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BESSEL POTENTIAL SPACES IN BEURLING'S DISTRIBUTIONS

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Abstract. We introduce the generalized Bessel potential spaces in the Beurling's distributions. We give the topological characterizations of the generalized Bessel potential spaces and consider multiplication and convolution operations in the generalized Bessel potential spaces.

1. Introduction

The Bessel potential spaces are of interest since they include the classical Sobolev spaces and they have close relationships with many other spaces, for example, Lorentz- Zygmund spaces [6], Orlicz spaces [5] and Besov spaces [9], etc. J. Barros-Neto and B. E. Petersen considered many topological properties of the Bessel potential spaces (Sobolev spaces) in the sense of Schwartz's distributions in [1] and [7], respectively.

In the mean time, A. Beurling presented the foundation of a more general theory of distributions in [2] and G. Björck developed the Beurling's generalized distribution theories in [3].

The purpose of the present paper is to extend Bessel potential spaces in Schwartz's distributions to Bessel potential spaces in the Beurling's tempered distributions. We will investigate the characterizations of the generalized Bessel potential spaces in the Beurling's tempered distributions. Also, we will consider multiplication and convolution operations in the generalized Bessel potential spaces in the Beurling's tempered distributions.

2. Beurling's Distributions and Preliminaries

Fistly, we review the Beurling's distribution spaces and related results which we need in this paper. We denote $\mathcal{M}_C$ the set of all real-valued functions ω on R^n

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satisfying the following conditions;

(α) $0 = \omega(0) \leq \omega(\xi + \eta) \leq \omega(\xi) + \omega(\eta), \quad \xi,\ \eta \in R^n$
(β) $J_n(\omega) = \int_{R^n} \frac{\omega(\xi)}{(1+|\xi|)^{n+1}} d\xi < \infty$
(γ) $\omega(\xi) \geq A + B\log(1 + |\xi|)$ for some constants A and $B > 0$
(δ) $\omega(\xi) = \sigma(|\xi|)$ for an increasing continuous concave function σ on $[0, \infty)$.

For example, $\omega(\xi) = \log(1 + |\xi|)$ and $\omega(\xi) = |\xi|^{\frac{1}{\alpha}}, \alpha > 1$, satisfy all conditions. Throughout this paper, ω represents an element in $\mathcal{M}_C$. Let $\mathcal{D}_\omega(U)$ be the set of all ϕ in $L^1(R^n)$ such that ϕ has a compact support in an open set U and

$$\|\phi\|_\lambda^{(\omega)} = \int_{R^n} |\hat{\phi}(\xi)| e^{\lambda\omega(\xi)}\, d\xi < \infty,$$

for any $\lambda > 0$. The topology on this space is given by the inductive limit topology of the Fréchet spaces $\mathcal{D}_\omega(K) = \{\phi \in \mathcal{D}_\omega \ :\ \text{supp}\ \phi \subset K\}$ induced by the above semi-norms where K is a compact set in U. From Proposition 1.3.6 in [3], if $\omega(\xi) = \log(1 + |\xi|)$, then $\mathcal{D}_\omega = C_c^\infty = \mathcal{D}$ (in the notation of Schwartz), where C_c^∞ is the set of all continuous functions with compact support in R^n.

Theorem 2.1. *Let $\omega_1, \omega_2 \in \mathcal{M}_C$. If for some real A and positive C we have*

$$\omega_2(\xi) \leq A + C\omega_1(\xi), \quad \xi \in R^n,$$

then $\mathcal{D}_{\omega_1} \subset \mathcal{D}_{\omega_2}$ and $\mathcal{D}_{\omega_1}(\Omega)$ is dense in $\mathcal{D}_{\omega_2}(\Omega)$ for each $\Omega \subset R^n$.

Proof. Theorem 1.3.18 in [3]. □

We denote by $\mathcal{E}_\omega(U)$ the set of all complex-valued functions ψ in U such that $\phi\psi$ is in $\mathcal{D}_\omega(U)$ for any $\phi \in \mathcal{D}_\omega(U)$. The topology in $\mathcal{E}_\omega(U)$ is given by the semi-norms $\psi \to \|\phi\psi\|_\lambda^{(\omega)}$ for any $\lambda > 0$ and any $\phi \in \mathcal{D}_\omega(U)$. The dual space of $\mathcal{D}_\omega(U)$ is denoted by $\mathcal{D}'_\omega(U)$ whose elements are called the Beurling's distributions. Because of ${\mathcal{D}_\omega}'(U) \supset \mathcal{D}'(U)$ by (γ), Beurling's distributions are generalized distributions. ${\mathcal{D}_\omega}'(U)$ is equal to $\mathcal{D}'(U)$ when $\omega(\xi) = \log(1 + |\xi|)$. The dual space $\mathcal{E}'_\omega(U)$ of $\mathcal{E}_\omega(U)$ can be identified with the set of all elements of $\mathcal{D}'_\omega(U)$ which has a compact support in U. $\mathcal{E}_\omega(U)$ is related to the Gevrey class when $\omega(\xi) = |\xi|^{\frac{1}{d}}, d > 1$.

