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WEIGHTED COMPOSITION OPERATORS ON THE LOGARITHMIC BLOCH SPACES WITH ITERATED WEIGHTS

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Abstract. We will characterize the boundedness and compactness of weighted composition operators on the logarithmic Bloch spaces with iterated weights on the open unit disk.

1. Introduction

Let $H(\mathbb{D})$ be the space of all analytic functions on the open unit disk $\mathbb{D}$ and $S(\mathbb{D})$ be the set of all analytic self-maps of $\mathbb{D}$. For $u \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$ we define the weighted composition operator $uC_\varphi : f \mapsto u \cdot f \circ \varphi$. Then uC_φ is a linear transformations on $H(\mathbb{D})$. We can regard this operator as a generalization of a multiplication operator M_u and a composition operator C_φ.

Let $H^\infty = H^\infty(\mathbb{D})$ be the set of all bounded analytic functions on $\mathbb{D}$. Then H^∞ is a Banach algebra with the supremum norm

$$\|f\|_\infty = \sup_{z\in\mathbb{D}} |f(z)|.$$

The Bloch space $\mathcal{B}$ is the set of all $f \in H(\mathbb{D})$ satisfying

$$\|\!|f\|\!| = \sup_{z\in\mathbb{D}} (1-|z|^2)|f'(z)| < \infty.$$

Then $\|\!|\cdot\|\!|$ defines a Möbius invariant complete semi-norm and $\mathcal{B}$ is a Banach space under the norm $\|f\|_{\mathcal{B}} = |f(0)| + \|\!|f\|\!|$. Note that $\|\!|f\|\!| \le \|f\|_\infty$ for any $f \in H^\infty$, hence $H^\infty \subset \mathcal{B}$.

Let the little Bloch space $\mathcal{B}_o$ denote the subspace of $\mathcal{B}$ consisting of those functions f such that

$$\lim_{|z|\to 1} (1-|z|^2)f'(z) = 0.$$

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The little Bloch space $\mathcal{B}_o$ is a closed subspace of $\mathcal{B}$. In particular, $\mathcal{B}_o$ is the closure in $\mathcal{B}$ of the polynomials.

For w, z in $\mathbb{D}$, the pseudo-hyperbolic distance $\rho(w, z)$ between z and w is given by

$$\rho(w, z) = \left| \frac{w - z}{1 - \overline{w}z} \right|,$$

and the hyperbolic metric $\beta(w, z)$ is given by

$$\beta(w, z) = \frac{1}{2} \log \frac{1 + \rho(w, z)}{1 - \rho(w, z)}.$$

We present a growth condition of the Bloch functions: for $f \in \mathcal{B}$,

$$\begin{aligned} |f(w)| &\leq |f(0)| + \|f\| \cdot \frac{1}{2} \log \frac{1 + |w|}{1 - |w|} \\ &= |f(0)| + \|f\| \, \beta(w, 0). \end{aligned}$$

Let $d_{\mathcal{B}}(w, z)$ denote the induced distance on $\mathcal{B}$ defined by

$$d_{\mathcal{B}}(w, z) = \sup_{\|f\|_{\mathcal{B}} \leq 1} |f(w) - f(z)|.$$

Then it is also known that the hyperbolic metric coincides with the induced distance on $\mathcal{B}$, that is,

$$d_{\mathcal{B}}(w, z) = \beta(w, z).$$

See [10] for more information on the Bloch space.

Madigan and Matheson [5] studied the composition operators on $\mathcal{B}$ and $\mathcal{B}_o$ in terms of the hyperbolic derivatives

$$\varphi^{\#}(z) = \frac{1 - |z|^2}{1 - |\varphi(z)|^2} \varphi'(z).$$

In [7], Ohno and Zhao characterized the boundedness and the compactness of uC_φ on $\mathcal{B}$ and $\mathcal{B}_o$. In [6] those results were generalized to the case of α-Bloch space $\mathcal{B}^\alpha$ induced by the α-Bloch semi-norm

$$\|f\|_\alpha = \sup_{z \in \mathbb{D}} (1 - |z|^2)^\alpha |f'(z)|$$

for $\alpha > 0$. On the other hand, Yoneda [9] studied C_φ on the logarithmic weighted Bloch space $\mathcal{LB}^1$ induced by

$$\begin{aligned} \|f\|_1 &= \sup_{z \in \mathbb{D}} (1 - |z|^2) \left(\frac{1}{2} \log \frac{1 + |z|}{1 - |z|} + 1 \right) |f'(z)| \\ &= \sup_{z \in \mathbb{D}} (1 - |z|^2)(\beta(z, 0) + 1)|f'(z)|. \end{aligned}$$

Here we review that M_u is bounded on $\mathcal{B}$ if and only if $u \in H^\infty \cap \mathcal{LB}^1$ (see [1] and [2]).

In this paper, we study the logarithmic Bloch space with iterated weights and the weighted composition operators acting on them. In Section 2, we define the logarithmic Bloch spaces with iterated weights and investigate their basic properties. In Section 3, we characterize the boundedness and the compactness of uC_φ on them. Moreover, we consider the case of the little logarithmic Bloch spaces in Section 4.

2. The logarithmic Bloch spaces

Here we define the logarithmic weighted Bloch spaces with iterated weights. Let n be any positive integer. Denote by $\ell(x) = (\log x) + 1$ and $\ell_n(x)$ be the n-th iteration of $\ell(x)$. Put $\ell_0(x) = x$ and

$$L_n(z) = \ell_{n-1}(\beta(z,0)+1) = \ell_{n-1}\left(\frac{1}{2}\log\frac{1+|z|}{1-|z|}+1\right).$$

For $f \in H(\mathbb{D})$, denote that

$$|||f|||_n = \sup_{z\in\mathbb{D}} (1-|z|^2)\prod_{k=1}^{n} L_k(z)|f'(z)|.$$

Then each $|||\cdot|||_n$ is a non-Möbius invariant semi-norm.

