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NONEXISTENCE RESULTS FOR HESSIAN INEQUALITY*

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Abstract. In this paper , the author proves a Liouville type theorem for some Hessian entire inequality with *sub-lower-critical* exponent, via suitable choices of test functions and the argument of integration by parts .

Key words. Hessian inequality, Liouville theorem, integration by parts.

AMS subject classifications. 35J60

1. Introduction. On a compact manifold with no boundary, one can integrates by parts freely without any obstacle. When the manifold is not compact or has some boundaries, the same argument can be done by using a suitable test function. Hence, the argument of integration by parts has been used widely for a long time in the study of partial differential equations and in differential geometry.

In this paper, via the argument of integration by parts, we first study the classical k-Hessian inequality (1.1) with the equality as the special case. We will deduce the Liouville type theorem of this inequality with *sub-lower-critical* exponent. Then we extend the result to the general case of *k-Hessian measure* by approximation.

Consider the following differential inequality:

$$\sigma_k(-D^2u) \geq u^\alpha \quad in \quad \mathbb{R}^n \tag{1.1}$$

where $\sigma_k(-D^2u)$ are the k-Hessian of $(-D^2u)$ as usual (see (2.1)).

When $k = 1$, then (1.1) coincides with the Laplacian inequality $-\triangle u \geq u^\alpha$ in $\mathbb{R}^n$, and some splendid results had been given by Gidas-Spruck [5] in case of equality. Inequality (1.1) had also been studied by many works, such as Phuc-Verbitsky [9, 10] and references there in.

When $2k < n$, denote

$$k^* := \frac{n(k+1)}{n-2k},\ k_* := \frac{nk}{n-2k}.$$

Then k^* is the *critical exponent* for Sobolev embedding in the sense of Wang [15], and we call k_* the *lower critical exponent*.

According to Caffarelli-Nirenberg-Spruck [1], we say u *k-admissible* (or *k-convex*) with respect to $\sigma_k(-D^2u)$ if $u \in \Gamma_k$, where Γ_k is defined by

$$\Gamma_k = \{u \in C^2(\mathbb{R}^n) : \sigma_s(-D^2u) \geq 0, s = 1, 2, \cdots, k\}.$$

Now we state our nonexistence result as follows:

THEOREM 1.1. *If $2k < n$, then (1.1) has no positive solution in Γ_k for any $\alpha \in (-\infty, k_*]$.*

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In fact, the result in Theorem 1.1 can be extended to *general k-convex* functions. Let Ω be a domain in $\mathbb{R}^n$, then an upper semi-continuous function u: $\Omega \to [-\infty, \infty)$ is called *general k-convex* in Ω if $-q \in \Gamma_k$ for all quadratic polynomials q for which the difference $u-q$ has a finite local maximum in Ω (see [3] or [13]). Denote by $\Phi_k(\Omega)$, the class of *general k-convex* functions in Ω which do not assume the value $-\infty$ identically on any component of Ω. Associated to the functions in $\Phi_k(\Omega)$, Trudinger-Wang [12, 13] introduced a Borel measure, called *k-Hessian measure*. In [12, 13], they also deduced some fundamental properties of the *general k-convex* functions and of the *k-Hessian measure*, especially, the followings will be needed in this paper:

Proposition 1.2. *A function u: $\Omega \to [-\infty, \infty)$ is general k-convex in Ω if and only if its restriction to any subdomain $\Omega' \subset\subset \Omega$ is the limit of a monotone decreasing sequence in $\Phi_k(\Omega') \cap C^2(\Omega')$.*

Proposition 1.3. *For any $u \in \Phi_k(\Omega)$, there exists a Borel measure $\mu_k[u]$ in Ω such that*

(a) $\mu_k[u] = \sigma_k(D^2u)$ for $u \in C^2(\Omega)$, and

(b) if $\{u_j\}$ is a sequence in $\Phi_k(\Omega)$ converging locally in measure to a function $u \in \Phi_k(\Omega)$, the sequence of Borel measure $\{\mu_k[u_j]\}$ converges weakly to $\mu_k[u]$.

