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SPECTRAL ASYMPTOTICS AND QUASICLASSICAL ANALYSIS OF SCHRÖDINGER TYPE OPERATORS*

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Abstract. In this work we consider a general class of Schrödinger type operators, associated to multi-quasi-elliptic symbols introduced by Buzano and Ziggioto in [9]. We develop their quasiclassical analysis and we obtain a uniform asymptotic formula for their counting function $\mathcal{N}_\epsilon(\tau)$, in the sense that it holds as $\tau \to +\infty$ and for all $0 < \epsilon \leq 1$.

Key words. Spectral Theory, counting function, elliptic operators, quasiclassical analysis.

AMS subject classifications. 35P20, 47B06

1. Introduction. Quasiclassical analysis and spectral asymptotics are strictly related to each other (this is particularly evident when dealing with homogeneous symbols, see [6],Remark A.2.2). In both of them, the object of study is the counting function (which we denote by $\mathcal{N}(\tau)$ in the case of spectral asymptotics and by $\mathcal{N}_\epsilon(\tau)$ in the case of quasiclassical analysis) associated to the operators we are dealing with.

In spectral asymptotics we analyze the behavior of $\mathcal{N}(\tau)$ as $\tau \to +\infty$, while in quasiclassical analysis we study the behavior of $\mathcal{N}_\epsilon(\tau)$ as $\epsilon \to 0$, where ϵ plays the role of the Planck constant in Quantum Mechanics.(1)

In this paper we take into consideration multi-quasi-elliptic operators of Schrödinger type h^w, introduced by Buzano and Ziggioto in [9]. We already obtained an asymptotic formula for their counting function $\mathcal{N}(\tau)$ as $\tau \to +\infty$ and in particular we proved an estimate of the remainder term, showing that it always goes to 0 as $\tau \to +\infty$.

Now we consider quasiclassical operators associated to h^w and their counting function $\mathcal{N}_\epsilon(\tau)$. Using the so called *Tauberian condition* (see condition 2. of Theorem 1 in Section 3), we manage to obtain a *uniform* asymptotic formula for $\mathcal{N}_\epsilon(\tau)$, in the sense that it is valid as $\tau \to +\infty$ and for all $0 < \epsilon \leq 1$.

We can make a comparison with the results obtained in one of our previous papers, see [8]. In that case we treated quasiclassical analysis of more general operators (hypoelliptic operators), but we didn't manage to obtain a *uniform* asymptotic formula, holding as $\tau \to +\infty$ and for all $0 < \epsilon \leq 1$. Moreover, in our uniform asymptotic formula obtained for multi-quasi-elliptic operators (see (10)) we don't need to exclude the critical values of the symbol $h(x, \xi)$ (i.e. the values τ for which $\operatorname{grad} h(x, \xi) = 0$ on the surface $\{(x, \xi) : h(x, \xi) = \tau\}$).

We employ the following notation: given two functions $f, g : X \to \mathbb{R}$, and a subset $A \subset X$, we write

$$f(x) \prec g(x), \qquad \forall\, x \in A,$$

if there exists a constant C such that

$$f(x) \leq C g(x), \qquad \forall\, x \in A.$$

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(1)In order to be consistent with the notations used in [10] here we denote the Planck constant by ϵ and not by h, since we use h to denote our operators.

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2. Multi-quasi-elliptic operators of Schrödinger type. We begin by recalling some basic notations and results about multi-quasi-elliptic weights and symbols. For references see [9], [1].

A convex polyhedron $\mathcal{P} \subset \mathbb{R}^n$ is the convex hull of a finite set of points in $\mathbb{R}^n$.

One can show that $\mathcal{P}$ can be obtained as the convex hull of a finite subset $V(\mathcal{P}) \subset \mathbb{R}^n$ of points, which are convex linearly independent, called the *vertices* of $\mathcal{P}$ and univoquely determined by $\mathcal{P}$. Moreover if $(0, 0, \dots, 0) \in \mathcal{P}$, then there exists a finite set $N(\mathcal{P}) = N_0(\mathcal{P}) \cup N_1(\mathcal{P}) \subset \mathbb{R}^n$ such that [2]

$$\begin{aligned}\mathcal{P} = &\{x \in \mathbb{R}^n \mid \nu \cdot x \geq 0,\ \forall\, \nu \in N_0(\mathcal{P})\} \cap \\ &\cap \{x \in \mathbb{R}^n \mid \nu \cdot x \leq 1,\ \forall\, \nu \in N_1(\mathcal{P})\}.\end{aligned}$$

The boundary of $\mathcal{P}$ is made of faces F_ν which are the convex hull of the vertices of $\mathcal{P}$ lying on the hyperplane H_ν orthogonal to $\nu \in N(\mathcal{P})$ and of equation

$$\begin{aligned}\nu \cdot x = 0, &\quad \text{if } \nu \in N_0(\mathcal{P}),\\ \nu \cdot x = 1, &\quad \text{if } \nu \in N_1(\mathcal{P}).\end{aligned}$$

