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Erratum

Rafał Czyż

Theorem 4.1 in [Cz] is false. The following example is due to Nguyen Quang Dieu [N].

Example. Let B be the unit ball in $\mathbb{C}^2$. We show that there exists a sequence of continuous functions $f_j \in \mathcal{C}(\partial B)$ such that (a) f_j converges pointwise to 0 as $j \to \infty$ but (b) the Perron–Bremermann envelope $U(f_j, 0)$ does not converge in capacity to 0.

Let a_j be an increasing sequence of positive numbers and b_j a decreasing sequence of positive numbers such that $a_j^2 + b_j^2 = 1$, with $\lim_{j\to\infty} a_j = a > 0$ and $\lim_{j\to\infty} b_j = b > 0$. Define the sets

$$T_j = \{(z,w) \in \partial B : |z| = a_j,\ |w| = b_j\}.$$

Observe that there exists a sequence of open sets $U_j \subset \partial B$ such that $T_j \subset U_j$ and $U_j \cap U_k = \emptyset$ for $j \neq k$. By the Tietze extension theorem, there exists a sequence of continuous functions $f_j \in \mathcal{C}(\partial B)$ such that $-1 \le f_j \le 0$, $f_j = -1$ on T_j, and $f_j = 0$ on $\partial B \setminus U_j$. Now it is easy to see that f_j converges pointwise to 0 as $j \to \infty$. Let

$$\Omega_j = \{(z,w) \in B : |z| < a_j,\ |w| < b_j\}.$$

The maximum principle for plurisubharmonic functions yields that $U(f_j, 0) \le -1$ on Ω_j and therefore $U(f_j, 0) = -1$ on Ω_j. Define the following open subset of B:

$$\Omega = \{(z,w) \in B : |z| < a_1,\ |w| < b\};$$

then we have $U(f_j, 0) = -1$ on Ω for all j, which implies that $U(f_j, 0)$ does not converge in capacity to 0.

The preceding example shows that pointwise convergence of boundary value is not enough to assure convergence in capacity of the Perron–Bremermann envelope. Following ideas from [N], we will give in Theorem 4.1′ some sufficient conditions that guarantee convergence in capacity of the Perron–Bremermann envelope. Let $\mathcal{MF}^a = \mathcal{MF}^a(\Omega)$ be the set of all positive finite measures μ on Ω such that μ vanishes on all pluripolar sets in Ω.

Theorem 4.1′. *Let $\Omega \subset \mathbb{C}^n$ be a bounded B-regular domain, let $\mu \in \mathcal{MF}^a$, let f_j be a uniformly bounded sequence of upper semicontinuous functions on $\partial\Omega$,*

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and let f be a bounded upper semicontinuous function on $\partial\Omega$. Let E be a subset of $\partial\Omega$ such that there exists a negative plurisubharmonic function ψ defined in Ω such that, for all $w \in E$,

$$\psi^*(w) = \limsup_{\Omega \ni z \to w} \psi(z) = -\infty.$$

If $f_j \to f$ locally uniformly on $\partial\Omega \setminus E$, then $U(f_j, \mu) \to U(f, \mu)$ in capacity as $j \to \infty$.

Proof. First we prove the theorem for $f = 0$, $\mu = 0$, and f_j a uniformly bounded sequence of measurable functions (not necessary upper semicontinuous) on $\partial\Omega$. Fix $\varepsilon > 0$. Since f_j a uniformly bounded and tends locally uniformly to 0 on $\partial\Omega \setminus E$, it follows that, for all j large enough, on $\partial\Omega$ we obtain

$$f_j + \varepsilon\psi^* \le \varepsilon \tag{1}$$

and

$$\varepsilon\psi^* - \varepsilon \le f_j. \tag{2}$$

Inequalities (1) and (2) imply that on Ω we have

$$\varepsilon\psi - \varepsilon \le U(-f_j, 0) \le -U(f_j, 0)$$

and

$$\varepsilon\psi - \varepsilon \le U(f_j, 0),$$

which implies that, on Ω,

$$|U(f_j, 0)| \le |\varepsilon - \varepsilon\psi|. \tag{3}$$

Fix a compact set $K \subset\subset \Omega' \subset\subset \Omega$, a plurisubharmonic function $-1 < u < 0$, and $\delta > 0$. Then by (3) it follows that