Now consider (1.1) in the sense of *k-Hessian measure*. For the convenience, we denote $\Phi_k = \{u : -u \in \Phi_k(\mathbb{R}^n)\}$. Then clearly $\Gamma_k = \Phi_k \cap C^2(\mathbb{R}^n)$. By employing Proposition 1.2, 1.3 and the argument of approximation, we can extend Theorem 1.1 to the following:

Theorem 1.4. *If $2k < n$ and $\alpha \in (-\infty, k_*]$, then (1.1) has no positive solution in Φ_k in the sense of k-Hessian measure.*

Remark 1.5. *Phuc-Verbitsky [9, 10] had proven Theorem 1.4 for $\alpha \in (k, k_*]$, where they employed the potential theory developed by Trudinger-Wang [12, 13, 14] and Labutin [7], and they also showed that the power $\alpha = k_*$ is sharp. But our method in this paper is different from theirs, since we only use the integration by parts via the careful choices of the test functions and the argument of approximation.*

The approach that we are going to describe is based on finding a priori sharp integral estimate. Our strategy to prove the nonexistence results is as follows: first we deduce some suitable local integral estimate, and then study the asymptotic behavior of this estimate with respect to the relevant parameter of the problem. As it is well known that this idea is widely used in partial differential equations, especially when no information is known on the possible behavior of the solutions, either near a possible singularity or at infinity. For the detail idea, history and its applications to parabolic and hyperbolic equations of this strategy, please see Mitidieri-Pohozaev [8]. To carry out our strategy, we will establish some iteration forms on the k-Hessian inequality (1.1), a technique first appeared in Chang-Gursky-Yang [2] and González [6].

We will prove Theorem 1.1 in section 3, to do this, some preparations of algebraic properties of σ_k are needed, which will be collected in section 2. In the last section, we will show that the proof Theorem 1.4 is just that of Theorem 1.1 combining with the argument of approximation.

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2. Notations and Algebraic properties of σ_k. For a general $n \times n$ symmetric matrix A, consider its eigenvalues $\lambda = (\lambda_1, \cdots, \lambda_n)$ and the elementary symmetric polynomial functions

$$\sigma_k(\lambda) = \sum_{1 \le i_1 < \cdots < i_k \le n} \lambda_{i_1} \cdots \lambda_{i_k}. \tag{2.1}$$

We also write $\sigma_k(\lambda)$ as $\sigma_k(A)$ or simply as σ_k without confusion.

Denote

$$T^k = \sigma_k I - \sigma_{k-1} A + \cdots + (-1)^k A^k = \sigma_k I - T^{k-1} A \tag{2.2}$$

for $k = 1, \cdots, n$. Here we take $\sigma_0 = 1$ and $T^0_{ij} = \delta_{ij}$.

The following properties are well known(see for examples [4], [11] or [6]):

Proposition 2.1. *For A and T^k as above:*
(a) $(n-k)\sigma_k = trace(T^k)$;
(b) $(k+1)\sigma_{k+1} = trace(AT^k)$;
(c) If $\sigma_1, \cdots, \sigma_k > 0$, then T^s is positive definite for $s = 1, \cdots, k-1$, and hence $\| T^s_{ij} \| \le C\sigma_s$;
(d) If $\sigma_1, \cdots, \sigma_k > 0$, then $\sigma_s \le C(\sigma_1)^s$, for $s = 1, \cdots, k$,
where the constant $C > 0$ depends only on n and s.

Proposition 2.2. *For $A = (-D^2u)$, the Hessian of a C^2 function u, and T^k as in (2.2), we have the divergence formulas:*
(a) $\partial_i T^k_{ij} = 0$;
(b) $\sigma_{k+1} = \frac{1}{k+1}\partial_j(u_i {T^k}_{ij})$.
Here and in the following, $\partial_i = \frac{\partial}{\partial x_i}$, $u_i = \partial_i u$ and repeated indices are summed, as usual.

3. Proof of Theorem 1.1. Assume $u > 0$ be a solution of (1.1) in Γ_k . In the following, we write $\sigma_k(-D^2u)$ simply as σ_k.

Let η be a C^2 cut-off function satisfying:

$$\begin{cases} \eta \equiv 1 & in\ B_R, \\ 0 \le \eta \le 1 & in\ B_{2R}, \\ \eta \equiv 0 & in\ \mathbb{R}^n \backslash B_{2R}, \\ |\nabla \eta| \lesssim \frac{1}{R} & in\ \mathbb{R}^n, \end{cases} \tag{3.1}$$

where and throughout this paper, B_R denotes a ball in $\mathbb{R}^n$ centered at the origin with radius R ; and we use "$\lesssim$" , "$\simeq$", etc. to drop out some positive constants independent of R and u.