We set

$$F(\mathcal{P}) = \bigcup_{\nu \in N_1(\mathcal{P})} F_\nu.$$

Definition 1. A *complete polyhedron* is a convex polyhedron $\mathcal{P} \subset \overline{\mathbb{R}^n_+}$ [3] with the following properties:

1. $V(\mathcal{P}) \subset \overline{\mathbb{R}^n_+}$;
2. $(0, \dots, 0) \in V(\mathcal{P})$;
3. $V(\mathcal{P}) \neq \{(0, \dots, 0)\}$;
4. $N_0(\mathcal{P}) = \{\mathsf{e}_1, \dots, \mathsf{e}_n\}$, with $\mathsf{e}_j = (0, \dots, 0, \underset{j-\text{entry}}{1}, 0, \dots, 0)$, for $j = 1, \dots, n$;
5. $N_1(\mathcal{P}) \subset \mathbb{R}^n_+$.

Consider now a complete polyhedron $\mathcal{P}$ with integer vertices:

$$V(\mathcal{P}) \subset \mathbb{N}^n.$$

To such a polyhedron we associate the *multi-quasi-elliptic weight function:*

$$\Lambda(\xi; \mathcal{P}) = \Bigg(\sum_{\alpha \in V(\mathcal{P}) \setminus 0} \xi^{2\alpha}\Bigg)^{1/2}.$$

Definition 2. Given a complete polyhedron $\mathcal{P}$, we set

$$m(\mathcal{P}) = \sup_{\nu \in N_1(\mathcal{P})} \max\left\{\frac{1}{\nu_j} \mid j = 1, \dots, n\right\}.$$

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[2] In $\mathbb{R}^n$ we always consider the norm $|x| = |x_1| + \cdots + |x_n|$.

[3] $\mathbb{R}_+ = \{x \in \mathbb{R} \mid x > 0\}$.

$m(\mathcal{P})$ is called the *formal order* of $\mathcal{P}$.

DEFINITION 3. *A multi-quasi-elliptic operator of Schrödinger type* is a *differential* operator h^w of domain $\mathcal{C}_0^\infty(\mathbb{R}^n)$ and *Weyl symbol*

$$h(x,\xi) = p(x,\xi) + q(x) = \sum_{\alpha\in A} a_\alpha(x)\xi^\alpha + q(x)$$

satisfying the following hypotheses.

1. The convex hull of A is a complete polyhedron $\mathcal{P}$.
2. The potential q is real valued and

$$q(x) \geq 1, \qquad \text{for all } x \in \mathbb{R}^n.$$

3. There exists $0 \leq \delta < 1/m(\mathcal{P})$ such that for all $\beta \in \mathbb{N}^n$ we have

$$|D^\beta q(x)| \prec q(x)^{1+\delta|\beta|}, \qquad \forall\, x \in \mathbb{R}^n. \tag{1}$$

4. The coefficients a_α are real valued.
5. There exists $0 \leq \rho < 1$, such that for all $\alpha \in A$ and $\beta \in \mathbb{N}^n$, we have

$$|D^\beta a_\alpha(x)| \prec q(x)^{(1-k(\alpha;\mathcal{P}))\rho+\delta|\beta|} \qquad \forall\, x \in \mathbb{R}^n, \tag{2}$$

   where

$$k(\alpha;\mathcal{P}) = \inf\{t > 0 \mid t^{-1}\alpha \in \mathcal{P}\} = \max_{\nu\in N_1(\mathcal{P})} \nu\cdot\alpha.$$

6. There exists $R_0 \geq 0$ such that

$$p_0(x,\xi) \succ \Lambda(\xi;\mathcal{P}), \qquad \forall\, |\xi| \geq R_0,\ \forall\, x \in \mathbb{R}^n, \tag{3}$$

   where

$$p_0(x,\xi) = \sum_{\alpha\in A\cap F(\mathcal{P})} a_\alpha(x)\xi^\alpha$$

   is the *principal symbol of* p^w.

REMARK. We can say that multi-quasi-elliptic symbols generalize elliptic and quasi-elliptic symbols. More specifically, limiting ourselves to dimension $n = 2$, we can represent the complete polyhedron $\mathcal{P}$ associated to a multi-quasi-elliptic symbol as a polygon with more than one face, as it is shown in Figure 1. The complete polyhedron $\mathcal{P}$ associated to an elliptic symbol can be represented instead as an isosceles triangle, as shown in Figure 2. Finally, the complete polyhedron $\mathcal{P}$ associated to a quasi-elliptic symbol can be represented as a right-angled triangle, as shown in Figure 3.