Theorem 2.2. *Let K be a compact convex set in R^n with support function H. The Fourier-Lapalace transform of $u \in \mathcal{E}'_\omega(K)$ is an entire function $U(\zeta)$ in $\zeta = \xi + i\eta = (\zeta_1, \zeta_2, ..., \zeta_n) \in C^n$ if and only if for some real λ and all positive ϵ there exists a constant $C_{\lambda,\epsilon}$ such that*

$$|U(\xi + i\eta)| \leq C_{\lambda,\epsilon} e^{H(\eta) + \epsilon|\eta| + \lambda\omega(\xi)}.$$

Proof. Theorem 1.8.14 in [3]. □

We denote by $\mathcal{S}_\omega$ the set of all functions $\phi \in L^1(R^n)$ with the property that (ϕ and $\hat{\phi} \in C^\infty$ and) for each multi-index α and each non-negative number λ we have

$$p_{\alpha,\lambda}(\phi) = \sup_{x \in R^n} e^{\lambda\omega(x)} |D^\alpha \phi(x)| < \infty$$

and

$$\pi_{\alpha,\lambda}(\phi) = \sup_{\xi \in R^n} e^{\lambda\omega(\xi)} |D^\alpha \hat{\phi}(\xi)| < \infty.$$

The topology of $\mathcal{S}_\omega$ is defined by the semi-norms $p_{\alpha,\lambda}$ and $\pi_{\alpha,\lambda}$. Then it can be seen that $\mathcal{D}_\omega \subset \mathcal{S}_\omega \subset \mathcal{E}_\omega$.

Theorem 2.3. *$\mathcal{D}_\omega$ is dense in $\mathcal{S}_\omega$.*

Proof. Theorem 1.8.7 in [3]. □

Theorem 2.4. *The Fourier transform is a continuous automorphism of $\mathcal{S}_\omega$.*

Proof. Proposition 1.8.2 in [3]. □

Theorem 2.5. *$\mathcal{S}_\omega$ is a topological algebra under point-wise miltiplication and also under convolution.*

Proof. Proposition 1.8.3 in [3]. □

A continuous linear form on $\mathcal{S}_\omega$ is called an Beurling's tempered distribution. The space of all Beurling's tempered distributions is given the weak topology and denoted by $\mathcal{S}'_\omega$. Also, we can see that $\mathcal{S}'_\omega \subset \mathcal{D}'_\omega$. If $u \in \mathcal{S}'_\omega$ and $\phi \in \mathcal{S}_\omega$, we define the Fourier transform $\hat{u}$ by $\hat{u}(\phi) = u(\hat{\phi})$ and the convolution $u * \phi$ as the function given by $(u * \phi)(x) = u_y(\phi(x - y))$.

Theorem 2.6. *The Fourier transform is a continuous automorphism of $\mathcal{S}'_\omega$.*

Proof. Remark of Definition 1.8.9 in [3]. □

Theorem 2.7. *If $u \in \mathcal{S}'_\omega$ and $\phi \in \mathcal{S}_\omega$, then $\phi * u \in \mathcal{S}'_\omega$ and $\widehat{\phi * u} = \hat{\phi} \cdot \hat{u}$ and $\widehat{\phi u}(\xi) = (2\pi)^{-n}(\hat{\phi} * \hat{u})(\xi)$.*

Proof. Theorem 1.8.12 in [3]. □

For details about the Beurling's distributions, we refer to [2] and [3].

3. Generalized Bessel Potential Spaces

We will extend Bessel potential spaces in the Schwartz's distributions to Bessel potential spaces in the Beurling's distributions, and investigate the topological properties of the generalized Bessel potential spaces.

The classical Bessel potential spaces in the Schwartz's distributions, (or, Sobolev spaces,) $L^2_s(R^n)$, is defined as

$$L^2_s(R^n) = \left\{ u \in \mathcal{S}' \ : \|u\|_{L^2_s} = \left(\int_{R^n} |\hat{u}(\xi)|^2 (1+|\xi|)^{2s} d\xi \right)^{\frac{1}{2}} < \infty \right\},$$

for $s \in R$.

In [8], Pahk and Kang defined Sobolev spaces in Beurling's distributions, H^s_ω, as follows;

$$H^s_\omega(R^n) = \left\{ u \in \mathcal{S}'_\omega : \|u\|_{H^s_\omega} = \left(\int_{R^n} |\hat{u}(\xi)|^2 e^{2s\omega(\xi)} d\xi \right)^{\frac{1}{2}} < \infty \right\},$$

for $s \in R$.

We are ready to introduce the generalized Bessel potential spaces in the Beurling's distributions.

Definition 3.1. Given $s \in R$ and $1 < p < \infty$, we define by the generalized Bessel potential spaces, $L^p_{s,\omega}(R^n)$, the set of all $u \in \mathcal{S}'_\omega$ such that

$$L^p_{s,\omega}(R^n) = \left\{ u : \int_{R^n} |\hat{u}(\xi)|^p e^{ps\omega(\xi)} d\xi < \infty \right\}.$$

The norm in $L^p_{s,\omega}$ is given by

$$\|u\|_{L^p_{s,\omega}} = \left(\int_{R^n} |\hat{u}(\xi)|^p e^{ps\omega(\xi)} d\xi \right)^{\frac{1}{p}}. \tag{3.1}$$

Among these spaces one has the classical Bessel potential spaces (or, Sobolev spaces) $L^2_s(R^n)$ when $\omega(\xi) = \log(1+|\xi|)$, $p = 2$ and the Segal algebra $S_0(R^n)$ when $s = 0, \omega = \log(1+|\xi)$ and $p = 1$. Only if $p = 2$, several results in this section reduce to the results in [7].