Denote for $s = 1, \cdots, k$:

$$\begin{array}{rcl} b_s &=& \frac{k+s}{s!2^s}\delta(\delta+1)\cdots(\delta+s-1), \\ B_s &=& \int \sigma_{k-s}|\nabla u|^{2s}u^{-\delta-s}\eta^\theta, \\ M_s &=& \int T^{k-s}_{ij}u_iu_j|\nabla u|^{2(s-1)}u^{-\delta-s}\eta^\theta, \\ E_s &=& \int T^{k-s}_{ij}u_i\eta_j|\nabla u|^{2(s-1)}u^{-\delta-s+1}\eta^{\theta-1}. \end{array}$$

Here and in the rest of the paper, δ, θ are constants to be determined, and we always dropout the domain in integration for the convenience unless otherwise stated , and one can think that all the integrations are taken over a suitable domain such as *supp*η with no confusion.

First, we have the following recursions:

LEMMA 3.1. *For* $s = 1, \cdots, k-1$*:*

$$m_s M_s = m_{s+1} M_{s+1} + b_s B_s - c_{s+1} E_{s+1} \tag{3.2}$$

where $m_i = \frac{2i}{k+i} b_i$ *and* $c_i = \frac{2ib_i}{(\delta+i-1)(k+i)}\theta$ *for* $i = 1, \cdots, k$*, and no summed with the repeated indices* s*.*

Proof. Using the above notations, by (2.2), Proposition 2.2(a) and integration by parts we have

$$\begin{aligned} m_s M_s =& \frac{2s}{k+s} b_s \int T_{ij}^{k-s} u_i u_j |\nabla u|^{2(s-1)} u^{-\delta-s} \eta^\theta \\ =& \frac{2s}{k+s} b_s \int (\sigma_{k-s}\delta_{ij} + T_{il}^{k-s-1} u_{lj}) u_i u_j |\nabla u|^{2(s-1)} u^{-\delta-s} \eta^\theta \\ =& \frac{2s}{k+s} b_s B_s + \frac{b_s}{k+s} \int u_i T_{il}^{k-s-1} \partial_l (|\nabla u|^{2s}) u^{-\delta-s} \eta^\theta \\ =& \frac{2s}{k+s} b_s B_s - \frac{b_s}{k+s} \int u_{il} T_{il}^{k-s-1} |\nabla u|^{2s} u^{-\delta-s} \eta^\theta \\ & + \frac{b_s(\delta+s)}{k+s} \int u_i u_j T_{ij}^{k-s-1} |\nabla u|^{2s} u^{-\delta-s-1} \eta^\theta \\ & - \frac{b_s}{k+s}\theta \int u_i \eta_j T_{ij}^{k-s-1} |\nabla u|^{2s} u^{-\delta-s} \eta^{\theta-1} \end{aligned} \tag{3.3}$$

Then, by Proposition 2.1(b) we arrive at (3.1) as desired. □

Now we give the proof of Theorem 1.1.

Proof of Theorem 1.1. Multiply both sides of (1.1) by $ku^{-\delta}\eta^\theta$ and integrate over $\mathbb{R}^n$ we have

$$k \int u^{\alpha-\delta} \eta^\theta \le k \int \sigma_k u^{-\delta} \eta^\theta. \tag{3.4}$$

Consider the integral on the right hand side of (3.4), integrate by parts once time we get

$$\begin{aligned} k \textstyle\int \sigma_k u^{-\delta} \eta^\theta &= \textstyle\int T_{ij}^{k-1} (-u_{ij}) u^{-\delta} \eta^\theta \\ &= -\delta \textstyle\int T_{ij}^{k-1} u_i u_j u^{-\delta-1} \eta^\theta + \theta \textstyle\int T_{ij}^{k-1} u_i \eta_j u^{-\delta} \eta^{\theta-1} \\ &= -\delta M_1 + \theta E_1. \end{aligned} \tag{3.5}$$

Iterating (3.2) into (3.5) step by step yields

$$k \int \sigma_k u^{-\delta} \eta^\theta = -\sum_{s=1}^k b_s B_s + \sum_{s=1}^k c_s E_s. \tag{3.6}$$

Next we estimate the error terms "E_s". By $|\nabla\eta| \lesssim \frac{1}{R}$ and Proposition 2.1(c), we have

$$|E_s| \lesssim \frac{1}{R}\int \sigma_{k-s}|\nabla u|^{2s-1}u^{-\delta-s+1}\eta^{\theta-1}.$$