We are going to use the Weyl-Hörmander calculus with locally temperate metrics and weights: see [2], and [4] for more details.

Let

$$\lambda(x,\xi) = \left\{\Lambda(\xi;\mathcal{P})^2 + q(x)^2\right\}^{1/2}.$$

Then the Riemannian metric

$$g_{x,\xi}(y,\eta) = \lambda(x,\xi)^{2\delta}|y|^2 + \lambda(x,\xi)^{-2/m(\mathcal{P})}|\eta|^2 \tag{4}$$

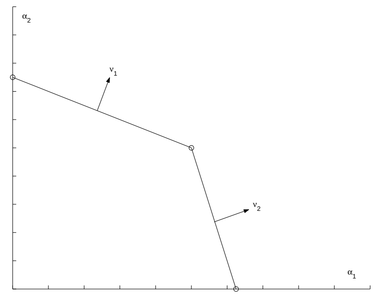

FIG. 1. *multi-quasi-elliptic case* $\xi_1^{12} + \xi_1^{10}\xi_2^{10} + \xi_2^{14}$

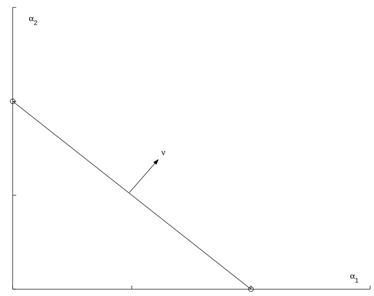

FIG. 2. *elliptic case* $\xi_1^2 + \xi_2^2$

(where δ is the same as in (1)) is locally σ temperate with respect to the slowly varying metric

$$G_x(t) = |t|^2 q(x)^{2\delta}.$$

In particular λ is locally σ, g temperate with respect to G. See [9], Proposition 3, for further details. Moreover we can show the following result:

PROPOSITION 1. *The so called* principle of indetermination *is satisfied by the metric g defined in* (4)*, that is*

$$\sup_{x,\xi} \frac{g_{x,\xi}(y,\eta)}{g^\sigma_{x,\xi}(y,\eta)} < +\infty.$$

Proof. In the case of the metric g defined in (4) it is standard to show that

$$\sup_{x,\xi} \frac{g_{x,\xi}}{g^\sigma_{x,\xi}} = \lambda(x,\xi)^{2\left(\delta - \frac{1}{m(\mathcal{P})}\right)}.$$

Then, since $0 \le \delta < \frac{1}{m(\mathcal{P})}$ and $q(x) \ge 1$ for all $x \in \mathbb{R}^n$, we have that

$$\sup_{x,\xi} \frac{g_{x,\xi}(y,\eta)}{g^\sigma_{x,\xi}(y,\eta)} \le 1.$$

$\square$

Finally, we define the *counting function* associated to the operator h^w:

$\mathcal{N}_h(\tau)$ = number of eigenfunctions of the closure of h^w corresponding to eigenvalues less or equal to τ.

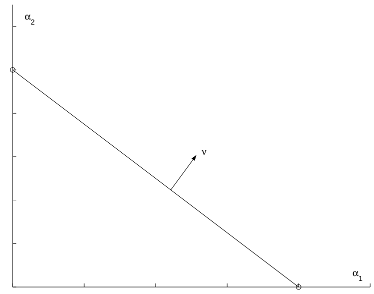

FIG. 3. *quasi-elliptic case* $\xi_1^4 + \xi_2^6$

3. Quasiclassical Analysis of Multi-Quasi-Elliptic Operators of Schrödinger Type. Consider a multi-quasi-elliptic operator of Schrödinger type h^w.

Let us introduce the operator h_ϵ^w whose Weyl symbol is

$$h_\epsilon(x,\xi) = h(\epsilon x, \epsilon\xi) = p(\epsilon x, \epsilon\xi) + q(\epsilon x),$$

where ϵ is a real parameter such that $0 < \epsilon \leq 1$.

Starting from the metrics defined in (4), let us define the following new Riemannian metrics ${}^\epsilon g_{x,\xi}(y,\eta)$ and ${}^\epsilon G_x(t)$ in this way:

$${}^\epsilon g_{x,\xi}(y,\eta) = g_{\epsilon x,\epsilon\xi}(\epsilon y, \epsilon\eta) = \epsilon^2 \left(\lambda(\epsilon x, \epsilon\xi)^{2\delta}|y|^2 + \lambda(\epsilon x, \epsilon\xi)^{-\frac{2}{m(\mathcal{P})}}|\eta|^2\right)$$

and

$${}^\epsilon G_x(t) = G_{\epsilon x}(\epsilon t) = \epsilon^2|t|^2 q(\epsilon x)^{2\delta}.$$

We know that $g_{x,\xi}(y,\eta)$ is locally σ temperate with respect to $G_x(t)$ and that λ is locally σ, g temperate with respect to $G_x(t)$. Therefore, it follows that also ${}^\epsilon g_{x,\xi}$ is slowly varying, locally σ temperate with respect to ${}^\epsilon G_x(t)$ for all $0 < \epsilon \leq 1$ and that also $\lambda(\epsilon x, \epsilon\xi)$ is locally σ, g temperate with respect to ${}^\epsilon G_x(t)$, for all $0 < \epsilon \leq 1$ (see [3], [8]).