In the mean time, Hörmander introduced the function spaces $\mathcal{B}_{p,k}$ in [4] as follow;

$\mathcal{K}$ is the set of all positive functions k in R^n with the following property. There exist positive constants C and N such that $k(\xi+\eta) \leq (1+C|\xi|)^N k(\eta)$ for all $\xi, \eta \in R^n$.

$$\mathcal{B}_{p,k}(R^n) = \left\{ u \in \mathcal{D}' : \left(\|u\|_{p,k} = (2\pi)^{-n} \int_{R^n} |k(\xi)\hat{u}(\xi)|^p d\xi \right)^{\frac{1}{p}} < \infty \right\},$$

for $k \in \mathcal{K}$ and $1 \leq p \leq \infty$.

The spaces $\mathcal{B}_{p,k}$ were extended to $\mathcal{B}^\omega_{p,k}$ by Björck in [3] as follow;

$\mathcal{K}_\omega$ is the set of all positive functions k in R^n with the following property. There exists positive constants λ such that $k(\xi+\eta) \leq e^{\lambda\omega(-\xi)} k(\eta)$ for all $\xi, \eta \in R^n$.

$$\mathcal{B}^\omega_{p,k}(R^n) = \left\{ u \in \mathcal{F}_\omega : \left(\|u\|_{p,k} = (2\pi)^{-n} \int_{R^n} |k(\xi)\hat{u}(\xi)|^p d\xi \right)^{\frac{1}{p}} < \infty \right\},$$

for $k \in \mathcal{K}$, $1 \le p \le \infty$ and $\mathcal{F}_\omega$ in Definition 1.8.10 in [3].

Clearly, we know that when $\omega(\xi) = \log(1 + |\xi|)$ and $k(\xi) = (1 + |\xi|)$, $L^p_{1,\omega}(R^n) = \mathcal{B}_{p,k}(R^n)$ and when $k(\xi) = e^{\omega(\xi)}$, $L^p_{1,\omega}(R^n) = \mathcal{B}^\omega_{p,k}(R^n)$ by the property of (δ) of ω.

In what follows, $L^p_{s,\omega}$ means $L^p_{s,\omega}(R^n)$.

Theorem 3.1. *$L^p_{s,\omega}$ is a Banach space with the norm $\|\cdot\|_{L^p_{s,\omega}}$ in (3.1). We have*

$$\mathcal{S}_\omega \subset L^p_{s,\omega} \subset \mathcal{S}'_\omega$$

in the topological sense. $\mathcal{D}_\omega$ (hence $\mathcal{S}_\omega$) is dence in $L^p_{s,\omega}$.

Proof. Let $L^p(e^{ps\omega(\xi)}d\xi)$ be the Banach space of all measurable functions v with norm $\|v\|^e_p$ such that

$$\|v\|^e_p = \left(\int |v(\xi)|^p e^{ps\omega(\xi)} d\xi\right)^{\frac{1}{p}} < \infty.$$

By the definition of $\mathcal{S}_\omega$, $\mathcal{S}_\omega \subset L^p(e^{ps\omega(\xi)}d\xi)$ in the topological sense. To prove that $L^p(e^{ps\omega(\xi)}d\xi) \subset \mathcal{S}'_\omega$, we note that Hölder inequality gives

$$\begin{aligned} \int |\varphi(\xi)v(\xi)|d\xi &= \int |\varphi(\xi)|e^{-s\omega(\xi)}|v(\xi)|e^{s\omega(\xi)}d\xi \\ &\le \|\varphi e^{-s\omega(\xi)}\|_q \cdot \|v(\xi)e^{s\omega(\xi)}\|_p, \end{aligned}$$

where $\varphi \in \mathcal{S}_\omega$ and $\frac{1}{p} + \frac{1}{q} = 1$. Since $\|\varphi e^{-s\omega(\xi)}\|_q$ is a continuous semi-norm in $\mathcal{S}_\omega$, we have $L^p(e^{ps\omega(\xi)}d\xi) \subset \mathcal{S}'_\omega$, so

$$\mathcal{S}_\omega \subset L^p(e^{ps\omega(\xi)}d\xi) \subset \mathcal{S}'_\omega. \tag{3.2}$$

From Theorem 2.1 with $\omega_1(\xi) = \omega(\xi)$ and $\omega_2(\xi) = \log(1 + |\xi|)$, $\mathcal{D}_\omega$ is dense in C^∞_c. Then since C^∞_c is dense in $L^p(e^{ps\omega(\xi)}d\xi)$, $\mathcal{D}_\omega$ is dense in $L^p(e^{ps\omega(\xi)}d\xi)$. From Theorem 2.3 and (3.2), $\mathcal{D}_\omega$ (hence $\mathcal{S}_\omega$) is dense in $L^p(e^{ps\omega(\xi)}d\xi)$. If we take the Fourier transform in (3.2) and use Theorems 2.4 and 2.6, we have the results. $\square$

Corollary 3.1. *If $s < t$, then $L^p_{t,\omega} \subset L^p_{s,\omega}$ and this inclusion is continuous.*

