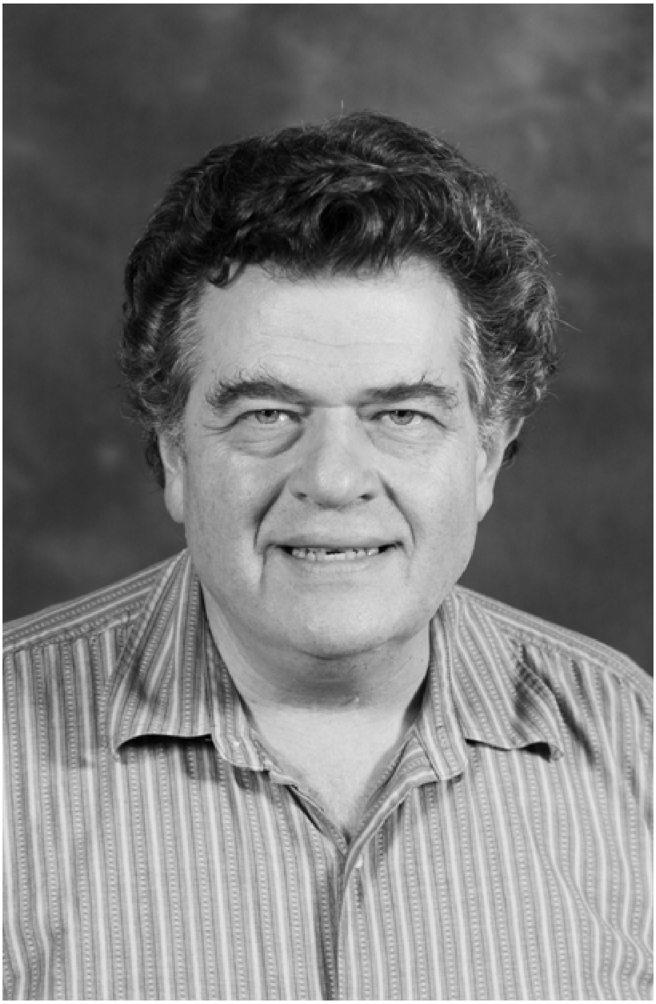
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Mel Hochster has had a transformative impact on the field of commutative algebra through his deep and enduring research as well as his generous and enthusiastic mentoring of the next generation. With our utmost respect and gratitude, we dedicate this special volume of the Michigan Mathematical Journal to Mel on the occasion of his 65th birthday.

Gopal Prasad, Managing Editor
Anurag Singh & Karen Smith

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Mathematical Biography of Melvin Hochster

Craig Huneke

Though commutative algebra had experienced two decades of strong growth during the 1950s and 1960s, it was still in its childhood. Notable successes—the proof of Auslander and Buchsbaum that regular local rings are unique factorization domains, the characterization of regularity as Noetherian local rings of finite global dimension by Auslander–Buchsbaum and Serre, and the characterization of rings of finite injective dimension by Bass—had paved the way for various homological conjectures, which largely came from the desire to understand modules of finite projective dimension or finite injective dimension. Among these conjectures were Bass's conjecture that a local Noetherian ring with a finitely generated module of finite injective dimension must be Cohen–Macaulay, and a particular favorite of the Chicago school headed by Irving Kaplansky, the zero-divisor conjecture, which stated that an element in a local Noetherian ring which was a nonzerodivisor on a finitely generated module of finite projective dimension was also necessarily a nonzerodivisor on the ring. Most of the young commutative algebraists in the United States at that time had some relationship to either Chicago or Brandeis, so it came as a bolt from the blue when a young researcher named Mel Hochster wrote a thesis on commutative algebra at Princeton and then became an assistant professor at the University of Minnesota.