Let us analyze the principle of indetermination in the case of the metric ${}^\epsilon g$.

PROPOSITION 2. *We have that the principle of indetermination is satisfied by the new metric* ${}^\epsilon g$ *, for all* $0 < \epsilon \leq 1$.

Proof. We have

$$\begin{aligned}\sup_{x,\xi} \frac{{}^\epsilon g_{x,\xi}}{({}^\epsilon g)^\sigma_{x,\xi}} &= \epsilon^4 \lambda(\epsilon x, \epsilon\xi)^{2\left(\delta - \frac{1}{m(\mathcal{P})}\right)} = \\ &= \left(\lambda(\epsilon x, \epsilon\xi)\epsilon^{\frac{2m(\mathcal{P})}{\delta m(\mathcal{P})-1}}\right)^{2\left(\delta - \frac{1}{m(\mathcal{P})}\right)} = \\ &= \lambda_\epsilon(x,\xi)^{2\left(\delta - \frac{1}{m(\mathcal{P})}\right)},\end{aligned}$$

for all $0 < \epsilon \leq 1$. Therefore, repeating the same arguments of the proof of Proposition 1, we obtain that the principle of indetermination is satisfied also by ${}^\epsilon g$. $\square$

Due to this proposition, from now on we will work with the following symbol:

$$H_\epsilon(x,\xi) = \epsilon^{\frac{2m(\mathcal{P})}{\delta m(\mathcal{P})-1}} h_\epsilon(x,\xi).$$

Now we formulate Proposition 4 of [9] in this new context:

PROPOSITION 3. *Consider a multi-quasi-elliptic operator of Schrödinger type* $h^w = p^w + q$. *If* $q(x) \to +\infty$ *then the operator* H_ϵ^w, *corresponding to the new symbol* H_ϵ, *is semi-bounded from below and essentially self-adjoint in* $L^2(\mathbb{R}^n)$, *for all* $0 < \epsilon \leq 1$.

Moreover its closure in $L^2(\mathbb{R}^n)$ *has discrete spectrum diverging to* $+\infty$.

Proof. Thanks to Proposition 2 of [9] (which is trivially satisfied also by H_ϵ and λ_ϵ), there exists $c_0 > 0$ such that

$$\lambda_\epsilon(x,\xi) \prec c_0 \epsilon^{\frac{2m(\mathcal{P})}{1-\delta m(\mathcal{P})}} + H_\epsilon(x,\xi) \prec \lambda_\epsilon(x,\xi), \quad \forall (x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n,$$

for all $0 < \epsilon \leq 1$. Then $\widetilde{H}_\epsilon = c_0 \epsilon^{\frac{2m(\mathcal{P})}{1-\delta m(\mathcal{P})}} + H_\epsilon$ is locally temperate for all $0 < \epsilon \leq 1$. Thanks to Proposition 2 of [9] again, it is easy to check that $\widetilde{H}_\epsilon$ belongs to the class of Weyl-Hörmander $S(\widetilde{H}_\epsilon, {}^\epsilon g)$ for all $0 < \epsilon \leq 1$ and the result is a special case of Proposition 6.1 of [2]. ☐

REMARK. Thanks to Proposition 3 we can define the *counting function* of the closure of the operator H_ϵ^w:

$\mathcal{N}_{H_\epsilon}(\tau) =$ number of eigenfunctions of the closure of H_ϵ^w, corresponding to eigenvalues less than or equal to τ.

It is clear that Proposition 3 also applies to

$$H_{0,\epsilon} = h_{0,\epsilon} \epsilon^{\frac{2m(\mathcal{P})}{\delta m(\mathcal{P})-1}},$$

where $h_{0,\epsilon}$ is the symbol of the principal part of h_ϵ^w, that is

$$h_{0,\epsilon}(x,\xi) = p_0(\epsilon x, \epsilon \xi) + q(\epsilon x).$$

In particular we have that $H_{0,\epsilon}^w$ is essentially self-adjoint and that its closure has a discrete spectrum diverging to $+\infty$ (see also Proposition 4 of [9]).