Proof. If $s < t$, then $e^{ps\omega(\xi)} \le e^{pt\omega(\xi)}$ and so $L^p_{t,\omega} \subset L^p_{s,\omega}$ and $\|u\|_{L^p_{s,\omega}} \le \|u\|_{L^p_{t,\omega}}$. $\square$

Corollary 3.2. *If P is a polynomial of degree m with constant coefficients, then $P(D)$ maps $L^p_{s,\omega}$ continuously into $L^p_{s-m,\omega}$.*

Proof. By the property (γ) of ω,

$$\begin{aligned} \|P(D)u\|^p_{L^p_{s,\omega}} &= \int |\widehat{P(D)u}|^p e^{ps\omega(\xi)}d\xi \\ &= \int |P(\xi)\hat{u}|^p e^{ps\omega(\xi)}d\xi \\ &\le C_1 \textstyle\sum_{|\alpha|\le m} \int |\hat{u}(\xi)|^p|\xi|^{pm}e^{ps\omega(\xi)}d\xi \\ &\le C_2 \int |\hat{u}(\xi)|^p e^{p(s+m)\omega(\xi)}d\xi \\ &= \|u\|^p_{L^p_{(s+m),\omega}}. \end{aligned}$$

$\square$

Theorem 3.2. *If $s_1 < s < s_2$, $\epsilon > 0$ and $u \in L^p_{s,\omega}$, then*

$$\|u\|^p_{L^p_{s,\omega}} \leq 2^p \epsilon \|u\|^p_{L^p_{s_2,\omega}} + 2^p \epsilon^{-m} \|u\|^p_{L^p_{s_1,\omega}},$$

where $m = \frac{(s-s_1)}{(s_2-s)}$.

Proof. Let $a, b > 0, 0 < t < 1$ and define $g(x) = tx^{1-t}a + (1-t)x^{-t}b$. Then g takes its minimum on $(0, \infty)$ at $x = \frac{b}{a}$. Thus $g(x) \geq a^t b^{1-t}$. Taking $a = e^{s_1\omega(\xi)}$ and $b = e^{s_2\omega(\xi)}$ and $s = ts_1 + (1-t)s_2$ we obtain

$$e^{s\omega(\xi)} \leq g(x) = tx^{1-t}e^{s_1\omega(\xi)} + (1-t)x^{-t}e^{s_2\omega(\xi)},$$

which implies that for any $x > 0$ we have

$$\begin{aligned} \|u\|^p_{L^p_{s,\omega}} &\leq 2^p t x^{1-t} \|u\|^p_{L^p_{s_1,\omega}} + 2^p (1-t) x^{-t} \|u\|^p_{L^p_{s_2,\omega}} \\ &\leq 2^p x^{1-t} \|u\|^p_{L^p_{s_1,\omega}} + 2^p x^{-t} \|u\|^p_{L^p_{s_2,\omega}}. \end{aligned}$$

By setting $x = \epsilon^{-\frac{1}{t}}$, we have the result. □

If $u \in \mathcal{S}'_\omega$ and $\phi \in \mathcal{S}_\omega$, we define

$$< u, \phi >= u(\overline{\phi}). \tag{3.3}$$

For $u \in L^p_{s,\omega}$, the map $v \to \int \hat{v}(\xi)\overline{\hat{u}}(\xi)d\xi$ is a linear functional on $L^q_{-s,\omega}$, where $\frac{1}{p} + \frac{1}{q} = 1$. In fact,

$$\begin{aligned} \int \hat{v}(\xi)\overline{\hat{u}}(\xi)d\xi &= \int \hat{v}(\xi)e^{-s\omega(\xi)}\overline{\hat{u}}(\xi)e^{s\omega(\xi)}d\xi \\ &\leq \left(\int |\hat{u}(\xi)|^p e^{ps\omega(\xi)}d\xi\right)^{\frac{1}{p}} \cdot \left(\int |\hat{v}(\xi)|^q e^{-qs\omega(\xi)}d\xi\right)^{\frac{1}{q}} \\ &= \|u\|_{L^p_{s,\omega}} \|v\|_{L^q_{-s,\omega}}. \end{aligned} \tag{3.4}$$

Hence the conjugate linear functional $< u, \cdot >$ on $\mathcal{S}_\omega$ in (3.3) can be extended uniquely to a conjugate linear functional on $L^q_{-s,\omega}$ such that

$$< v, u >= \int \hat{v}(\xi)\overline{\hat{u}}(\xi)d\xi, \tag{3.5}$$

for $u \in L^p_{s,\omega}$ and $v \in L^q_{-s,\omega}$. Hence $< v, \cdot >$ is a conjugate linear functional on $L^p_{s,\omega}$.