Definition 2.1. For any positive integer n, let $\mathcal{LB}^n$ be the set of all functions $f \in H(\mathbb{D})$ such that $|||f|||_n < \infty$. The little logarithmic Bloch space, denoted by $\mathcal{LB}^n_o$, is the closed subspace of $\mathcal{LB}^n$ consisting of analytic functions f on $\mathbb{D}$ with

$$\lim_{|z|\to 1} (1-|z|^2)\prod_{k=1}^{n} L_k(z)|f'(z)| = 0.$$

For the case of $n = 0$, denote that $|||\cdot|||_0 = |||\cdot|||$, $\mathcal{LB}^0 = \mathcal{B}$, and $\mathcal{LB}^0_o = \mathcal{B}_o$. Then $\mathcal{LB}^n$ and $\mathcal{LB}^n_o$ are Banach spaces with the norm $\|f\|_{\mathcal{LB}^n} = |f(0)| + |||f|||_n$ for any non-negative integer n.

The following are basic properties of the logarithmic Bloch spaces.

Proposition 2.2. *Let n be any non negative integer.*

(i) $\mathcal{LB}^{n+1} \subsetneqq \mathcal{LB}^n$.

(ii) $\mathcal{LB}^n_o \subsetneqq \mathcal{LB}^n$.

(iii) *For $r \in (0,1)$, let $f_r(z) = f(rz)$. Then $f \in \mathcal{LB}^n_o$ if and only if $|||f_r - f|||_n \to 0$ as $r \to 1$.*

(iv) *$\mathcal{LB}^n_o$ is the closure in $\mathcal{LB}^n$ of the set of all polynomials. So $\mathcal{LB}^n_o$ is a separable Banach space.*

Proof. Let

$$g(z) = \ell_n\left(\frac{1}{2}\log\frac{2}{1-z}+1\right).$$

At first, we prove that $g \in \mathcal{LB}^n$. Observe that

$$(1-|z|^2)\prod_{k=1}^{n} L_k(z)|g'(z)| = \frac{1-|z|^2}{2|1-z|}\prod_{k=0}^{n-1}\frac{\ell_k(\frac{1}{2}\log\frac{1+|z|}{1-|z|}+1)}{\left|\ell_k(\frac{1}{2}\log\frac{2}{1-z}+1)\right|}.$$

If $g \notin \mathcal{LB}^n$, we can choose a sequence $\{z_j\} \subset \mathbb{D}$ such that for some $0 \le k \le n-1$

$$\lim_{j\to\infty}\frac{(1-|z_j|)^{\frac{1}{n}}}{|1-z_j|^{\frac{1}{n}}}\frac{\ell_k(\frac{1}{2}\log\frac{2}{1-|z_j|}+1)}{\left|\ell_k(\frac{1}{2}\log\frac{2}{1-z_j}+1)\right|} = \infty. \tag{2.1}$$

After passing to a subsequence we may achieve that $z_j \to 1$. Note that for any k,

$$\left|\ell_k\Big(\frac{1}{2}\log\frac{2}{1-z}+1\Big)\right| \ge \ell_k\Big(\frac{1}{2}\log\frac{2}{|1-z|}+1\Big). \tag{2.2}$$

Indeed, since $\operatorname{Re}\frac{2}{1-z} = \operatorname{Re}(\frac{1+z}{1-z}+1) \ge 1$ on $\mathbb{D}$, we have that

$$\left|\frac{1}{2}\log\frac{2}{1-z}+1\right| \ge \frac{1}{2}\log\frac{2}{|1-z|}+1 \ge 1.$$

Moreover, if $\operatorname{Re} w \ge 1$, we have that $|\ell(w)| \ge \ell(|w|)$. By the induction, we get (2.2).

For $x > 0$, consider the function

$$h(x) = x^{\frac{1}{n}}\,\ell_k\Big(\frac{1}{2}\log\frac{2}{x}+1\Big). \tag{2.3}$$

Then we have that

$$h'(x) = \frac{x^{\frac{1}{n}-1}}{2n}\Big(2\ell_k\Big(\frac{1}{2}\log\frac{2}{x}+1\Big) - n\,\ell_k'\Big(\frac{1}{2}\log\frac{2}{x}+1\Big)\Big).$$

Notice that

$$\frac{\ell_k'(t)}{\ell_k(t)} = \left(\prod_{j=0}^{k-1}\ell_j(t)\right)^{-1} \to 0$$

as $t \to \infty$. This implies that h is increasing in a small neighborhood of 0. It follows that

$$h(1-|z_j|) \le h(|1-z_j|) \tag{2.4}$$

for j large enough. Combining (2.2) and (2.4), we get a contradiction to (2.1). Thus g is a function in $\mathcal{LB}^n$.

Next we prove that g belongs to neither $\mathcal{LB}_o^n$ nor $\mathcal{LB}^{n+1}$. Let r be a positive number in $(0,1)$. It is easy to check that

$$\lim_{r\to 1}(1-r^2)\prod_{k=1}^{n} L_k(r)|g'(r)| = 1. \tag{2.5}$$

This means that $g \notin \mathcal{LB}_o^n$. By (2.5), we have

$$\lim_{r\to 1}(1-r^2)\prod_{k=1}^{n+1} L_k(r)|g'(r)| = \infty.$$

We conclude that $g \notin \mathcal{LB}^{n+1}$. Here we get (i) and (ii).

For the proofs of (iii) and (iv), we can apply the same proofs of Theorem 5.9 and Corollary 5.10 of [10]. $\square$

Next we give the growth condition of $\mathcal{LB}^n$.