Before claiming our main result, we have to state the following theorem, which is a direct consequence of Proposition 5 of [9]:

PROPOSITION 4. *Consider a multi-quasi-elliptic operator of Schrödinger type* h^w *and assume that* $q(x) \to +\infty$ *as* $|x| \to +\infty$. *If there exists* $k > 0$ *such that*

$$h_0^{-k} \in L^1(\mathbb{R}^{2n}), \tag{5}$$

then there exists τ_0 *such that*

$$\mathcal{N}_{H_{0,\epsilon}}(\tau) = \mathcal{W}(\tau; H_{0,\epsilon})\{1 + O(\mathcal{R}_{\epsilon,\mu_0})\},$$

for all $\tau \geq \tau_0$, *uniformly with respect to* $0 < \epsilon \leq 1$, *where*

$$\mathcal{W}(\tau; H_{0,\epsilon}) = (2\pi)^{-n} \iint\limits_{H_{0,\epsilon} \leq \tau} dx\, d\xi,$$

$$\mathcal{R}_{\epsilon,\mu_0}(\tau) = \frac{\mathcal{W}(\tau + \tau^{1-\mu_0}; H_{0,\epsilon}) - \mathcal{W}(\tau - \tau^{1-\mu_0}; H_{0,\epsilon})}{\mathcal{W}(\tau; H_{0,\epsilon})},$$

and

$$0 < \mu_0 < \frac{2}{3}\,\frac{1-\delta m(\mathcal{P})}{m(\mathcal{P})}.$$

Proof. By means of a change of coordinates we immediately obtain that

$$\begin{aligned}
\|H_{0,\epsilon}^{-k}\|_{L^1} &= \epsilon^{\frac{2km(\mathcal{P})}{1-\delta m(\mathcal{P})}} \int |h_{0,\epsilon}(x,\xi)|^{-k}\,dx\,d\xi = \\
&= \epsilon^{\frac{2km(\mathcal{P})}{1-\delta m(\mathcal{P})}} \int |h_0(\epsilon x,\epsilon\xi)|^{-k}\,dx\,d\xi = \\
&= \epsilon^{\frac{2km(\mathcal{P})}{1-\delta m(\mathcal{P})}-2n} \int |h_0(x,\xi)|^{-k}\,dx\,d\xi \le \\
&\le \int |h_0(x,\xi)|^{-k}\,dx\,d\xi = \|h_0^{-k}\|_{L^1},
\end{aligned}$$

if we take $\frac{2km(\mathcal{P})}{1-\delta m(\mathcal{P})} - 2n \ge 0$, that is

$$\frac{1}{m(\mathcal{P})} - \frac{k}{n} \le \delta < \frac{1}{m(\mathcal{P})}.$$

Therefore we obtain that the integrability of $h_0^{-k}(x,\xi)$ implies the integrability of $H_{0,\epsilon}^{-k}(x,\xi)$ and that the L^1 norm of $H_{0,\epsilon}^{-k}$ is uniformly bounded with respect to $0 < \epsilon \le 1$. The remaining part of the proof is an immediate consequence of Proposition 5 in [9]. □

Now we can state and prove our main result.

THEOREM 1. *Let $\mathcal{N}_\epsilon(\tau)$ be the counting function associated to the operator h_ϵ^{w}. Assume that*

1. $q(x) \to +\infty$, *as* $|x| \to +\infty$,
2. *there exists* $\tau_0 \ge 0$, *such that*

$$\mathcal{V}(2\tau) \prec \mathcal{V}(\tau), \qquad \forall\,\tau \ge \tau_0,$$

where

$$\mathcal{V}(\tau) = \int_{q(x)\le\tau} dx.$$

3. *for all* $r > 0$ *we have*

$$\inf_{\substack{x\in\mathbb{R}^n\\ |\xi|\ge r}} p_0(x,\xi) > 0, \tag{6}$$

4. *there exist* $t_0 > 0$, $\omega \in \mathbb{R}^n_+$ *and* $C_0 > 0$, *such that*

$$p_0(x,(1+t)^\omega\xi) \ge (1+C_0 t)p_0(x,\xi),$$

for all $0 < t < t_0$ *and all* $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n$, *where*

$$(1+t)^\omega\xi = \left((1+t)^{\omega_1}\xi_1,\ldots,(1+t)^{\omega_n}\xi_n\right).$$