Theorem 3.3. *The pairing in (3.5) identifies $L^q_{-s,\omega}$ isometrically with the antidual of $L^p_{s,\omega}$. If $u \in \mathcal{D}'_\omega$, then $u \in L^p_{s,\omega}$ if and only if there is a constant C such that $| < u, \phi > | \leq C\|\phi\|_{L^q_{-s,\omega}}$, for $\phi \in \mathcal{D}_\omega$ and $\frac{1}{p} + \frac{1}{q} = 1$. Moreover, the best value of C is $\|u\|_{L^p_{s,\omega}}$, that is $\|u\|_{L^p_{s,\omega}} \leq C$.*

Proof. If we let $(L^p_{s,\omega})^*$ be the antidual of $L^p_{s,\omega}$, that is, the space of continuous conjugate linear functionals on $L^p_{s,\omega}$, we define $G : L^q_{-s,\omega} \to (L^p_{s,\omega})^*$ by

$$G(v)u =< v, u >= \int \hat{v}(\xi)\overline{\hat{u}}(\xi)d\xi.$$

Firstly, we will show that G is an isometric isomorphism from $L^q_{-s,\omega}$ onto $(L^p_{s,\omega})^*$. From (3.4), we have $\|G(v)\|_{(L^p_{s,\omega})^*} \le \|v\|_{L^q_{-s,\omega}}$ for all $v \in L^q_{-s,\omega}$, which implies $\|G\|_{\mathcal{L}((L^q_{-s,\omega})\to(L^p_{s,\omega}))} \le 1$. To show the isometry, let $v \in L^q_{-s,\omega}$ and put $u \in L^p_{s,\omega}$ such that

$$\hat{u}(\xi) = \frac{|\hat{v}(\xi)|}{\overline{\hat{v}(\xi)}}|\hat{v}(\xi)|^{\frac{q}{p}}e^{-qs\omega(\xi)} = \frac{\hat{v}(\xi)}{|\hat{v}(\xi)|}|\hat{v}(\xi)|^{q-1}e^{-qs\omega(\xi)}.$$

Since $\|u\|^p_{L^p_{s,\omega}} = \|v\|^q_{L^q_{-s,\omega}}$,

$$\begin{aligned} G(v)u &= \int |\hat{v}(\xi)|^{\frac{q+p}{p}} e^{-qs\omega(\xi)} d\xi \\ &= \|v\|^q_{L^q_{-s,\omega}} \\ &= \|v\|_{L^q_{-s,\omega}} \cdot \|v\|^{\frac{q}{p}}_{L^q_{-s,\omega}} \\ &= \|v\|_{L^q_{-s,\omega}} \cdot \|u\|_{L^p_{s,\omega}}. \end{aligned}$$

Then $\|G(v)\|_{(L^p_{s,\omega})^*} \ge \|v\|_{L^q_{-s,\omega}}$, which implies $\|G\|_{\mathcal{L}((L^q_{-s,\omega})\to(L^p_{s,\omega}))} \ge 1$. Hence

$$\|G\|_{\mathcal{L}((L^q_{-s,\omega})\to(L^p_{s,\omega}))} = 1.$$

Now, if $G(v) = 0$, then $G(v)u =< v, u >= \int \hat{v}(\xi)\overline{\hat{u}}(\xi)d\xi = 0$ for all $u \in \mathcal{S}_\omega$. Hence $v = 0$ in $\mathcal{S}'_\omega$, i.e., $v = 0$ in $L^q_{-s,\omega}$, which implies the injectivity of G. Next, we will show the surjectivity of G. We note that $(L^p_{s,\omega})^* \subset (\mathcal{S}_\omega)^*$ by Theorem 3.1. We can identify $L^p_{s,\omega}$ with the closed subspaces of L^p, $\{f \in L^p : V(u) = f, u \in L^p_{s,\omega}\}$, by the isometric isomorphic map $V : u(x) \to \hat{u}(\xi)e^{s\omega(\xi)}$. Then, by Riesz representation theorem for L^p, if $F \in (L^p_{s,\omega})^*$, there exist a $u_1 \in L^q$ such that

$$F(u) = \int \hat{u}_1(\xi)\overline{\hat{u}(\xi)e^{s\omega(\xi)}}d\xi,$$

for all $u \in L^p_{s,\omega}$. Since $\phi \to \int \hat{u}_1(\xi)\overline{\hat{\phi}(\xi)}e^{s\omega(\xi)}d\xi$ is a continuous linear functional on $\mathcal{S}_\omega$, there is a distribution $u_2 \in L^q_{-s,\omega}$ such that $\hat{u}_2(\xi) = \hat{u}_1(\xi)e^{s\omega(\xi)}$ a.e. Then we have

$$\begin{aligned} F(u) &= \int \hat{u}_1(\xi)\overline{\hat{u}(\xi)e^{s\omega(\xi)}}d\xi \\ &= \int \hat{u}_2(\xi)e^{-s\omega(\xi)}\overline{\hat{u}(\xi)}e^{s\omega(\xi)}d\xi \\ &=< u_2, u >= G(u_2)(u), \end{aligned}$$

for all $u \in L^p_{s,\omega}$. Hence $F = G(u_2)$, which implies the surjectivity of G.

For the last statements, let $u \in L^p_{s,\omega}$. Then,

$$\|u\|_{L^p_{s,\omega}} = \|G(u)\| = \sup\left\{\frac{|u(\phi)|}{\|\phi\|_{L^q_{-s,\omega}}} : \phi \in L^q_{-s,\omega}\right\} \ge \sup\left\{\frac{|u(\phi)|}{\|\phi\|_{L^q_{-s,\omega}}} : \phi \in \mathcal{D}_\omega\right\},$$

which implies $| < u, \phi > | \le C\|\phi\|_{L^q_{-s,\omega}}$.