Proposition 2.3. *Let f be in $\mathcal{LB}^n$. Then for $w \in \mathbb{D}$,*

$$|f(w)| \leq |f(0)| + \|\|f\|\|_n \log L_n(w). \tag{2.6}$$

Proof. We have that

$$\begin{aligned} |f(w) - f(0)| &= \left| w \int_0^1 f'(wt)dt \right| \leq |w| \int_0^1 |f'(wt)|dt \\ &\leq \|\|f\|\|_n \int_0^1 \frac{|w|dt}{(1-|w|^2t^2)\prod_{k=1}^n L_k(wt)} \\ &= \|\|f\|\|_n \log L_n(w). \end{aligned}$$

Hence we get (2.6). $\square$

Let $d_{\mathcal{LB}^n}(w,z)$ denote the induced distance on $\mathcal{LB}^n$ defined by

$$d_{\mathcal{LB}^n}(w,z) = \sup_{\|\|f\|\|_n \leq 1} |f(w) - f(z)|.$$

Lemma 2.4. *For $p \in \mathbb{D}$ and positive integer m, put*

$$f_{m,p}(z) = \left(\ell_n\left(\frac{1}{2}\log\frac{(1+|p|)^2}{1-\overline{p}z} + 1\right)\right)^m \Big/ \big(L_{n+1}(p)\big)^{m-1}. \tag{2.7}$$

Then $\{f_{m,p} : p \in \mathbb{D}\}$ is a bounded subset of $\mathcal{LB}^n$ for each m. Moreover

$$\lim_{|p|\to 1} \|f_{m,p}\|_{\mathcal{LB}^n} = \lim_{|p|\to 1} \|\|f_{m,p}\|\|_n = \frac{m}{2}.$$

Proof. We have that

$$\begin{aligned} &(1-|z|^2)\prod_{k=1}^n L_k(z)\,|f'_{m,p}(z)| \\ = \;& \frac{m|p|}{2}\frac{1-|z|^2}{|1-\overline{p}z|}\prod_{k=0}^{n-1}\frac{\ell_k(\frac{1}{2}\log\frac{1+|z|}{1-|z|}+1)}{\left|\ell_k(\frac{1}{2}\log\frac{(1+|p|)^2}{1-\overline{p}z}+1)\right|}\left(\frac{\left|\ell_n(\frac{1}{2}\log\frac{(1+|p|)^2}{1-\overline{p}z}+1)\right|}{\ell_n(\frac{1}{2}\log\frac{1+|p|}{1-|p|}+1)}\right)^{m-1}. \end{aligned}$$

Assume that $\{f_{m,p} : p \in \mathbb{D}\}$ is not a bounded subset of $\mathcal{LB}^n$. Then there exist sequences $\{z_j\}$ and $\{p_j\}$ in $\mathbb{D}$ such that

$$(1-|z_j|^2)\prod_{k=1}^n L_k(z_j)\,|f'_{m,p_j}(z_j)| \to \infty$$

as $j \to \infty$. By passing to a subsequence we may assume that $|z_j| \to 1$ and $\overline{p_j} z_j \to 1$ as j goes to ∞. Notice that there exist a constant $C > 0$ and a positive integer N such that for $j > N$,

$$\Big|\ell_n\Big(\frac{1}{2}\log\frac{(1+|p_j|)^2}{1-\overline{p_j}z_j}+1\Big)\Big| \le \ell_n\Big(\frac{1}{2}\log\frac{1+|p_j|}{1-|p_j|}+1\Big)+C. \tag{2.8}$$

This implies that there exists some $0 \le k \le n-1$ satisfying

$$\lim_{j\to\infty}\frac{(1-|z_j|)^{\frac{1}{n}}}{|1-\overline{p_j}z_j|^{\frac{1}{n}}}\frac{\ell_k(\frac{1}{2}\log\frac{1+|z_j|}{1-|z_j|}+1)}{\big|\ell_k(\frac{1}{2}\log\frac{(1+|p_j|)^2}{1-\overline{p_j}z_j}+1)\big|} = \infty. \tag{2.9}$$

Thus for all j sufficiently large such that $(1+|p_j|)^2 \ge 2$, we have

$$\ell_k\Big(\frac{1}{2}\log\frac{1+|z_j|}{1-|z_j|}+1\Big) \le \ell_k\Big(\frac{1}{2}\log\frac{2}{1-|z_j|}+1\Big) \tag{2.10}$$

and

$$\Big|\ell_k\Big(\frac{1}{2}\log\frac{(1+|p_j|)^2}{1-\overline{p_j}z_j}+1\Big)\Big| \ge \ell_k\Big(\frac{1}{2}\log\frac{2}{|1-\overline{p_j}z_j|}+1\Big). \tag{2.11}$$

Combining (2.9), (2.10) and (2.11), we get

$$\lim_{j\to\infty}\frac{(1-|z_j|)^{\frac{1}{n}}\,\ell_k(\frac{1}{2}\log\frac{2}{1-|z_j|}+1)}{|1-\overline{p_j}z_j|^{\frac{1}{n}}\,\ell_k(\frac{1}{2}\log\frac{2}{|1-\overline{p_j}z_j|}+1)} = \infty. \tag{2.12}$$

For $x > 0$, consider $h(x)$ defined by (2.3). Then it follows that

$$h(1-|z_j|) \le h(|1-\overline{p_j}z_j|) \tag{2.13}$$

for every j big enough. This is a contradiction to (2.12). Thus the family $\{f_{m,p}\}$ is uniformly bounded in $\mathcal{LB}^n$ for every m.

Next we prove $\|f_{m,p}\|_{\mathcal{LB}^n} \to m/2$ as $|p| \to 1$. Since $f_{m,p}$ converges to 0 uniformly on every compact subset of $\mathbb{D}$ as $|p| \to 1$, $|f_{m,p}(0)| \to 0$. It is enough to prove that $\|\!|f_{m,p}|\!\|_n \to m/2$ as $|p| \to 1$. Here we put

$$M = \lim_{|p|\to 1} \|\!|f_{m,p}|\!\|_n.$$

Since $\{f_{m,p}\}$ is a bounded subset of $\mathcal{LB}^n$, we have $M < \infty$. Note that

$$\|\!|f_{m,p}|\!\|_n \ge (1-|p|^2)\prod_{k=1}^{n} L_k(p)\,|f'_{m,p}(p)| = \frac{m}{2}.$$