Using Young's inequality with exponent pair $(\frac{2s}{2s-1}, 2s)$ and $\varepsilon > 0$ small, the last inequality turns into

$$|E_s| \lesssim \varepsilon \int \sigma_{k-s}|\nabla u|^{2s}u^{-\delta-s}\eta^{\theta} + \frac{C(\varepsilon)}{R^{2s}}\int \sigma_{k-s}u^{-\delta+s}\eta^{\theta-2s}. \tag{3.7}$$

For the last term of (3.7), we have

$$\begin{aligned}
&\frac{1}{R^{2s}}\int \sigma_{k-s}u^{-\delta+s}\eta^{\theta-2s}\\
\backsimeq& -\frac{1}{R^{2s}}\int T_{ij}^{k-s-1}u_{ij}u^{-\delta+s}\eta^{\theta-2s}\\
=&\frac{-\delta+s}{R^{2s}}\int T_{ij}^{k-s-1}u_iu_ju^{-\delta+s-1}\eta^{\theta-2s} + \frac{\theta-2s}{R^{2s}}\int T_{ij}^{k-s-1}u_i\eta_ju^{-\delta+s}\eta^{\theta-2s-1}\\
\lesssim&\varepsilon\int \sigma_{k-s-1}|\nabla u|^{2(s+1)}u^{-\delta-s-1}\eta^{\theta} + \frac{C(\varepsilon)}{R^{2(s+1)}}\int \sigma_{k-s-1}u^{-\delta+s+1}\eta^{\theta-2(s+1)}.
\end{aligned}\tag{3.8}$$

Going through the same process again in (3.8) gives

$$\frac{1}{R^{2s}}\int \sigma_{k-s}u^{-\delta+s}\eta^{\theta-2s} \lesssim \varepsilon\sum_{i=s+1}^{k}B_i + \frac{1}{R^{2k}}\int u^{-\delta+k}\eta^{\theta-2k}. \tag{3.9}$$

Substituting (3.9) and (3.7) into (3.6) we reach

$$k\int \sigma_k u^{-\delta}\eta^{\theta} + \sum_{s=1}^{k}(b_s-\varepsilon)B_s \lesssim \frac{1}{R^{2k}}\int u^{-\delta+k}\eta^{\theta-2k}. \tag{3.10}$$

Now, for $\alpha \in (-\infty, k_*]$ we split into four cases with suitable choice of δ respectively:

(i) Let $\delta = \alpha$ for $\alpha = k$;

(ii) Let $\delta > \frac{n-2k}{2k}(k_* - \alpha)$ for $\alpha \in (-\infty, k)$

(iii) Let $0 < \delta < \frac{n-2k}{2k}(k_* - \alpha)$ for $\alpha \in (k, k_*)$

(iv) Let $\delta = 0$ first and then $0 < \delta < 1$ for $\alpha = k_*$.

In all cases of (i)-(iii), we see that $b_s > 0$ for $s = 1, \cdots, k$.

For case (i), by Young's inequality once again, (3.10) can be rewritten as

$$k\int \sigma_k u^{-\delta}\eta^{\theta} + \sum_{s=1}^{k}(b_s-\varepsilon)B_s \lesssim \varepsilon\int \eta^{\theta} + R^{n-\theta}. \tag{3.11}$$

Combining this with (3.4) we have

$$k\int \eta^\theta + \sum_{s=1}^{k}(b_s-\varepsilon)B_s \lesssim \varepsilon\int \eta^\theta + R^{n-\theta}. \tag{3.12}$$

Now choosing ε small, setting $\theta > n$ and let $R \to +\infty$ we get a contradiction in (3.12).

For cases (ii)-(iii), we always have $\frac{\alpha-\delta}{-\delta+k} > 1$ and $n - 2k \times \frac{\alpha-\delta}{\alpha-k} < 0$. Using Young's inequality with exponent pair $(\frac{\alpha-\delta}{-\delta+k}, \frac{\alpha-\delta}{\alpha-k})$ to the last term in (3.10) we get

$$k\int \sigma_k u^{-\delta}\eta^\theta + \sum_{s=1}^{k}(b_s-\varepsilon)B_s \lesssim \varepsilon\int u^{\alpha-\delta}\eta^\theta + R^{n-\frac{2k(\alpha-\delta)}{\alpha-k}}. \tag{3.13}$$

Combining this with (3.4) we have

$$k\int u^{\alpha-\delta}\eta^\theta + \sum_{s=1}^{k}(b_s-\varepsilon)B_s \lesssim \varepsilon\int u^{\alpha-\delta}\eta^\theta + R^{n-\frac{2k(\alpha-\delta)}{\alpha-k}}. \tag{3.14}$$

Again, we reach a contradiction if $R \to +\infty$ in (3.14).