Hochster's education was as rigorous as one can obtain in this country. He graduated from one of the three special schools in New York City, Stuyvesant High School, long known for its top-notch program in science (among its graduates are several Nobel Prize winners and Fields Medalists). There he captained the math team in 1960, and later that same year he became an undergraduate at Harvard with a prestigious Westinghouse Science Scholarship. He was a Putnam Fellow at Harvard in 1960—one of the top five finishers on the Putnam exam. He graduated in 1964 and went to Princeton for his graduate work. He received his Ph.D. in 1967 at the age of 23; his official advisor was Goro Shimura, and he talked some with Gerard Washnitzer, but he basically worked independently while at Princeton. His interest in commutative algebra can perhaps be traced back to courses at Harvard taught by John Tate. In particular, Hochster took a reading course from Tate on Nagata's *Local Rings.* Even earlier, Hochster recalls working problems with George Bergman from the book of Zariski and Samuel during his high-school years while on a camping trip, with the aid of a lantern in their tent.

Hochster's thesis contained results on two connected themes: symbolic powers of ideals and the structure of the Zariski topology of a commutative ring. His work on symbolic powers had as a corollary the well-known theorem of Nagata and Zariski that in a regular local ring the nth symbolic power of a prime ideal is contained in the nth power of the maximal ideal. He returned to the study of symbolic powers several times during the next thirty years—in fact, his most recent published paper is on this topic. His thesis contained a pathbreaking characterization of the spectrum of a commutative ring in the category of topological spaces; this work appeared in the *Transactions of the American Mathematical Society* in 1969 and remains one of his most cited papers. He has remarked that his work on the topology of the spectrum of a commutative ring is possibly his most difficult work.

At Minnesota, Hochster met Jack Eagon, who served as his mentor and sparked Hochster's interest in Cohen–Macaulay rings. This interest became a passion with several classic papers in the early 1970s: a paper with Eagon proving that generic determinantal ideals define Cohen–Macaulay rings (this appeared in the *American Journal of Mathematics* in 1971 and introduced a new method to prove ideals are self-radical, called *principal radical systems*); a paper in the *Annals of Mathematics* in 1972 that proved the now famous theorem that rings of invariants of tori are Cohen–Macaulay; and a paper in the *Journal of Algebra* in 1973 proving that Grassmannians and their Schubert subvarieties are (arithmetically) Cohen–Macaulay. These results can be rephrased, at least in equicharacteristic 0, as special cases of a more general and far-reaching result, that rings of invariants of linearly reductive linear algebraic groups over a field k acting on a regular Noetherian k-algebra are Cohen–Macaulay, a result proved by Hochster and Joel Roberts and appearing in *Advances in Mathematics* in 1974. The proof is a tour de force of commutative algebra and is done via reduction to characteristic p.

Hochster had three students while at Minnesota. Stanley–Reisner rings, which relate square-free monomial ideals to simplicial complexes, come from Reisner's thesis (and, of course, Stanley's work), done in Minnesota under Hochster's direction. His career at Minnesota, while short (from 1967 to 1973), was meteoric: he received tenure and rose to associate professor in three years. He left in 1973 to become a full professor at Purdue University, where he stayed, at least officially, until 1977. However, he spent a year in 1973–74 at the Mathematical Institute at Aarhus, Denmark, a year that turned out to be of enormous importance.