$$\begin{aligned}\int_{K\cap\{|U(f_j,0)|>\delta\}} (dd^c u)^n &\le \frac{1}{\delta}\int_K |U(f_j,0)|(dd^c u)^n \le \frac{1}{\delta}\int_K |\varepsilon - \varepsilon\psi|(dd^c u)^n \\ &\le \frac{\varepsilon}{\delta}\left(\int_K (dd^c u)^n + \int_K (-\psi)(dd^c u)^n\right) \\ &\le \frac{\varepsilon}{\delta}(\mathrm{cap}(K) + C(K,\Omega')\|\psi\|_{L^1(\Omega')}\|u\|_\infty^n) \\ &\le \frac{\varepsilon}{\delta}(\mathrm{cap}(K) + C(K,\Omega')\|\psi\|_{L^1(\Omega')}),\end{aligned}$$

where $C(K, \Omega')$ is a constant depending only on K and Ω' (see [Ce; D]). This implies that $U(f_j, 0) \to 0$ in capacity as $j \to \infty$. We know (see [BT]) that $U(f_j, 0) = U(f_j, 0)^*$ outside a pluripolar set, so we have proved also that $U(f_j, 0)^* \to 0$ in capacity as $j \to \infty$.

In the general case, observe that

$$U(f_j - f, 0) \le U(f_j, \mu) - U(f, \mu) \le -U(f - f_j, 0); \tag{4}$$

by the first part of the proof we know that $U(f - f_j, 0)^*, U(f_j - f, 0)^* \to 0$ in capacity as $j \to \infty$, which implies that $U(f_j, \mu) \to U(f, \mu)$ in capacity as $j \to \infty$. This ends the proof. □

Remark. In [N] Nguyen proved that, under the assumptions of Theorem 4.1′, $U(f_j, \mu) \to U(f, \mu)$ pointwise as $j \to \infty$ outside some pluripolar set in Ω. Theorem 4.1′ generalizes that result to convergence in capacity. It is possible to get even more. Observe that by (3) and (4) one can obtain that $U(f_j, \mu) \to U(f, \mu)$ locally uniformly on the set $\{\psi > -\infty\}$ as $j \to \infty$.

We can also reformulate the statement of Theorem 4.5 in [Cz] as follows.

Theorem 4.5′. *Let $\Omega \subset \mathbb{C}^n$ be a bounded strictly pseudoconvex domain, and let $\mu \in \mathcal{MF}^a$. Let f_j, f be uniformly bounded upper semicontinuous functions on the boundary, where $f_j \to f$ locally uniformly on $\partial\Omega \setminus E$ as $j \to \infty$ (with E as in Theorem 4.1′). Suppose the functions g_j, g are μ-measurable with values in $[0, 1]$ and that they satisfy $g_j d\mu \to g d\mu$ for $j \to \infty$. Then $U(f_j, g_j d\mu) \to U(f, g d\mu)$ in capacity as $j \to \infty$.*

Proof. The proof is the same as the proof of Theorem 4.5 in [Cz], but instead of using Theorem 4.1 we here use Theorem 4.1′. □

Theorem 4.4 and Corollary 4.6 in [Cz] are false.

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References

[BT] E. Bedford and B. A. Taylor, *A new capacity for plurisubharmonic functions,* Acta Math. 149 (1982), 1–40.

[Ce] U. Cegrell, *An estimate of the complex Monge–Ampère operator,* Analytic functions (Błażejewko, 1982), Lecture Notes in Math., 1039, pp. 84–87, Springer-Verlag, Berlin, 1983.

[Cz] R. Czyż, *Convergence in capacity of the Perron–Bremermann envelope,* Michigan Math. J. 53 (2005), 497–509.

[D] J.-P. Demailly, *Monge–Ampère operators, Lelong numbers and intersection theory,* Complex analysis and geometry, pp. 115–193, Univ. Ser. Math., Plenum, New York, 1993.

[N] Q. D. Nguyen, *Convergence of Perron–Bremermann envelope,* preprint, 2006.

Department of Mathematics
Jagiellonian University
Reymonta 4
30-059 Cracow
Poland

rafal.czyz@im.uj.edu.pl