Thus $M \ge m/2$. For every positive integer j, we can choose p_j and z_j such that

$$M - \frac{1}{j} \le (1-|z_j|^2)\prod_{k=1}^{n} L_k(z_j)\,|f'_{m,p_j}(z_j)|.$$

By passing to a subsequence we may assume that $\overline{p_j} z_j \to 1$. Combining (2.8), (2.10), (2.11) and (2.13),

$$\begin{aligned} M - \frac{1}{j} &\leq \frac{m|p_j|}{2} \frac{1-|z_j|^2}{|1-\overline{p_j}z_j|} \prod_{k=0}^{n-1} \frac{\ell_k(\frac{1}{2}\log\frac{2}{1-|z_j|}+1)}{\ell_k(\frac{1}{2}\log\frac{2}{|1-\overline{p_j}z_j|}+1)} \\ &\quad \times \left(\frac{\ell_n(\frac{1}{2}\log\frac{1+|p_j|}{1-|p_j|}+1)+C}{\ell_n(\frac{1}{2}\log\frac{1+|p_j|}{1-|p_j|}+1)} \right)^{m-1} \\ &\leq \frac{m|p_j|}{2} \left(\frac{\ell_n(\frac{1}{2}\log\frac{1+|p_j|}{1-|p_j|}+1)+C}{\ell_n(\frac{1}{2}\log\frac{1+|p_j|}{1-|p_j|}+1)} \right)^{m-1}. \end{aligned}$$

Taking $j \to \infty$, we get $M \leq m/2$. The proof is complete. □

We give some estimate on the induced distance $d_{\mathcal{LB}^n}$.

Proposition 2.5. (i) *There exists a constant $C > 0$ such that for $w \in \mathbb{D}$*

$$C\Big(\log L_n(w) - \log \ell_{n-1}(\log 2 + 1)\Big) \leq d_{\mathcal{LB}^n}(w,0) \leq \log L_n(w).$$

Moreover, we have

$$\lim_{|w|\to 1} \frac{d_{\mathcal{LB}^n}(w,0)}{\log L_n(w)} = 1.$$

(ii) *For every $w \in D \setminus \{0\}$ and $C > 1$, there exists a small neighborhood U of w such that for every $z \in U$*

$$d_{\mathcal{LB}^n}(w,z) \leq C \frac{\log\left(\ell_{n-1}(\beta(w,z)+1)\right)}{\prod_{k=0}^{n-1} L_k(w)}.$$

Proof. (i) From the proof of Proposition 2.3, we have that

$$d_{\mathcal{LB}^n}(w,0) \leq \log L_n(w).$$

Using a test function $f_{2,w}/\|\!|f_{2,w}|\!\|_n$, we obtain that

$$d_{\mathcal{LB}^n}(w,0) \geq \frac{1}{\|\!|f_{2,w}|\!\|_n} \frac{L_{n+1}(w)^2 - \left(\ell_n\left(\frac{1}{2}\log(1+|w|)^2+1\right)\right)^2}{L_{n+1}(w)}.$$

Since $(1-|w|)^2 \leq (1+|w|)/(1-|w|)$, we get

$$\begin{aligned} d_{\mathcal{LB}^n}(w,0) &\geq \frac{1}{\|\!|f_{2,w}|\!\|_n}\Big(L_{n+1}(w) - \ell_n(\log(1+|w|)+1)\Big) \\ &\geq \frac{1}{\|\!|f_{2,w}|\!\|_n}\Big(\log L_n(w) - \log \ell_{n-1}(\log 2+1)\Big). \qquad (2.14) \end{aligned}$$

Put

$$C = \left(\sup_{w\in\mathbb{D}} \|\!|f_{2,w}|\!\|_n\right)^{-1}.$$

By Lemma 2.4, we have that $C > 0$. Thus we get

$$d_{\mathcal{LB}^n}(w,0) \leq C\Big(\log L_n(w) - \log \ell_{n-1}(\log 2+1)\Big).$$

From (2.14), we have that

$$\frac{1}{\|\!|f_{2,w}\|\!|_n}\left(1 - \frac{\log \ell_{n-1}(\log 2+1)}{\log L_n(w)}\right) \leq \frac{d_{\mathcal{LB}^n}(w,0)}{\log L_n(w)} \leq 1.$$

Lemma 2.4 implies that $\|\!|f_{2,w}\|\!|_n \to 1$ as $|w| \to 1$. Therefore we get

$$\lim_{|w|\to 1} \frac{d_{\mathcal{LB}^n}(w,0)}{\log L_n(w)} = 1.$$

(ii) Let f be a function of $\mathcal{LB}^n$ with $\|\!|f\|\!|_n \leq 1$. For $z \neq w$, denote by $[w,z]$ the segment between w and z. Notice that

$$\frac{|f(w)-f(z)|}{|w-z|} \leq \sup_{\xi\in[w,z]} |f'(\xi)| \leq \sup_{\xi\in[w,z]} \frac{1}{(1-|\xi|^2)\prod_{k=1}^n L_k(\xi)}.$$

It implies that

$$\limsup_{z\to w} \frac{d_{\mathcal{LB}^n}(w,z)}{|w-z|} \leq \frac{1}{(1-|w|^2)\prod_{k=1}^n L_k(w)}. \tag{2.15}$$

On the other hand,

$$\lim_{z\to w} \frac{\log \ell_{n-1}(\beta(w,z)+1)}{|w-z|} = \lim_{z\to w}\left(\prod_{k=1}^{n-1} \frac{\ell_k(\beta(w,z)+1)}{\ell_{k-1}(\beta(w,z)+1)}\,\frac{\beta(w,z)}{|w-z|}\right) = \frac{1}{1-|w|^2}. \tag{2.16}$$

Combining (2.15) and (2.16) we get the desired conclusion. $\square$

3. Weighted composition operator on $\mathcal{LB}^n$

In this section, we characterize the boundedness and the compactness of uC_φ on $\mathcal{LB}^n$. We define the generalized logarithmic hyperbolic derivative corresponding to $\mathcal{LB}^n$.

Definition 3.1. For $\varphi \in S(\mathbb{D})$, define that

$$\varphi_n^{\#}(z) = \frac{(1-|z|^2)\prod_{k=1}^n L_k(z)}{(1-|\varphi(z)|^2)\prod_{k=1}^n L_k(\varphi(z))}\,\varphi'(z).$$

Now we get the following.