As mentioned above, the homological conjectures mostly had concentrated on properties of modules of finite projective or injective dimension. The breakthrough 1973 paper of Peskine and Szpiro solved many of the conjectures for rings of characteristic p or essentially of finite type over a field of characteristic 0. Hochster's fundamental insight was the importance of the existence of a module that is Cohen–Macaulay, even if it is not finitely generated. In a remarkable unpublished preprint *Deep Local Rings*, which was completed at Aarhus, Hochster established many of the homological conjectures for equicharacteristic local Noetherian rings, including Bass's conjecture and the zero-divisor conjecture, by proving the existence of a big Cohen–Macaulay module and showing how many of the other homological conjectures followed from the existence of such a module. The proof of the existence of a big Cohen–Macaulay module is a brilliant example of deep insight combined with elegant simplicity, a trademark of Hochster's work. Hochster's research in the early 1970s sent commutative algebra in new directions. Several new homological conjectures were added by Hochster during this time. These new conjectures included the direct summand conjecture, which stated that a module-finite extension algebra of a regular Noetherian ring R must split as a map of R-modules, and the monomial conjecture. A goal set by him was to understand constraints on systems of parameters, that is, to understand the concept of height in terms of equations that cannot be satisfied by any system of parameters. The monomial conjecture is a perfect example of this underlying philosophy. It states that if R is a local Noetherian ring and $x_1, \ldots, x_d$ is a system of parameters, then, for every positive integer t, one has $x_1^t \cdots x_d^t \notin (x_1^{t+1}, \ldots, x_d^{t+1})$. An important event during this time was the CBMS conference held in Lincoln, Nebraska, in the summer of 1974, in which Hochster gave ten lectures. The resulting publication, *Topics in the Homological Theory of Commutative Rings,* is a far-ranging synthesis of what Hochster knew at that time. It served as a bible to many researchers at that time. The direct summand and monomial conjectures still remain open in mixed characteristic (Hochster proved them in equicharacteristic), though Ray Heitmann achieved a major breakthrough in 2002 by proving these for mixed characteristic rings of dimension up to three. In the 1980s, Paul Roberts proved some of the original homological conjectures in mixed characteristic; both Bass's conjecture and the zero-divisor conjecture follow from Roberts's work. It is probably fair to say that the direct summand conjecture, or equivalently the monomial conjecture, has occupied a central role in Hochster's career. He has often expressed the desire that these conjectures be settled during his lifetime.

Though it is true that the homological conjectures occupied Hochster's attention, his work ranged well beyond those conjectures. In 1976, he followed up his work with Joel Roberts on invariants of reductive groups with an important paper concentrating on the purity of the Frobenius map, presaging the now well-known list of classifying singularities via the Frobenius map. In fact, this work was used by Reisner in his characterization of the Cohen–Macaulay property of the Stanley–Reisner ring. Hochster expanded on Reisner's work in his Oklahoma conference paper of 1976. In that paper he gave a (much-used) formula for the Betti numbers of the Stanley–Reisner ring, showed that the projective dimension depends on the characteristic of the ground field, and characterized when the Stanley–Reisner ring is Gorenstein. In 1977 he proved the Zariski–Lipman conjecture for graded rings. A short but influential paper with David Eisenbud in 1979 eventually led to work on uniform Artin–Rees theorems by O'Carroll, Duncan, and Huneke. Many honors came with Hochster's high level of research. He was an invited speaker at the International Congress of Mathematicians in Helsinki in 1978, and he received the Cole Prize in algebra in 1980 for his work on big Cohen–Macaulay modules and on rings of invariants.

After moving to Michigan, Hochster began a frequently repeating pattern of having a large number of graduate students. In 1978, Craig Huneke joined him as a postdoc, becoming one of the first of many postdocs supervised by Hochster at Michigan. Hochster was awarded a Guggenheim Fellowship in 1982, which he took at MIT. Among other results in his early years at Michigan was one on the multiplicity conjectures of Serre; a paper in 1985, in *Inventiones Mathematicae,* together with Sankar Dutta and Jack McLaughlin, gave a remarkable example of a module of finite projective dimension with negative intersection multiplicity. In 1983 he introduced the canonical element conjecture, proving it in equicharacteristic and showing its equivalence to the direct summand conjecture.

In 1986 Hochster's research took an unexpected and dramatic turn, one that has grown into a lasting legacy. He realized that many of the characteristic p proofs had one common feature that could be isolated and studied. This insight was truly profound: for example, if one looks back at the proof of the existence of big Cohen–Macaulay modules, or his proof with Joel Roberts that rings of invariants of linearly reductive groups are Cohen–Macaulay, or Paul Roberts's proof of the new intersection theorem, all of which use reduction to characteristic p, one would be hard-pressed to see any common method without the benefit of hindsight. The common feature found by Hochster became the theory of tight closure, developed first by Hochster and Huneke over a series of about fifteen papers and later by many other researchers, including several of Hochster's students. The late 1980s and early 1990s were an exciting time of rapid development. The newly discovered notion proved extremely fruitful. It provided quick proofs and generalizations of many of the previous results, including the Hochster–Roberts theorem on invariants (tight closure was used to prove that direct summands of equicharacteristic regular local rings are Cohen–Macaulay), many of the homological conjectures, the Briançon–Skoda theorem, and the syzygy theorem of Graham Evans and Phil Griffith. Not only did the theory provide much easier proofs, but it offered new insight and yielded new theorems as well.