Then for

$$0 < d < \frac{4}{3}\,\frac{|\nu|}{1+|\nu|} \tag{7}$$

$$0 < \mu < \frac{2}{3}\,\frac{1-\delta m(\mathcal{P})}{m(\mathcal{P})}\,\frac{|\nu|}{1+|\nu|}, \tag{8}$$

$$\mu \leq (1-\rho)(1-\zeta)\frac{|\nu|}{1+|\nu|}, \tag{9}$$

with

$$\frac{|\nu|}{1+|\nu|} = \max_{\tilde{\nu}\in N_1(\mathcal{P})} \frac{|\tilde{\nu}|}{1+|\tilde{\nu}|}$$

and

$$\zeta = \max_{\alpha \in A\setminus F(\mathcal{P})} k(\alpha;\mathcal{P}),$$

we have that

$$\mathcal{N}_\epsilon(\tau) = \epsilon^{-2n}\mathcal{W}(\tau;h_0)\{1+O(\epsilon^d\tau^{-\mu})\}, \tag{10}$$

as $\tau \to +\infty$ *and for all* $0 < \epsilon \leq 1$*, where*

$$\mathcal{W}(\tau;h_0) = (2\pi)^{-n}\int\limits_{h_0\leq\tau} dx\,d\xi.$$

Remark. As already remarked in the introduction, in order to obtain the result of Theorem 1, we don't need to avoid the critical values of the symbol $h(x,\xi)$ from our asymptotic formula, as instead we are compelled to do in the case of hypoelliptic operators (see [10]). Moreover, in the case of hypoelliptic operators we don't have a *uniform* asymptotic formula, in the sense that it holds only as $\epsilon \to 0$ (see [8]).

4. Proof of Theorem 1. In order to prove Theorem 1, let us begin to estimate the remainder term $\mathcal{R}_{\epsilon,\mu_0}(\tau)$ for the counting function associated to $h^w_{0,\epsilon}$, as $\tau \to +\infty$ and for all $0 < \epsilon \leq 1$.

Proposition 5. *Under the same hypotheses of Theorem 1 we have that*

$$\mathcal{N}_{h_{0,\epsilon}}(\tau) = \epsilon^{-2n}\mathcal{W}(\tau;h_0)\{1+O(\epsilon^d\tau^{-\mu})\}, \tag{11}$$

as $\tau \to \infty$ *and for all* $0 < \epsilon \leq 1$*, where*

$$0 < \mu < \frac{2}{3}\,\frac{1-\delta m(\mathcal{P})}{m(\mathcal{P})}\,\frac{|\nu|}{1+|\nu|}$$

and

$$0 < d < \frac{4}{3}\,\frac{|\nu|}{1+|\nu|}.$$

Proof. Since $\mathcal{N}_{h_{0,\epsilon}}(\tau)$ is the counting function associated to the operator $h^w_{0,\epsilon}$, then it is clear that $H^w_{0,\epsilon}$ has exactly $\mathcal{N}_{h_{0,\epsilon}}(\epsilon^{\frac{2m(\mathcal{P})}{1-\delta m(\mathcal{P})}}\tau)$ eigenvalues less than or equal to τ and that

$$\mathcal{W}(\tau; H_{0,\epsilon}) = (2\pi)^{-n} \iint\limits_{h_0(\epsilon x,\epsilon\xi)\epsilon^{\frac{2m(\mathcal{P})}{\delta m(\mathcal{P})-1}} \le \tau} dx\, d\xi = \epsilon^{-2n}\mathcal{W}(\epsilon^{\frac{2m(\mathcal{P})}{1-\delta m(\mathcal{P})}}\tau; h_0).$$

Thanks to Proposition 4, we obtain that for all $0 < \mu_0 < \frac{2}{3}\frac{1-\delta m(\mathcal{P})}{m(\mathcal{P})}$ there exists a real number $C_{\mu_0} > 0$ such that

$$\begin{aligned}&\left|\mathcal{N}_{h_{0,\epsilon}}(\epsilon^{\frac{2m(\mathcal{P})}{1-\delta m(\mathcal{P})}}\tau) - \epsilon^{-2n}\mathcal{W}(\epsilon^{\frac{2m(\mathcal{P})}{1-\delta m(\mathcal{P})}}\tau; h_0)\right| \le \\ &\qquad \le C_{\mu_0}\epsilon^{-2n}\left(\mathcal{W}(\epsilon^{\frac{2m(\mathcal{P})}{1-\delta m(\mathcal{P})}}(\tau+\tau^{1-\mu_0}); h_0) - \mathcal{W}(\epsilon^{\frac{2m(\mathcal{P})}{1-\delta m(\mathcal{P})}}(\tau-\tau^{1-\mu_0}); h_0)\right),\end{aligned}$$

as $\tau \to +\infty$, for all $0 < \epsilon \le 1$.