Theorem 3.2. *Let u be in $H(\mathbb{D})$ and φ be in $S(\mathbb{D})$. Then uC_φ is bounded on $\mathcal{LB}^n$ if and only if $\|u\,\varphi_n^{\#}\|_\infty < \infty$ and*

$$\sup_{z\in\mathbb{D}} (1-|z|^2)\prod_{k=1}^n L_k(z)\, L_{n+1}(\varphi(z))|u'(z)| < \infty.$$

Proof. Since $L_n(w) \geq 1$ for any $w \in \mathbb{D}$, the growth condition (2.6) implies that

$$\begin{aligned} \|uC_\varphi\|_{\mathcal{LB}^n} &= \sup_{\|f\|_{\mathcal{LB}^n} \leq 1} \Big(|u(0)\, f(\varphi(0))| + \|\!|uC_\varphi f\|\!|_n \Big) \\ &\leq |u(0)| \log L_n(\varphi(0)) + \sup_{\|f\|_{\mathcal{LB}^n} \leq 1} \|\!|uC_\varphi f\|\!|_n. \end{aligned}$$

So we can see that uC_φ is bounded on $\mathcal{LB}^n$ if and only if

$$\sup_{\|f\|_{\mathcal{LB}^n} \leq 1} \|\!|uC_\varphi f\|\!|_n < \infty.$$

Suppose that $\|u\,\varphi_n^{\#}\|_\infty < \infty$ and

$$\sup_{z\in\mathbb{D}} (1-|z|^2) \prod_{k=1}^{n} L_k(z)\, L_{n+1}(\varphi(z)) |u'(z)| < \infty.$$

Let $f \in \mathcal{LB}^n$ with $\|f\|_{\mathcal{LB}^n} \leq 1$. By (2.6) again, we have that

$$\begin{aligned} \|\!|uC_\varphi f\|\!|_n &= \sup_{z\in\mathbb{D}} (1-|z|^2) \prod_{k=1}^{n} L_k(z) \big| u'(z)\, f(\varphi(z)) + u(z)\, \varphi'(z)\, f'(\varphi(z)) \big| \\ &\leq \sup_{z\in\mathbb{D}} \big| u(z)\, \varphi_n^{\#}(z) \big| \, (1-|\varphi(z)|^2) \prod_{k=1}^{n} L_k(\varphi(z))\, f'(\varphi(z)) \\ &\quad + \sup_{z\in\mathbb{D}} (1-|z|^2) \prod_{k=1}^{n} L_k(z) \big| u'(z)\, f(\varphi(z)) \big| \\ &\leq \sup_{z\in\mathbb{D}} \big| u(z)\, \varphi_n^{\#}(z) \big| + \sup_{z\in\mathbb{D}} (1-|z|^2) \prod_{k=1}^{n} L_k(z) L_{n+1}(\varphi(z)) \big| u'(z) \big|. \end{aligned}$$

Hence we have that uC_φ is bounded on $\mathcal{LB}^n$.

To prove the converse, we use the test functions $f_{m,p}$ of (2.7). Remark that $f_{m,p}(p) = L_{n+1}(p)$ and

$$f'_{m,p}(p) = \frac{m}{2} \cdot \frac{\overline{p}}{(1-|p|^2) \prod_{k=1}^{n} L_k(p)}.$$

Let C denote a positive constant independent of any point $w \in \mathbb{D}$, whose value is not necessarily the same at each occurrence. By Lemma 2.4, there exists a constant $C > 0$ such that for any $w \in \mathbb{D}$

$$\begin{aligned} C &\geq \|\!|uC_\varphi f_{2,\varphi(w)}\|\!|_n \\ &\geq (1-|w|^2) \prod_{k=1}^{n} L_k(w) \\ &\quad \times \big| u'(w) f_{2,\varphi(w)}(\varphi(w)) + u(w)\varphi'(w) f'_{2,\varphi(w)}(\varphi(w)) \big| \\ &= \Big| (1-|w|^2) \prod_{k=1}^{n} L_k(w) L_{n+1}(\varphi(w)) u'(w) + \overline{\varphi(w)}\, u(w)\, \varphi_n^{\#}(w) \Big|. \end{aligned}$$

Next, by the estimation of $\|uC_\varphi f_{4,\varphi(w)}\|_n$, we get

$$C \geq \left|(1-|w|^2)\prod_{k=1}^{n} L_k(w)L_{n+1}(\varphi(w))u'(w) + 2\overline{\varphi(w)}\,u(w)\,\varphi_n^{\#}(w)\right|.$$

By the triangle inequality, we obtain that $|u(w)\,\varphi_n^{\#}(w)| < C$ and

$$(1-|w|^2)\prod_{k=1}^{n} L_k(w)\,L_{n+1}(\varphi(w))|u'(w)| < C.$$

Since w is arbitrary, we get the assertion. □

In the case that $\varphi(z) = z$, the theorem above gives the answer to the multiplier problem on $\mathcal{LB}^n$.

Corollary 3.3. *Let u be in $H(\mathbb{D})$. Then u is a multiplier on $\mathcal{LB}^n$, that is, M_u is bounded on $\mathcal{LB}^n$ if and only if $u \in H^\infty \cap \mathcal{LB}^{n+1}$.*

On the other hand, in the case that $u(z) \equiv 1$, we get the characterization of the boundedness of C_φ.

Corollary 3.4. *Let φ be in $S(\mathbb{D})$. Then C_φ is bounded on $\mathcal{LB}^n$ if and only if $\|\varphi_n^{\#}\|_\infty < \infty$.*

Next we consider the compactness of uC_φ on $\mathcal{LB}^n$. We prepare a lemma which is a generalization of Proposition 3.11 in [3].