Finally, if $u \in \mathcal{D}'_\omega$ and $| < u, \phi > | \le C\|\phi\|_{L^q_{-s,\omega}}$ for $\phi \in \mathcal{D}_\omega$, then the map $\phi \to u(\overline{\phi})$ extends to an element of $(L^q_{-s,\omega})^*$ with norm $\le C$. Thus there exists a unique $w \in L^p_{s,\omega}$ such that $G(w)(\phi) =< u, \phi >$ for each $\phi \in \mathcal{D}_\omega$. But, then $< w, \phi >=< u, \phi >$ for each $\phi \in \mathcal{D}_\omega$, i.e., $u = w$. $\square$

Corollary 3.3. *Let $A : \mathcal{D}_\omega \to \mathcal{D}'_\omega$ be a linear map. Then A extends uniquely to a continuous linear map $A : L^p_{s,\omega} \to L^p_{t,\omega}$ if and only if*

$$|< Au, \overline{v} >| \leq C\|u\|_{L^p_{s,\omega}} \|v\|_{L^q_{-t,\omega}},$$

for $u, v \in \mathcal{D}_\omega$. Moreover, the best value of C is the norm $\|A\|_{s,t}$ of the operator $A : L^p_{s,\omega} \to L^p_{t,\omega}$.

Proof. If $u \in \mathcal{D}_\omega$ and $Au \in L^p_{t,\omega}$, we have

$$\begin{aligned} |(Au)(v)| &= |< Au, \overline{v} >| \leq \|Au\|_{L^p_{t,\omega}} \|v\|_{L^q_{-t,\omega}} \\ &= \|A\|\|u\|_{L^p_{s,\omega}} \|v\|_{L^q_{-t,\omega}} \\ &= C\|u\|_{L^p_{s,\omega}} \|v\|_{L^q_{-t,\omega}}, \end{aligned}$$

for $v \in \mathcal{D}_\omega$ from Theorem 3.3. The converse comes from Theorem 3.3. $\square$

It is convenient to introduce the intersection and the union of the spaces $L^p_{s,\omega}$. We let

$$L^p_{\infty,\omega} = \cap_{s\in R} L^p_{s,\omega}, \quad L^p_{-\infty,\omega} = \cup_{s\in R} L^p_{s,\omega}.$$

We provide $L^p_{\infty,\omega}$ with the weakest topology so that the inclusion map $L^p_{\infty,\omega} \to L^p_{s,\omega}$ is continuous for each s (projective topology). We provide $L^p_{-\infty,\omega}$ with the strongest locally convex topology so that the inclusion map $L^p_{s,\omega} \to L^p_{-\infty,\omega}$ is continuous for each s (locally convex inductive topology).

From the definition of $\mathcal{E}_\omega$ by

$$\mathcal{E}_\omega = \left\{ \psi : \|\phi\psi\|^{(\omega)}_\lambda = \int_{R^n} |\widehat{\phi\psi}(\xi)| e^{\lambda\omega(\xi)} d\xi < \infty \right\},$$

for all $\lambda > 0$, we can define $\mathcal{E}^k_\omega$, as follow:

We denote by $\mathcal{E}^k_\omega(\Omega)$ the vector space of all locally integrable functions u on Ω such that

$$\int |\widehat{\phi u}(\xi)| e^{k\omega(\xi)} < \infty,$$

for all $\phi \in \mathcal{D}_\omega(\Omega)$ and a non-negative integer k.

We note that $\cap_{k>0} \mathcal{E}^k_\omega(\Omega) = \mathcal{E}_\omega(\Omega)$ and $\mathcal{E}^k_\omega(\Omega) \subset C^k(\Omega)$, for any non-negative k, by Proposition 3.1 in [8]. Now we are ready to give an imbedding theorem for $L^p_{s,\omega}$.

Theorem 3.4. *If $s > (\frac{n}{q}) + k$, then $L^p_{s,\omega} \subset \mathcal{E}^k_\omega$, where $\frac{1}{p} + \frac{1}{q} = 1$.*

Proof. Let $u \in L^p_{s,\omega}$ and $\phi \in \mathcal{D}_\omega$. Then

$$\begin{aligned} \textstyle\int |\widehat{\phi u}(\xi)| e^{k\omega(\xi)} d\xi &= \textstyle\int |\widehat{\phi u}(\xi)| e^{s\omega(\xi)} e^{(k-s)\omega(\xi)} d\xi \\ &\leq \left(\textstyle\int |\widehat{\phi u}(\xi)|^p e^{ps\omega(\xi)} d\xi\right)^{\frac{1}{p}} \cdot \left(\textstyle\int e^{q(k-s)\omega(\xi)} d\xi\right)^{\frac{1}{q}} \\ &\leq C \|\phi u\|_{L^p_{s,\omega}}, \end{aligned}$$

for $s > (\frac{n}{q}) + k$. Hence $u \in \mathcal{E}^k_\omega$. $\square$

Theorem 3.5. *$L^p_{\infty,\omega} \subset \mathcal{E}_\omega$ and $\mathcal{E}'_\omega \subset L^p_{-\infty,\omega}$.*