For case (iv), we first choose $\delta = 0$, then we see that all the $b_s (s = 1, \cdots, k)$ are zero, hence we must be careful to deal with the error terms "E_s". In fact, this time we will start at (3.5) which becomes

$$k\int \sigma_k u^{-\delta}\eta^\theta = \theta E_1. \tag{3.15}$$

First we have

$$|E_1| \lesssim \frac{1}{R}\int \sigma_{k-1}|\nabla u|\eta^{\theta-1}$$

or that

$$\begin{aligned} R^{\frac{n}{\alpha}\delta}|E_1| &\lesssim R^{\frac{n}{\alpha}\delta-1}\int \sigma_{k-1}|\nabla u|\eta^{\theta-1} \\ &\lesssim \int \sigma_{k-1}|\nabla u|^2 u^{-\delta-1}\eta^\theta + R^{\frac{2n}{\alpha}\delta-2}\int \sigma_{k-1}u^{\delta+1}\eta^{\theta-2} \end{aligned} \tag{3.16}$$

by Cauchy inequality, where $0 < \delta < 1$ is fixed.

To deal with the last term in (3.16), we denote:

$$V_s = R^{2s\frac{n}{\alpha}\delta-2s}\int \sigma_{k-s}u^{(2s-1)\delta+s}\eta^{\theta-2s}$$

and

$$W_s = R^{\frac{n}{\alpha}\delta-2s}\int \sigma_{k-s}u^s\eta^{\theta-2s}.$$

Then we can prove the following:

LEMMA 3.2.

$$V_s \lesssim B_{s+1} + V_{s+1} + W_{s+1} \tag{3.17}$$

for $s = 1, \cdots, k-1$.

Proof. First we have, by integrating by parts:

$$\begin{aligned}
V_s =& R^{2s\frac{n}{\alpha}\delta-2s} \int \sigma_{k-s} u^{(2s-1)\delta+s} \eta^{\theta-2s} \\
\backsimeq& - R^{2s\frac{n}{\alpha}\delta-2s} \int T_{ij}^{k-s-1} u_{ij} u^{(2s-1)\delta+s} \eta^{\theta-2s} \\
\backsimeq& R^{2s\frac{n}{\alpha}\delta-2s} \int T_{ij}^{k-s-1} u_i u_j u^{(2s-1)\delta+s-1} \eta^{\theta-2s} \\
&+ R^{2s\frac{n}{\alpha}\delta-2s} \int T_{ij}^{k-s-1} u_i \eta_j u^{(2s-1)\delta+s} \eta^{\theta-2s-1} \\
\lesssim& R^{2s\frac{n}{\alpha}\delta-2s} \int \sigma_{k-s-1} |\nabla u|^2 u^{(2s-1)\delta+s-1} \eta^{\theta-2s} \\
&+ R^{2s\frac{n}{\alpha}\delta-2s-1} \int \sigma_{k-s-1} |\nabla u| u^{(2s-1)\delta+s} \eta^{\theta-2s-1} \\
\lesssim& \int \sigma_{k-s-1} |\nabla u|^{2(s+1)} u^{-\delta-s-1} \eta^{\theta} \\
&+ R^{2(s+1)\frac{n}{\alpha}\delta-2(s+1)} \int \sigma_{k-s-1} u^{(2(s+1)-1)\delta+s+1} \eta^{\theta-2(s+1)} \\
&+ R^{2s\frac{2(s+1)}{2s+1}\frac{n}{\alpha}\delta-2(s+1)} \int \sigma_{k-s-1} u^{(2s\frac{2(s+1)}{2s+1}-1)\delta+s+1} \eta^{\theta-2(s+1)}
\end{aligned} \tag{3.18}$$

where in the last step we have used the Young's inequality with exponent pairs $(s+1, \frac{s+1}{s})$ and $(2(s+1), \frac{2(s+1)}{2s+1})$ respectively.

