2 H^2 \otimes [e_2]$ and $\mathrm{Ker}\mathbb{T}_2 \cap M_2' \subseteq (q_3 H^2 \ominus q_1 H^2) \otimes [e_2 - x e_1]$ where $q_j (1 \le j \le 3)$ is inner. Hence

$$M_0 \subseteq \{(H^2 \ominus q_1 H^2) \otimes [e_1]\} \oplus \{(H^2 \ominus q_2 H^2) \otimes [e_2]\}$$

and

$$\dim M_0 = \dim P_1 M_0 = \dim P_2 M_0 \le \min(\deg q_1, \deg q_2).$$

$M_0 = [0]$ if and only if $M = (K_1 \otimes [e_2 - x e_1]) \oplus (q_1 H^2 \otimes [e_1]) \oplus (q_2 H^2 \otimes [e_2])$ where $K_1 \subseteq q_3 H^2 \ominus q_1 H^2$, q_j $(1 \le j \le 3)$ is inner, $\bar{q}_1 q_3 \in H^2$ and $\bar{q}_1 q_2 \in H^2$. $A = 0$ if and only if $M = qH^2 \otimes [e_1, e_2]$. $AB = 0$ if and only if $M = qH^2 \otimes [e_1]$, $M = qH^2 \otimes [e_1, e_2]$. Here q is inner.

If $q_1 = z$ then $\dim M_0 \le 1$. If $M_0 = [0]$ then $M = zH^2 \otimes [e_1, e_2]$ or $M = (zH^2 \otimes [e_1, e_2]) \oplus ([1] \otimes [e_2 - x e_1])$. If $M_0 \ne [0]$ then $M = (zH^2 \otimes [e_1]) \oplus M_2$ and $M_0 = [1 \otimes e_1 + g \otimes e_2]$ where $g(0) = -1/x$, $g \perp q_2 H^2$ and $q_2(0) = 0$. Moreover $M_2 = [1 \otimes e_1 + g \otimes e_2] \oplus (q_2 H^2 \otimes [e_2])$ or $M_2 = [1 \otimes e_1 + g \otimes e_2] \oplus ([1] \otimes [e_2 - x e_1]) \oplus (q_2 H^2 \otimes [e_2])$.

Example 4.7 Suppose $H_1 = H^2$ and T_1 is a unilateral backward shift on H^2. Suppose $\mathbb{T}_2^2 = 0$ or $\mathbb{T}_2^2 = \mathbb{T}_2$. Then we can apply the results in §2 and §3

Example 4.8 Let $\mathbb{H}^2$ be the Hardy space on the torus in $\mathbb{C}^2$, and let z and w be coordinate functions on $\mathbb{C}^2$. Put

$$D_z f = zf \quad \text{and} \quad D_w f = wf \quad (f \in \mathbb{H}^2).$$

It is an important problem to describe $\mathrm{Lat} D_z \cap \mathrm{Lat} D_w$ or $\mathrm{Lat} D_z^* \cap \mathrm{Lat} D_w^*$.

When $\mathbb{H}^2 = M \oplus N$, $N \in \mathrm{Lat}D_z \cap \mathrm{Lat}D_\omega$ if and only if $M \in \mathrm{Lat}D_z^* \cap \mathrm{Lat}D_w^*$. Hence we consider only about $\mathrm{Lat}D_z^* \cap \mathrm{Lat}D_w^*$. For $M \in \mathrm{Lat}D_z^* \cap \mathrm{Lat}D_w^*$ with $M \subseteq \mathbb{H}^2 \ominus w^2\mathbb{H}^2 = H^2 \oplus wH^2$, put $H = H_1 \otimes H_2$ where $H_1 = H^2$ and $H_2 = [1, w] = \mathbb{C}^2$, and suppose

$$\mathbb{T}_1 = D_z^* \mid H \quad \text{and} \quad \mathbb{T}_2 = D_w^* \mid H.$$

Then M belongs to $\mathrm{Lat}\mathbb{T}_1 \cap \mathrm{Lat}\mathbb{T}_2$. Hence we can apply M the results in §2.

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(Takahiko Nakazi) Hokusei Gakuen University, 2–3–1, Ohyachi-Nishi, Atsubetu-ku, Sapporo 004–8631, Japan

(Michio Seto) Department of Mathematics, Shimane University, Matsue 690–8504, Japan

E-mail address: `z00547@hokusei.ac.jp` (T. Nakazi),
`mseto@shimane-u.ac.jp` (M. Seto)

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