Lemma 3.5. *Let u be an analytic function and φ be an analytic self-maps of $\mathbb{D}$. Then the following are equivalent:*

(i) *uC_φ is compact on $\mathcal{LB}^n$.*
(ii) *$\|uC_\varphi f_j\|_{\mathcal{LB}^n} \to 0$ for any bounded sequence $\{f_j\}$ in $\mathcal{LB}^n$ that converges to 0 uniformly on every compact subset of $\mathbb{D}$.*
(iii) *$\|uC_\varphi f_j\|_n \to 0$ for any sequence $\{f_j\}$ as in (ii).*

We have also the following.

Theorem 3.6. *Let u be in $H(\mathbb{D})$ and φ be in $S(\mathbb{D})$. Suppose that uC_φ is bounded on $\mathcal{LB}^n$. Then uC_φ is compact on $\mathcal{LB}^n$ if and only if the following hold:*

(i) $\displaystyle\lim_{|\varphi(z)|\to 1} |u(z)\,\varphi_n^{\#}(z)| = 0,$

(ii) $\displaystyle\lim_{|\varphi(z)|\to 1} (1-|z|^2)\prod_{k=1}^{n} L_k(z)\,L_{n+1}(\varphi(z))\,|u'(z)| = 0.$

Proof. First, we suppose that uC_φ is compact on $\mathcal{LB}^n$. Let $\{w_j\}$ be a sequence in $\mathbb{D}$ such that $|\varphi(w_j)| \to 1$ as $j \to \infty$. By Lemma 2.4, we have that both $\{f_{2,\varphi(w_j)}\}$

and $\{f_{4,\varphi(w_j)}\}$ are bounded sequences in $\mathcal{LB}^n$ that converges to 0 uniformly on every compact subset of $\mathbb{D}$. Here Lemma 3.5 implies that $\|uC_\varphi f_{2,\varphi(w_j)}\|_n \to 0$ and $\|uC_\varphi f_{4,\varphi(w_j)}\|_n \to 0$ as $j \to \infty$. We have that

$$\left|(1-|w_j|^2)\prod_{k=1}^{n} L_k(w_j)L_{n+1}(\varphi(w_j))u'(w_j) + \overline{\varphi(w_j)}\, u(w_j)\, \varphi_n^{\#}(w_j)\right| \to 0$$

and

$$\left|(1-|w_j|^2)\prod_{k=1}^{n} L_k(w_j)L_{n+1}(\varphi(w_j))u'(w_j) + 2\overline{\varphi(w_j)}\, u(w_j)\, \varphi_n^{\#}(w_j)\right| \to 0.$$

Thus we get the conditions (i) and (ii).

Conversely, for any bounded sequences $\{f_j\} \subset \mathcal{LB}^n$ that converges to 0 uniformly on every compact subset of $\mathbb{D}$,

$$\begin{aligned} \|uC_\varphi f_j\|_n \;\le\; & \sup_{z\in\mathbb{D}} \left|u(z)\, \varphi_n^{\#}(z)\right| (1-|\varphi(z)|^2) \prod_{k=1}^{n} L_k(\varphi(z))\, f_j'(\varphi(z)) \\ & + \sup_{z\in\mathbb{D}} (1-|z|^2) \prod_{k=1}^{n} L_k(z) \left|u'(z)\, f_j(\varphi(z))\right|. \end{aligned}$$

By the uniform convergence of $\{f_j\}$ and the conditions (i) and (ii), we get $\|uC_\varphi f_j\|_n \to 0$. Lemma 3.5 concludes that uC_φ is compact on $\mathcal{LB}^n$. □

If we assume the compactness of M_u on either $\mathcal{LB}^n$ or $\mathcal{LB}_o^n$, it would be required that $u \in H^\infty \cap \mathcal{LB}^{n+1}$ and $|u(z)| \to 0$ as $|z| \to 1$. Thus we get the following corollary.

Corollary 3.7. *Let $u \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$.*

(i) *M_u is compact on $\mathcal{LB}^n$ if and only if $u = 0$.*
(ii) *C_φ is compact on $\mathcal{LB}^n$ if and only if $\varphi_n^{\#}(z) \to 0$ as $|\varphi(z)| \to 1$.*

We give some examples.

Example 3.8. Let

$$u(z) = \ell_n\left(\frac{1}{2}\log\frac{2}{1-z} + 1\right),$$

$v(z) = 1+z$ and $\varphi(z) = (1-z)/2$. Then we have the following:

(i) M_u is not bounded on $\mathcal{LB}^n$.
(ii) C_φ is bounded on $\mathcal{LB}^n$, but is not compact on $\mathcal{LB}^n$.
(iii) uC_φ is bounded on $\mathcal{LB}^n$, but is not compact on $\mathcal{LB}^n$.
(iv) M_v is bounded on $\mathcal{LB}^n$, but is not compact on $\mathcal{LB}^n$.
(v) vC_φ is compact on $\mathcal{LB}^n$.

Proof. (i) Since $u \notin H^\infty$, M_u is not bounded on $\mathcal{LB}^n$.

(ii) We have that

$$|\varphi_n^\#(z)| = \frac{1}{2}\,\frac{1-|z|^2}{1-|\varphi(z)|^2}\prod_{k=0}^{n-1}\frac{\ell_k(\frac{1}{2}\log\frac{1+|z|}{1-|z|}+1)}{\ell_k(\frac{1}{2}\log\frac{1+|\varphi(z)|}{1-|\varphi(z)|}+1)}.$$

For $x > 0$, consider the function

$$h(x) = x^{\frac{1}{n}}\,\ell_k\Big(\frac{1}{2}\log\frac{2-x}{x}+1\Big).$$

Reasoning as in the proof of Proposition 2.2, we can show that there exists a constant $\delta > 0$ such that h is an increasing function on $(0,\delta)$. We prove that $\varphi_n^\#$ is bounded on $\mathbb{D}$. Put $\Delta = \{z \in \mathbb{D} : |z| \geq 1-\delta, |\varphi(z)| \geq 1-\delta\}$. Since $\varphi_n^\#(z)$ is bounded off Δ, it is enough to prove that $\varphi_n^\#(z)$ is bounded on Δ. For $z \in \Delta$ such that $|z| \geq |\varphi(z)|$, we have that $h(1-|z|) \leq h(1-|\varphi(z)|)$. Then we get $|\varphi_n^\#(z)| \leq 1$. For $z \in \Delta$ such that $|z| \leq |\varphi(z)|$, we have that $L_k(z) \leq L_k(\varphi(z))$ for each k. Then the Schwarz-Pick lemma implies that

$$|\varphi_n^\#(z)| \leq \frac{1}{2}\frac{1-|z|^2}{1-|\varphi(z)|^2} \leq 1.$$

Hence we obtain that $\varphi_n^\#$ is bounded on $\mathbb{D}$, that is, C_φ is bounded on $\mathcal{LB}^n$.