For the last term in (3.18), we need the following Young's inequality with exponent pair $\left(\frac{(2s+1)^2}{(2s+1)^2-2(s+1)}, \frac{(2s+1)^2}{2(s+1)}\right)$:

$$\begin{aligned}
& R^{2s\frac{2(s+1)}{2s+1}\frac{n}{\alpha}\delta-2(s+1)} u^{(2s\frac{2(s+1)}{2s+1}-1)\delta+s+1} \\
=& R^{2(s+1)\frac{n}{\alpha}\delta-2(s+1)} u^{s+1} \left[u^{(2s\frac{2(s+1)}{2s+1}-1)\delta} \cdot R^{-\frac{2(s+1)}{2s+1}\frac{n}{\alpha}\delta} \right] \\
\lesssim& R^{2(s+1)\frac{n}{\alpha}\delta-2(s+1)} u^{s+1} \left[u^{(2(s+1)-1)\delta} + R^{-(2s+1)\frac{n}{\alpha}\delta} \right] \\
=& R^{2(s+1)\frac{n}{\alpha}\delta-2(s+1)} u^{(2(s+1)-1)\delta+s+1} + R^{\frac{n}{\alpha}\delta-2(s+1)} u^{s+1}.
\end{aligned} \tag{3.19}$$

Hence by using (3.19), (3.18) can be rewritten as:

$$\begin{aligned}
V_s \lesssim& \int \sigma_{k-s-1} |\nabla u|^{2(s+1)} u^{-\delta-s-1} \eta^{\theta} \\
&+ R^{2(s+1)\frac{n}{\alpha}\delta-2(s+1)} \int \sigma_{k-s-1} u^{(2(s+1)-1)\delta+s+1} \eta^{\theta-2(s+1)} \\
&+ R^{\frac{n}{\alpha}\delta-2(s+1)} \int \sigma_{k-s-1} u^{s+1} \eta^{\theta-2(s+1)} \\
=& B_{s+1} + V_{s+1} + W_{s+1}.
\end{aligned} \tag{3.20}$$

This is just (3.17) and lemma 3.2 is proved. □

To go forward, similarly we have the following:

LEMMA 3.3.

$$W_s \lesssim B_{s+1} + V_{s+1} + W_{s+1} \tag{3.21}$$

for $s = 1, \cdots, k-1$.

Proof. Similar to (3.18) we compute:

$$\begin{aligned}
W_s =& R^{\frac{n}{\alpha}\delta-2s}\int \sigma_{k-s}u^s\eta^{\theta-2s}\\
\simeq& -R^{\frac{n}{\alpha}\delta-2s}\int T_{ij}^{k-s-1}u_{ij}u^s\eta^{\theta-2s}\\
\simeq& R^{\frac{n}{\alpha}\delta-2s}\int T_{ij}^{k-s-1}u_iu_ju^{s-1}\eta^{\theta-2s} + R^{\frac{n}{\alpha}\delta-2s}\int T_{ij}^{k-s-1}u_i\eta_ju^s\eta^{\theta-2s-1}\\
\lesssim& R^{\frac{n}{\alpha}\delta-2s}\int \sigma_{k-s-1}|\nabla u|^2u^{s-1}\eta^{\theta-2s} + R^{\frac{n}{\alpha}\delta-2s-1}\int \sigma_{k-s-1}|\nabla u|u^s\eta^{\theta-2s-1}\\
\lesssim& \int \sigma_{k-s-1}|\nabla u|^{2(s+1)}u^{-\delta-s-1}\eta^\theta + R^{\frac{s+1}{s}\frac{n}{\alpha}\delta-2(s+1)}\int \sigma_{k-s-1}u^{\frac{1}{s}\delta+s+1}\eta^{\theta-2(s+1)}\\
&+ R^{\frac{2(s+1)}{2s+1}\frac{n}{\alpha}\delta-2(s+1)}\int \sigma_{k-s-1}u^{\frac{1}{2s+1}\delta+s+1}\eta^{\theta-2(s+1)}.
\end{aligned} \tag{3.22}$$