Letting $\epsilon^{\frac{2m(\mathcal{P})}{1-\delta m(\mathcal{P})}}\tau = \lambda$ and provided that also $\lambda\epsilon^{\frac{2m(\mathcal{P})}{\delta m(\mathcal{P})-1}}$ is sufficiently large, we obtain:

$$\begin{aligned}&|\mathcal{N}_{h_{0,\epsilon}}(\lambda) - \epsilon^{-2n}\mathcal{W}(\lambda; h_0)| \le \\ &\le C_{\mu_0}\epsilon^{-2n}\left(\mathcal{W}(\lambda(1+\epsilon^{\frac{2m(\mathcal{P})\mu_0}{1-\delta m(\mathcal{P})}}\lambda^{-\mu_0}); h_0) - \mathcal{W}(\lambda(1-\epsilon^{\frac{2m(\mathcal{P})\mu_0}{1-\delta m(\mathcal{P})}}\lambda^{-\mu_0}; h_0))\right)\end{aligned} \tag{12}$$

for λ large enough, $0 < \mu_0 < \frac{2}{3}\frac{1-\delta m(\mathcal{P})}{m(\mathcal{P})}$ and $0 < \epsilon \le 1$.

Recalling Lemma 2 of [9] and letting $\theta = \epsilon^{\frac{2m(\mathcal{P})\mu_0}{1-\delta m(\mathcal{P})}}\lambda^{-\mu_0}$ we obtain that

$$\begin{aligned}\mathcal{W}(\lambda(1+\epsilon^{\frac{2m(\mathcal{P})\mu_0}{1-\delta m(\mathcal{P})}}\lambda^{-\mu_0}); h_0) - \mathcal{W}(\lambda(1-\epsilon^{\frac{2m(\mathcal{P})\mu_0}{1-\delta m(\mathcal{P})}}\lambda^{-\mu_0}); h_0) \prec \\ \prec \lambda^{-\mu_0\frac{|\nu|}{1+\nu|}}\epsilon^{\frac{2m(\mathcal{P})\mu_0}{1-\delta m(\mathcal{P})}\frac{|\nu|}{1+|\nu|}}\mathcal{W}(\lambda; h_0),\end{aligned}$$

for all ϵ and λ such that $\lambda \ge \tau_1$ and $0 < \epsilon \le \theta_0^{\frac{1-\delta m(\mathcal{P})}{2m(\mathcal{P})\mu_0}}$ (we can suppose $\tau_1 \ge 1$ without any restriction). Therefore, if we let

$$\begin{aligned}\mu &= \mu_0\,\frac{|\nu|}{1+|\nu|},\\ d &= \frac{2m(\mathcal{P})\mu_0}{1-\delta m(\mathcal{P})}\,\frac{|\nu|}{1+|\nu|},\end{aligned}$$

we have that formula (11) holds with

$$\begin{aligned}0 < \mu &< \frac{2}{3}\,\frac{1-\delta m(\mathcal{P})}{m(\mathcal{P})}\,\frac{|\nu|}{1+|\nu|},\\ 0 < d &< \frac{4}{3}\,\frac{|\nu|}{1+|\nu|}.\end{aligned}$$

□

We still need the following result:

PROPOSITION 6. *Under the same hypotheses of Theorem 1, there exist $C_2 \ge 1$ and $\tau_2 \ge 0$ such that*

$$\mathcal{N}_{h_{0,\epsilon}}(\tau - C_2\tau^{1-(1-\zeta)(1-\rho)}) \le \mathcal{N}_\epsilon(\tau) \le \mathcal{N}_{h_{0,\epsilon}}(\tau + C_2\tau^{1-(1-\zeta)(1-\rho)}), \tag{13}$$

for all $\tau \geq \tau_2$, *for all* $0 < \epsilon \leq 1$.

Proof. The operators h_ϵ^w and $h_{0,\epsilon}^w$ have the same domain D, because they have the same principal symbol. Then we have the following variational characterizations of the counting function. If $\mathcal{L}$ is the set of all linear subspaces of D, then

$$\mathcal{N}_\epsilon(\tau) = \inf\{\mathrm{codim} L | L \in \mathcal{L} : (h_\epsilon^w u, u) > \tau \|u\|_{L^2}, \forall u \in L\} \tag{14}$$

and

$$\mathcal{N}_{0,\epsilon}(\tau) = \inf\{\mathrm{codim} L | L \in \mathcal{L} : (h_{0,\epsilon}^w u, u) > \tau \|u\|_{L^2}, \forall u \in L\}. \tag{15}$$

In the proof of Proposition 5.1 of [5], it is shown that (14) and (15) together with Proposition 7 in [9] imply (13) and the constant C_2 can be chosen *uniformly* with respect to $0 < \epsilon \leq 1$. □

Now we are ready to prove our main theorem.