Proof. The first statement follows from Theorem 3.4. Let $u \in \mathcal{E}'_\omega$. By Theorem 2.2, there exist some constant $\lambda > 0$ and C_λ such that $|\hat{u}(\xi)| \leq C_\lambda e^{\lambda\omega(\xi)}$. Then

$$\begin{aligned} \textstyle\int |\hat{u}(\xi)|^p e^{ps\omega(\xi)} d\xi &\leq C^p_\lambda \textstyle\int e^{p(s+\lambda)\omega(\xi)} d\xi \\ &\leq C \textstyle\int (1+|\xi|)^{p(s+\lambda)} d\xi \\ &< \infty, \end{aligned}$$

for $p(s+\lambda) < -2n$. Hence $u \in L^p_{(-\lambda-\frac{n}{p}),\omega}$, which implies $u \in L^p_{-\infty,\omega}$. $\square$

4. Mutiplication and convolution operations in $L^p_{s,\omega}$

Theorem 4.1. *If $\phi \in \mathcal{S}_\omega$ and $u \in L^p_{s,\omega}$, then the product ϕu belongs to $L^q_{s,\omega}$, where $\frac{1}{p} + \frac{1}{q} = 1$. Furthermore, the bilinear map*

$$\mathcal{S}_\omega \times L^p_{s,\omega} \ni (\phi, u) \to \phi u \in L^q_{s,\omega}$$

is separately continuous.

The proof of Theorem 4.1 is based upon the following Lemma.

Lemma 4.1. *Let $K(x,y)$ be a continuous function on $R^n \times R^n$ and suppose that there is a constant $C > 0$ such that $\int_{R^n} |K(x,y)| dx \leq C$ uniformly on Y and $\int_{R^n} |K(x,y)| dy \leq C$ uniformly on X. Then*

$$AF(x) = \int_{R^n} K(x,y) f(y) dy$$

defines a continuous linear operator from L^p into L^q, where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Let $f \in L^p$ and $g \in L^q$. We have

$$
\begin{aligned}
| < Af, g > | &= \left| \int_{R^n} Af(x)\overline{g(x)}dx \right| \\
&= \left| \int_{R^n} \int_{R^n} K(x,y)f(y)\overline{g(x)}dxdy \right| \\
&\leq \int_{R^n} \int_{R^n} |K(x,y)|^{\frac{1}{p}}|f(y)||K(x,y)|^{\frac{1}{q}}|g(x)|dxdy \\
&\leq \left(\int_{R^n} \int_{R^n} |K(x,y)||f(x)|^p dxdy \right)^{\frac{1}{p}} \cdot \left(\int_{R^n} \int_{R^n} |K(x,y)||g(x)|^q dxdy \right)^{\frac{1}{q}} \\
&\leq C\|f\|_p \cdot \|g\|_q, \qquad (4.1)
\end{aligned}
$$

which implies that $A : L^p \to L^q$ is a continuous linear operator. □

Remark 4.1. It follows from (4.1) that the norm, $\|A\|$, is at most equal to C.

Proof. (Proof of Theorem 4.1) 1. Let $\phi \in \mathcal{S}_\omega$ and $u \in L^p_{s,\omega}$. By Theorem 2.7, we have

$$\widehat{\phi u}(\xi) = (2\pi)^{-n}(\hat{\phi} * \hat{u})(\xi) = (2\pi)^{-n} \int_{R^n} \hat{\phi}(\xi - \eta)\hat{u}(\eta)d\eta. \qquad (4.2)$$

In order to prove that $\phi u \in L^q_{s,\omega}$, it suffices to show that $\widehat{\phi u}(\xi)e^{s\omega(\xi)} \in L^q$.

By (4.2), we have

$$
\begin{aligned}
\widehat{\phi u}(\xi)e^{s\omega(\xi)} &= (2\pi)^{-n} \int_{R^n} \hat{\phi}(\xi - \eta)\hat{u}(\eta)e^{s\omega(\xi)}d\eta. \\
&= (2\pi)^{-n} \int_{R^n} \hat{\phi}(\xi - \eta)e^{s\omega(\xi)}e^{-s\omega(\eta)}\hat{u}(\eta)e^{s\omega(\eta)}d\eta.
\end{aligned}
$$

Set

$$K(\xi, \eta) = \hat{\phi}(\xi - \eta)e^{s\omega(\xi)}e^{-s\omega(\eta)}.$$

Since $\phi \in \mathcal{S}_\omega$, we have

$$
\begin{aligned}
|K(\xi, \eta)| &= |\hat{\phi}(\xi - \eta)|e^{s\omega(\xi)}e^{-s\omega(\eta)} \\
&\leq |\hat{\phi}(\xi - \eta)|e^{s\omega(\xi-\eta)} \\
&\leq |\hat{\phi}(\xi - \eta)|e^{(s+2n)\omega(\xi-\eta)}e^{-2n\omega(\xi-\eta)} \\
&\leq C_1 e^{-2n\omega(\xi-\eta)} \leq C_2(1 + |\xi - \eta|)^{-2n},
\end{aligned} \qquad (4.3)
$$

by the property (α) and (γ) of ω. Hence $K(\xi, \eta)$ satisfies the assumption of Lemma 4.1. Since $u \in L^p_{s,\omega}$, we have $\hat{u}(\eta)e^{s\omega(\eta)} \in L^p$. Thus $\widehat{\phi u}(\xi)e^{s\omega(\xi)} \in L^q$ by Lemma 4.1, which implies $\phi u \in L^q_{s,\omega}$.