The following two Yung's inequalities are obvious:

$$\begin{aligned}
&R^{\frac{s+1}{s}\frac{n}{\alpha}\delta-2(s+1)}u^{\frac{1}{s}\delta+s+1}\\
=&R^{2(s+1)\frac{n}{\alpha}\delta-2(s+1)}u^{s+1}\left[u^{\frac{1}{s}\delta}\cdot R^{(\frac{s+1}{s}-2(s+1))\frac{n}{\alpha}\delta}\right]\\
\lesssim&R^{2(s+1)\frac{n}{\alpha}\delta-2(s+1)}u^{s+1}\left[u^{(2(s+1)-1)\delta} + R^{\frac{s(2s+1)}{s(2s+1)-1}(\frac{s+1}{s}-2(s+1))\frac{n}{\alpha}\delta}\right]\\
=&R^{2(s+1)\frac{n}{\alpha}\delta-2(s+1)}u^{(2(s+1)-1)\delta+s+1} + R^{\frac{n}{\alpha}\delta-2(s+1)}u^{s+1},
\end{aligned} \tag{3.23}$$

$$\begin{aligned}
&R^{\frac{2(s+1)}{2s+1}\frac{n}{\alpha}\delta-2(s+1)}u^{\frac{1}{2s+1}\delta+s+1}\\
=&R^{2(s+1)\frac{n}{\alpha}\delta-2(s+1)}u^{s+1}\left[u^{\frac{1}{2s+1}\delta}\cdot R^{(\frac{2(s+1)}{2s+1}-2(s+1))\frac{n}{\alpha}\delta}\right]\\
\lesssim&R^{2(s+1)\frac{n}{\alpha}\delta-2(s+1)}u^{s+1}\left[u^{(2(s+1)-1)\delta} + R^{-(2s+1)\frac{n}{\alpha}\delta}\right]\\
=&R^{2(s+1)\frac{n}{\alpha}\delta-2(s+1)}u^{(2(s+1)-1)\delta+s+1} + R^{\frac{n}{\alpha}\delta-2(s+1)}u^{s+1}.
\end{aligned} \tag{3.24}$$

Hence by using (3.23) and (3.24) to the last two terms of (3.22), we can deduce:

$$\begin{aligned}
W_s \lesssim& \int \sigma_{k-s-1}|\nabla u|^{2(s+1)}u^{-\delta-s-1}\eta^\theta\\
&+ R^{2(s+1)\frac{n}{\alpha}\delta-2(s+1)}\int \sigma_{k-s-1}u^{(2(s+1)-1)\delta+s+1}\eta^{\theta-2(s+1)}\\
&+ R^{\frac{n}{\alpha}\delta-2(s+1)}\int \sigma_{k-s-1}u^{s+1}\eta^{\theta-2(s+1)}\\
=&B_{s+1} + V_{s+1} + W_{s+1}.
\end{aligned} \tag{3.25}$$

This is just (3.21) and lemma 3.3 is proved. ∎

Using (3.17) and (3.21) alternatively we deduce immediately

$$V_s \lesssim \sum_{i=s+1}^{k} B_i + W_k \tag{3.26}$$

for $s = 1, 2, \cdots, k-1$. Especially, for $s = 1$ we have:

$$R^{\frac{2n}{\alpha}\delta-2} \int \sigma_{k-1} u^{\delta+1} \eta^{\theta-2} = V_1 \lesssim \sum_{i=2}^{k} B_i + W_k. \tag{3.27}$$

Submitting this into (3.16) yields

$$R^{\frac{n}{\alpha}\delta} |E_1| \lesssim \sum_{i=1}^{k} B_i + W_k. \tag{3.28}$$

Now we choose $\delta \in (0,1)$, then (3.10) is still valid and $b_s > 0$. Hence (3.28) and (3.10) show that

$$R^{\frac{n}{\alpha}\delta} |E_1| \lesssim W_k + \frac{1}{R^{2k}} \int u^{-\delta+k} \eta^{\theta-2k}, \tag{3.29}$$

i.e.

$$|E_1| \lesssim R^{-2k} \int u^k \eta^{\theta-2k} + R^{-\frac{n}{\alpha}\delta-2k} \int u^{-\delta+k} \eta^{\theta-2k}. \tag{3.30}$$

Next, by Hölder inequality we have

$$\begin{aligned} R^{-2k} \int u^k \eta^{\theta-2k} \leq & R^{-2k} \Big(\int u^\alpha \eta^\theta \Big)^{\frac{k}{\alpha}} \Big(\int \eta^{\theta-2\alpha} \Big)^{\frac{\alpha-k}{\alpha}} \\ \lesssim & R^{\frac{\alpha-k}{\alpha}(n-\frac{2k\alpha}{\alpha-k})} \Big(\int u^\alpha \eta^\theta \Big)^{\frac{k}{\alpha}} \end{aligned} \tag{3.31}$$