Let $x \in (-1,0)$. Then

$$\begin{aligned}\varphi_n^\#(x) &\geq \frac{1+|z|}{1+|\varphi(z)|}\prod_{k=0}^{n-1}\frac{\ell_k(\frac{1}{2}\log\frac{1-x}{1+x}+1)}{\ell_k(\frac{1}{2}\log\frac{4}{1+x}+1)}\\ &\to 1\end{aligned}$$

as $x \to -1$. Hence we have that C_φ is not compact on $\mathcal{LB}^n$.

(iii) To check that $|u(z)\varphi_n^\#(z)|$ is bounded on $\mathbb{D}$, it is enough to see that

$$\lim_{z\to 1}|u(z)\varphi_n^\#(z)| < \infty.$$

Since $\varphi(z) \to 0$ as $z \to 1$, we have that

$$\lim_{z\to 1}|u(z)\varphi_n^\#(z)| \leq \lim_{z\to 1} C(1-|z|^2)\prod_{k=0}^{n}\ell_k\left(\frac{1}{2}\log\frac{2}{1-|z|}+1\right) < \infty.$$

Next we have that

$$\begin{aligned}&\sup_{z\in\mathbb{D}}(1-|z|^2)\prod_{k=0}^{n}L_k(z)L_{n+1}(\varphi(z))|u'(z)|\\ \leq\ & C\sup_{z\in\mathbb{D}}(1-|z|^2)^{\frac{1}{2}}\prod_{k=0}^{n}L_k(z)\cdot\left(\frac{1-|z|^2}{1-|\varphi(z)|}\right)^{\frac{1}{2}}(1-|\varphi(z)|)^{\frac{1}{2}}L_{n+1}(\varphi(z))\\ <\ & \infty.\end{aligned}$$

Thus uC_φ is bounded on $\mathcal{LB}^n$.

On the other hand, we have that

$$\lim_{z \to -1} |u(z)\varphi_n^{\#}(z)| = 1.$$

This implies that uC_φ is not compact on $\mathcal{LB}^n$.

(iv) It is easy to see that M_v is bounded on $\mathcal{LB}^n$, but is not compact on $\mathcal{LB}^n$.

(v) We have that $vC_\varphi = M_v C_\varphi$ is bounded on $\mathcal{LB}^n$. We can see that

$$\lim_{z \to -1} |v(z)\varphi_n^{\#}(z)| = 0$$

and

$$\lim_{z \to -1} (1 - |z|^2) \prod_{k=0}^{n} L_k(z) L_{n+1}(\varphi(z)) |v'(z)| = 0.$$

Thus we have that vC_φ is compact on $\mathcal{LB}^n$. □

4. Weighted composition operator on $\mathcal{LB}_o^n$

In this section, we characterize the boundedness and the compactness of uC_φ on $\mathcal{LB}_o^n$.

Theorem 4.1. *Let u be in $H(\mathbb{D})$ and φ be in $S(\mathbb{D})$. Then the following are equivalent:*

(i) *uC_φ is bounded on $\mathcal{LB}_o^n$.*

(ii) *uC_φ is bounded on $\mathcal{LB}^n$ and $uC_\varphi z^k \in \mathcal{LB}_o^n$ for any non-negative integer k.*

(iii) *uC_φ is bounded on $\mathcal{LB}^n$ and $uC_\varphi z^k \in \mathcal{LB}_o^n$ for $k = 0, 1$.*

Proof. Suppose that uC_φ is bounded on $\mathcal{LB}_o^n$. Since the sequences $\{f_{m,p}\}$ defined in Lemma 2.4 are included in $\mathcal{LB}_o^n$, we obtain the same conditions as Theorem 3.2, that is, $\|u\,\varphi_n^{\#}\|_\infty < \infty$ and

$$\sup_{z \in \mathbb{D}} (1 - |z|^2) \prod_{k=1}^{n} L_k(z)\, L_{n+1}(\varphi(z)) |u'(z)| < \infty.$$

Therefore we get (ii).

Conversely, by the density of the set of all polynomials in $\mathcal{LB}_o^n$, (ii) implies (i).

The implication (ii) $\Rightarrow$ (iii) is trivial. Here we assume the condition (iii). Then we have that u and $u\varphi$ are in $\mathcal{LB}_o^n$, that is,

$$\lim_{|z| \to 1} (1 - |z|^2) \prod_{k=1}^{n} L_k(z)\, |u'(z)| = 0 \tag{4.1}$$

and

$$\lim_{|z| \to 1} (1 - |z|^2) \prod_{k=1}^{n} L_k(z)\, |u'(z)\varphi(z) + u(z)\varphi'(z)| = 0$$

Since $|\varphi(z)| \leq 1$, we get

$$\lim_{|z|\to 1} (1-|z|^2) \prod_{k=1}^{n} L_k(z)\, |u(z)\varphi'(z)| = 0 \tag{4.2}$$

We claim that $u\varphi^m \in \mathcal{LB}_o^n$ for any positive integer m. Indeed, by (4.1) and (4.2),

$$\begin{aligned}
&\lim_{|z|\to 1} (1-|z|^2) \prod_{k=1}^{n} L_k(z)\, \left|(u\varphi^m)'(z)\right| \\
=\ &\lim_{|z|\to 1} (1-|z|^2) \prod_{k=1}^{n} L_k(z)\, \left|u'(z)\varphi(z)^m + m\, u(z)\varphi'(z)\varphi(z)^{m-1}\right| \\
\leq\ &\lim_{|z|\to 1} (1-|z|^2) \prod_{k=1}^{n} L_k(z)\, \big(|u'(z)| + m|u(z)\varphi'(z)|\big) \\
=\ &0.
\end{aligned}$$

This means $u\varphi^m \in \mathcal{LB}_o^n$. Our proof is accomplished. □

From the definition, we can see that $\mathcal{LB}^{n+1} \subset \mathcal{LB}_o^n$. Hence we obtain the following corollary.