2. We will mention the results on the convolution in [1, p.90]. Let r, s, and t be real numbers such that $1 \leq r, s, t \leq \infty$ and $\frac{1}{t} = \frac{1}{r} + \frac{1}{s} - 1$. If $f \in L^r$ and $g \in L^s$, then $f * g \in L^t$ and

$$\|f * g\|_t = \|f\|_r \cdot \|g\|_s.$$

Since $\phi \in \mathcal{S}_\omega$, we have $\hat{\phi}(\cdot)e^{s\omega(\cdot)} \in L^l$ for any real number $l \geq 1$. Hence, for $\frac{1}{q} = \frac{1}{l} + \frac{1}{p} - 1$,

$$\begin{aligned}
\|\phi u\|_{L^q_{s,\omega}} &= \left(\int_{R^n} |\widehat{\phi u}(\xi)|^q e^{qs\omega(\xi)} d\xi\right)^{\frac{1}{q}} \\
&= \left(\int_{R^n} \left((2\pi)^{-n} \int_{R^n} \hat{\phi}(\xi-\eta)\hat{u}(\eta)d\eta\right)^q e^{qs\omega(\xi)} d\xi\right)^{\frac{1}{q}} \\
&= \left(\int_{R^n} \left((2\pi)^{-n} \int_{R^n} \hat{\phi}(\xi-\eta)e^{s\omega(\xi-\eta)}\hat{u}(\eta)e^{s\omega(\eta)}d\eta\right)^q d\xi\right)^{\frac{1}{q}} \\
&\leq \|\hat{\phi}e^{s\omega} * \hat{u}e^{s\omega}\|_q \\
&\leq \|\hat{\phi}e^{s\omega}\|_l \cdot \|\hat{u}e^{s\omega}\|_p \\
&\leq C\|u\|_{L^p_{s,\omega}},
\end{aligned} \tag{4.4}$$

which implies the continuity of the map $(\phi, u) \to \phi u$ with respect to $u \in L^p_{s,\omega}$.

Now, suppose that $\phi_j \to 0$ in $\mathcal{S}_\omega$ and let $C_j = \sup_{\zeta \in R^n} |\phi_j(\zeta)|e^{(s+2n)\omega(\zeta)}$ be the corresponding constant in (4.3). By the remark followed the proof of Lemma 4.1 and (4.4), we have

$$\|\phi_j u\|_{L^q_{s,\omega}} = \|\widehat{\phi_j u}(\zeta)e^{s\omega(\zeta)}\|_q \leq C_j C'\|u\|_{L^p_{s,\omega}}. \tag{4.5}$$

Since $\phi_j \to 0$ in $\mathcal{S}_\omega$, hence $C_j \to 0$, the last inequality (4.5) implies the continuity of the product ϕu with respect to $\phi \in \mathcal{S}_\omega$. $\square$

Combined with Corollary 3.2, we have following useful result:

Corollary 4.1. *Let $P(x, D) = \sum_{|\alpha| \leq m} a_\alpha(x)D^\alpha$ be a partial differential operator of order $\leq m$ and $a_\alpha(x) \in \mathcal{S}_\omega$. For every real number s, $P(x, D)$ defines a continuous linear map from $L^p_{s,\omega}$ into $L^q_{s-m,\omega}$.*

Theorem 4.2. *If $\phi \in \mathcal{S}_\omega$ and $u \in L^p_{s,\omega}$, the convolution $\phi * u \in L^p_{s,\omega}$ and the bilinear map*

$$\mathcal{S}_\omega \times L^p_{s,\omega} \ni (\phi, u) \to \phi * u \in L^p_{s,\omega},$$

is separately continuous. Furthermore, $\phi * u \in L^p_{\infty,\omega}$.

Proof. If $\phi \in \mathcal{S}_\omega$ and $u \in L^p_{s,\omega}$, then $\phi * u \in \mathcal{S}'_\omega$ by Theorem 2.7. Since $\hat{u}(\xi)e^{s\omega(\xi)} \in L^p$ and $\hat{\phi}(\xi) \in \mathcal{S}_\omega$, we have $\hat{u}(\xi)\hat{\phi}(\xi)e^{s\omega(\xi)} \in L^p$ by the property $\widehat{\phi * u} = \hat{\phi} \cdot \hat{u}$ in Theorem 2.7. Hence $\phi * u \in L^p_{s,\omega}$

The inequality

$$\begin{aligned}
\|\phi * u\|_{L^p_{s,\omega}} &= \left(\int_{R^n} |\hat{u}(\xi)|^p |\hat{\phi}(\xi)|^p e^{ps\omega(\xi)} d\xi\right)^{\frac{1}{p}} \\
&\leq \sup_{\xi \in R^n} |\hat{\phi}(\xi)| \cdot \|u\|_{L^p_{s,\omega}}.
\end{aligned}$$

implies the separate continuity of the convolution product $\phi * u$. Finally, if $\phi \in \mathcal{S}_\omega$, for every real number k, $\hat{\phi}(\xi)e^{k\omega(\xi)} \in \mathcal{S}_\omega$. Hence

$$\hat{\phi}(\xi)\hat{u}(\xi)e^{(k+s)\omega(\xi)} \in L^p,$$

for every k, which implies that $\phi * u \in L^p_{\infty,\omega}$. □

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