Tight closure theory was fully introduced in Hochster and Huneke's most cited paper in the *Journal of the American Mathematical Society* in 1990. Besides work already mentioned, this paper gave a tight closure version of the Buchsbaum–Eisenbud acyclicity criterion, and a tight closure analogue of David Rees's characterization of integral closure, by replacing multiplicity with the Hilbert–Kunz multiplicity. Other highlights followed, including the proof in 1992, by Hochster and Huneke in the *Annals of Mathematics,* that the integral closure of a complete local Noetherian domain of positive characteristic in an algebraic closure of its fraction field is a big Cohen–Macaulay algebra, and the thesis of Karen Smith that among other things proved that the tight closure and plus closure agreed for parameter ideals in rings of positive characteristic. Other notable papers during this period include the introduction of strong F-regularity (1989), work on the tight closure of parameters and F-rationality (1994), the existence of test ideals in great generality (1994), and the existence and application of big Cohen–Macaulay algebras (1995). Hochster was elected to both the National Academy of Sciences and the American Academy of Arts and Sciences in 1992.

During the 1990s, it became clear that tight closure methods and characteristic 0 methods, coming from resolution of singularities and variants of the Kodaira vanishing theorem, were somehow related. The Japanese school headed by Kei-ichi Watanabe played a leading role in classifying singularities through properties of the Frobenius homomorphism in characteristic p. An early highlight was the proof— one direction due to Karen Smith, the other direction independently to Nobuo Hara and to Vikram Mehta and Vasudevan Srinivas—that rational singularities correspond to rings of F-rational type. With the advent of asymptotic multiplier ideals as described by Lawrence Ein, Rob Lazarsfeld, and Karen Smith in 2001, and the development of test ideals for pairs by Nobuo Hara, Shunsuke Takagi, and Ken-ichi Yoshida a few years later, this relationship became much better understood. Many applications followed, including important work on symbolic powers by Ein, Lazarsfeld, and Smith. In 2002, Hochster and Huneke generalized this work in a paper in *Inventiones* using tight closure theory.

One of the ideas suggested by tight closure theory is to find similar ideal closures in either equicharacteristic or mixed characteristic. This has proved to be very difficult. Hochster proposed one such theory in his work on solid closure (1994). Although it did not totally capture what is needed to make a similar theory, it did provide a foundation for Holger Brenner's definitive study of tight closure in two-dimensional graded rings, which in turn led to the recent example (2007) by Brenner and Paul Monsky showing that tight closure does not commute with localization, answering in the negative a twenty-year-old question.

Very few people revolutionize their fields; it is fair to say that Hochster has done so twice. It is almost impossible even to imagine what the state of commutative algebra would be without him, other than to realize it would be a much poorer subject. He devotes much of his time to his many students and postdocs; at the latest count he has had thirty-five students, with more on the way. He has written more than eighty papers with fourteen co-authors. Throughout his career he has been active in his department, his university, and in national organizations such as the American Mathematical Society and various mathematics institutes, including MSRI and the IMA. He has been an editor of several journals, including the *Journal of Algebra, Advances in Mathematics,* and the *Michigan Mathematical Journal*. In recent years, through his participation in the ADVANCE program of NSF, he has become an active educator for mathematicians and other scholars concerning the problems faced by women in the sciences. Not slowing down at 65, he is becoming chair of the mathematics department at Michigan. This volume is dedicated to his generosity of spirit, to his deep insight into commutative algebra, and to the multitude of beautiful theorems he has given us.

Department of Mathematics
University of Kansas
Lawrence, KS 66045

huneke@math.ku.edu