Corollary 4.2. *Let u be in $H(\mathbb{D})$. Then the following are equivalent:*

(i) *M_u is bounded on $\mathcal{LB}^n$.*
(ii) *M_u is bounded on $\mathcal{LB}_o^n$.*
(iii) *$u \in H^\infty \cap \mathcal{LB}^{n+1}$.*

To characterize the compactness of uC_φ on $\mathcal{LB}_o^n$, we use the following lemma which is a generalization of Lemma 1 of [5].

Lemma 4.3. *A closed set $K \in \mathcal{LB}_o^n$ is compact if and only if it is bounded and satisfies that*

$$\lim_{|z|\to 1} \sup_{f\in K} (1-|z|^2) \prod_{k=1}^{n} L_k(z)\, |f'(z)| = 0.$$

Theorem 4.4. *Let u be in $H(\mathbb{D})$ and φ be in $S(\mathbb{D})$. Then uC_φ is compact on $\mathcal{LB}_o^n$ if and only if the following hold:*

(i) $\displaystyle\lim_{|z|\to 1} |u(z)\, \varphi_n^{\#}(z)| = 0,$

(ii) $\displaystyle\lim_{|z|\to 1} (1-|z|^2) \prod_{k=1}^{n} L_k(z)\, L_{n+1}(\varphi(z))\, |u'(z)| = 0.$

Proof. First, we suppose the conditions (i) and (ii). Then it is easy to see that (i) implies (4.1). Also, since $L_{n+1}(\varphi(z)) \geq 1$, (4.2) follows from (ii). Hence, Theorem

4.1 implies that uC_φ is bounded on $\mathcal{LB}^n_o$. By Lemma 4.3, to prove the compactness of uC_φ on $\mathcal{LB}^n_o$, it suffices to check

$$\lim_{|z|\to 1} \sup_{\|f\|_n \le 1} (1-|z|^2) \prod_{k=1}^{n} L_k(z) \left|u'(z)f(\varphi(z)) + u(z)\varphi'(z)f'(\varphi(z))\right| = 0. \tag{4.3}$$

By the growth condition (2.3) and the definition of $\varphi^{\#}_n$,

$$\begin{aligned}
&\lim_{|z|\to 1} \sup_{\|f\|_n \le 1} (1-|z|^2) \prod_{k=1}^{n} L_k(z) \left|u'(z)f(\varphi(z)) + u(z)\varphi'(z)f'(\varphi(z))\right| \\
\le\ & \lim_{|z|\to 1} \sup_{\|f\|_n \le 1} \Bigg((1-|z|^2) \prod_{k=1}^{n} L_k(z) \Big(|f(0)| + \|f\|_n \log L_n(\varphi(z))\Big)|u(z)| \\
&\qquad + (1-|\varphi(z)|^2) \prod_{k=1}^{n} L_k(\varphi(z))\, |f'(\varphi(z))|\, |u(z)\varphi^{\#}_n(z)| \Bigg) \\
\le\ & \lim_{|z|\to 1} (1-|z|^2) \prod_{k=1}^{n} L_k(z)\, L_{n+1}(\varphi(z))|u(z)| + \lim_{|z|\to 1} |u(z)\varphi^{\#}_n(z)| \\
=\ & 0.
\end{aligned}$$

Hence we conclude that uC_φ is compact on $\mathcal{LB}^n_o$.

Conversely, we suppose that uC_φ is compact on $\mathcal{LB}^n_o$. We remark that uC_φ is also bounded on $\mathcal{LB}^n_o$. Let $\{z_j\}$ be a sequence in $\mathbb{D}$ such that $|z_j| \to 1$ but $|\varphi(z_j)| \to r < 1$ as $j \to \infty$. Then (4.1) and (4.2) yield that

$$\lim_{j\to\infty} |u(z_j)\, \varphi^{\#}_n(z_j)| = 0$$

and

$$\lim_{j\to\infty} (1-|z_j|^2) \prod_{k=1}^{n} L_k(z_j)\, L_{n+1}(\varphi(z_j))\, |u'(z_j)| = 0.$$

On the other hand, using the sequences of the test functions $\{f_{m,p}\}$ for $m = 2, 4$, we have that

$$\lim_{|\varphi(z)|\to 1} |u(z)\, \varphi^{\#}_n(z)| = 0$$

and

$$\lim_{|\varphi(z)|\to 1} (1-|z|^2) \prod_{k=1}^{n} L_k(z)\, L_{n+1}(\varphi(z))\, |u'(z)| = 0.$$

Hence we have (i) and (ii). $\square$

Corollary 4.5. *Let $u \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$.*

- (i) *The following are equivalent:*
  - (a) *M_u is compact on $\mathcal{LB}^n$.*
  - (b) *M_u is compact on $\mathcal{LB}^n_o$.*

(c) $u = 0$.

(ii) C_φ *is compact on* $\mathcal{LB}^n_o$ *if and only if* $|\varphi^{\#}_n(z)| \to 0$ *as* $|z| \to 1$.

Addendum. *After we prepared this paper, we found P. Galanopoulos' paper* [4] *in which the same results as Theorem 3.2 and Theorem 3.6 for the case of* $\mathcal{LB}^1$ *have been established. In* [4] *Galanopoulos has studied the family of the analytic function spaces* $Q^p_{\log}$ *including* $\mathcal{LB}^1$ *and the weighted composition operators from* $\mathcal{LB}^1$ *to* $Q^p_{\log}$.

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