Proof of Theorem 1. From (11) we have

$$\begin{aligned} &\mathcal{N}_{h_{0,\epsilon}}(\tau + C_2 \tau^{1-(1-\zeta)(1-\rho)}) \leq \\ &\qquad \leq \epsilon^{-2n} \mathcal{W}(\tau + C_2 \tau^{1-(1-\zeta)(1-\rho)}; h_0) \big\{1 + O(\epsilon^d \tau^{-\mu})\big\}, \end{aligned}$$

as $\tau \to +\infty$ and for all $0 < \epsilon \leq 1$. From (42) in Lemma 2 of [9], we have

$$\begin{aligned} &\mathcal{W}(\tau + C_2 \tau^{1-(1-\zeta)(1-\rho)}; h_0) \leq \\ &\qquad \leq \mathcal{W}(\tau; h_0) \left\{1 + O(\tau^{-(1-\zeta)(1-\rho)\frac{|\nu|}{1+|\nu|}})\right\} \end{aligned}$$

and therefore

$$\begin{aligned} &\mathcal{N}_{h_{0,\epsilon}}(\tau + C_2 \tau^{1-(1-\zeta)(1-\rho)}) \leq \\ &\qquad \leq \epsilon^{-2n} \mathcal{W}(\tau; h_0) \big\{1 + O(\epsilon^d \tau^{-\mu})\big\} \left\{1 + O(\tau^{-(1-\zeta)(1-\rho)\frac{|\nu|}{1+|\nu|}})\right\} \\ &\qquad \leq \epsilon^{-2n} \mathcal{W}(\tau; h_0) \big\{1 + O(\epsilon^d \tau^{-\mu})\big\}, \end{aligned} \tag{16}$$

with d satisfying (7) and μ satisfying (8) and (9). Using (43) in Lemma 2 of [9], we obtain in the same way the other estimate:

$$\mathcal{N}_{h_{0,\epsilon}}(\tau - C_2 \tau^{1-(1-\zeta)(1-\rho)}) \geq \epsilon^{-2n} \mathcal{W}(\tau; h_0) \big\{1 + O(\epsilon^d \tau^{-\mu})\big\}. \tag{17}$$

Formula (10) now follows from (16), (17) and (13). The proof of Theorem 1 is complete. □

5. An example: the quasi-elliptic case. At the end of our paper we take into exam the special case in which $p_0(x,\xi)$ is quasi-elliptic with respect to ξ, that is

$$p_0(x, t^\omega \xi) = t p_0(x,\xi) \qquad \forall\, t \in \mathbb{R}_+,\ \forall\, x, \xi \in \mathbb{R}^n, \tag{18}$$

where $\omega \in \mathbb{R}_+^n$ is defined in assumption 4 of Theorem 1.

In the quasi-elliptic case, (42) and (43) in Lemma 2 of [9] become (see [9], Section 6)

$$\mathcal{W}((1+\theta)\tau; h_0) \leq (1 + K\theta) \mathcal{W}(\tau; h_0), \tag{19}$$

$$\mathcal{W}((1-\theta)\tau; h_0) \geq (1 - K\theta) \mathcal{W}(\tau; h_0), \tag{20}$$

for a suitable $K > 0$. Then, repeating the same arguments as in the proof of Proposition 5 and of Theorem 1, we obtain the following result:

THEOREM 2. *Let p_0 be quasi-elliptic with respect to ξ. Let $\mathcal{N}_\epsilon(\tau)$ be the counting function associated to the operator h_ϵ^w. Under the hypotheses of Theorem 1, for*

$$0 < d < \frac{4}{3},$$
$$0 < \mu < \frac{2}{3}\frac{1-\delta m(\mathcal{P})}{m(\mathcal{P})},$$
$$\mu \le (1-\rho)(1-\zeta),$$

with

$$\zeta = \max_{\alpha \in A \setminus F(\mathcal{P})} k(\alpha;\mathcal{P}),$$

we have that

$$\mathcal{N}_\epsilon(\tau) = \epsilon^{-2n}\mathcal{W}(\tau;h_0)\{1 + O(\epsilon^d\tau^{-\mu})\}, \tag{21}$$

as $\tau \to +\infty$ and for all $0 < \epsilon \le 1$.

REMARK. From Theorem 2 it is clear that estimates (7),(8) and (9) can be improved with the following ones:

$$0 < d < \frac{4}{3},$$
$$0 < \mu < \frac{2}{3}\frac{1-\delta m(\mathcal{P})}{m(\mathcal{P})},$$
$$\mu \le (1-\rho)(1-\zeta).$$

Moreover, in the quasi-elliptic case, the *Weyl term* $\mathcal{W}(\tau;h_0)$ can be expressed in a more explicit form:

$$\mathcal{W}(\tau;h_0) = (2\pi)^{-n}\int (\tau - q(x))_+^{|\omega|}\,\sigma_\omega(x)\,dx$$

where $(\tau - q(x))_+$ is the positive part of $\tau - q(x)$,

$$\sigma_\omega(x) = \frac{1}{|\omega|}\int\limits_\Psi p_0(x,\zeta(\psi))^{-|\omega|}|J_\omega(\psi)|\,d\psi, \tag{22}$$

and $|J_\omega(\psi)|$ is the *Jacobian* of a suitable matrix (see [9], Section 4, for further details).

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