and

$$\begin{aligned} R^{-\frac{n}{\alpha}\delta-2k} \int u^{-\delta+k} \eta^{\theta-2k} \leq & R^{-\frac{n}{\alpha}\delta-2k} \Big(\int u^\alpha \eta^\theta \Big)^{\frac{k-\delta}{\alpha}} \Big(\int \eta^{\theta-2k\frac{\alpha}{\alpha-k+\delta}} \Big)^{\frac{\alpha-k+\delta}{\alpha}} \\ \lesssim & R^{\frac{\alpha-k}{\alpha}(n-\frac{2k\alpha}{\alpha-k})} \Big(\int u^\alpha \eta^\theta \Big)^{\frac{k-\delta}{\alpha}}. \end{aligned} \tag{3.32}$$

Since $\alpha = k_*$, i.e. $n - \frac{2k\alpha}{\alpha-k} = 0$, inserting (3.31) and (3.32) into (3.30) we can see

$$|E_1| \lesssim \Big(\int u^\alpha \eta^\theta \Big)^{\frac{k}{\alpha}} + \Big(\int u^\alpha \eta^\theta \Big)^{\frac{k-\delta}{\alpha}}. \tag{3.33}$$

Recall the definition of "E_s", all the integrations in (3.33) are taken over the domain $U := supp\nabla\eta = \{R < |x| < 2R\}$. Hence combining (3.15) with (3.33) we can get:

$$k\int_{\mathbb{R}^n} \sigma_k \eta^\theta \lesssim \big(\int_U u^\alpha \eta^\theta\big)^{\frac{k}{\alpha}} + \big(\int_U u^\alpha \eta^\theta\big)^{\frac{k-\delta}{\alpha}}. \tag{3.34}$$

Combining this with (3.4) we have

$$\int_{\mathbb{R}^n} u^\alpha \eta^\theta \lesssim \big(\int_U u^\alpha \eta^\theta\big)^{\frac{k}{\alpha}} + \big(\int_U u^\alpha \eta^\theta\big)^{\frac{k-\delta}{\alpha}}. \tag{3.35}$$

Since $0 < \frac{k}{\alpha}, \frac{k-\delta}{\alpha} < 1$, (3.35) shows that

$$\int_{\mathbb{R}^n} u^\alpha \eta^\theta \leq constant < \infty. \tag{3.36}$$

This implies

$$\int_U u^\alpha \eta^\theta \to 0 \quad \text{as} \quad R \to +\infty. \tag{3.37}$$

Return to (3.35) again, we deduce

$$\int_{\mathbb{R}^n} u^\alpha \eta^\theta \to 0 \quad \text{as} \quad R \to +\infty. \tag{3.38}$$

This is a contradiction, and hence the proof of Theorem 1.1 goes to the end. □

4. Proof of Theorem 1.4. We can get Theorem 1.4 by the similar process as in the proof of Theorem 1.1 in the last section combining with the argument of approximation. In fact, let $u > 0$ be a solution of (1.1) in Φ_k. Then by Proposition 1.2, we may assume $\{-u_j\}$ be a decreasing sequence of negative functions in $\Phi_k(B_{2R}) \cap C^2(B_{2R})$, which converges to $-u$ in B_{2R} for any given $R > 0$. Then (3.10) will also be valid for u_j for all j, namely, we have

$$k\int \sigma_k(-D^2 u_j) u_j^{-\delta} \eta^\theta \lesssim \frac{1}{R^{2k}} \int u_j^{-\delta+k} \eta^{\theta-2k}. \tag{4.1}$$

Now for case (i)-(iii), first we see that $u^{-\delta} \leq u_j^{-\delta}$ by our choices of δ, and hence

$$k\int \sigma_k(-D^2 u_j) u^{-\delta} \eta^\theta \lesssim \frac{1}{R^{2k}} \int u_j^{-\delta+k} \eta^{\theta-2k}. \tag{4.2}$$

When $j \to \infty$, (4.2) will converges to, by Proposition 1.2,1.3,

$$k\int \sigma_k(-D^2 u) u^{-\delta} \eta^\theta \lesssim \frac{1}{R^{2k}} \int u^{-\delta+k} \eta^{\theta-2k}. \tag{4.3}$$

Then by the inequality (1.1) and the arbitrariness of R, we can get contradiction as before.

For case (iv), we see that (3.34) is also valid for u_j for all j. Then by a similar argument we can get the result as desired, and hence the proof of Theorem 1.4 is completed. □

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