METHODS AND APPLICATIONS OF ANALYSIS.
Vol. 14, No. 4, pp. 387–404, December 2007
© 2007 International Press
005

HYPOELLIPTIC CONVOLUTION EQUATIONS IN $\mathcal{S}'(\mathbb{R})$ FOR THE DUNKL THEORY ON $\mathbb{R}$*

SLAIM BEN FARAH† AND KAMEL MOKNI‡

Abstract. The aim of this paper is to characterize hypoelliptic convolution-equations in $\mathcal{S}'(\mathbb{R})$ for the Dunkl theory on the real line. For this we determine the spaces of convolution and multiplication operators in $\mathcal{S}'(\mathbb{R})$ for the Dunkl convolution and we show that the Fourier-Dunkl transform is a topological isomorphism between them.

Key words. Dunkl-Convolution, tempered distributions, hypoelliptic equations.

AMS subject classifications. 46F10, 42A85, 42A38

1. Introduction.

Let $\mathcal{H}'$ be one of the spaces $\mathcal{D}'(\mathbb{R}^d)$ or $\mathcal{S}'(\mathbb{R}^d)$ of distributions and tempered distributions on $\mathbb{R}^d$. We denote by $\mathcal{O}'_C(\mathcal{H}')$ the space of usual convolution operators on $\mathcal{H}'$ and by $\mathcal{O}_C(\mathcal{H}')$ its dual space. We consider convolution equations of the form

$$S * U = F,$$

where $S \in \mathcal{O}'_C(\mathcal{H}')$, $F \in \mathcal{H}'$ and the indeterminate $U \in \mathcal{H}'$. Such equations or S are said to be hypoelliptic, if any solution U belongs to $\mathcal{O}_C(\mathcal{H}')$ whenever F belongs to $\mathcal{O}_C(\mathcal{H}')$.

In the case $\mathcal{H}' = \mathcal{D}'(\mathbb{R}^d)$, we note that $\mathcal{O}'_C(\mathcal{H}') = \mathcal{E}'(\mathbb{R}^d)$ the space of compact support distributions on $\mathbb{R}^d$, and $\mathcal{O}_C(\mathcal{H}') = \mathcal{E}(\mathbb{R}^d)$ the space of $\mathcal{C}^\infty$-functions on $\mathbb{R}^d$. L. Ehrempreis [7] and next L. Hörmander [9] have characterized hypoelliptic distributions by giving necessary and sufficient conditions on their usual Fourier transforms. Analogous result is obtained by Trimèche [18] for the Dunkl convolution on $\mathbb{R}^d$. Similar characterizations are established for other convolutions, see for example [1], [11] and [19].

In the case $\mathcal{H}' = \mathcal{S}'(\mathbb{R}^d)$, and according to Schwartz [16], $\mathcal{O}'_C(\mathcal{H}')$ is the space of rapidly decreasing distributions on $\mathbb{R}^d$ which we denote by $\mathcal{O}'_C(\mathbb{R}^d)$, and $\mathcal{O}_C(\mathcal{H}')$ is the space of very slowly increasing $\mathcal{C}^\infty$-functions and is denoted by $\mathcal{O}_C(\mathbb{R}^d)$. Let us also denote by $\mathcal{O}_M(\mathbb{R}^d)$ the space of multiplication operators on $\mathcal{S}'(\mathbb{R}^d)$. For this situation, Zieleźny [20] has characterized hypoelliptic convolution equations. He uses essentially the properties of the spaces $\mathcal{O}'_C(\mathbb{R}^d)$ and $\mathcal{O}_M(\mathbb{R}^d)$ developed in [16], and the fact that the usual Fourier transform is a continuous isomorphism between them. We note that such equations has been studied in [3], for the Jacobi convolution on $\mathbb{R}$ instead of the usual convolution.

In this work, we are interested by hypoelliptic Dunkl-convolution equations in $\mathcal{S}'(\mathbb{R})$, so we have to investigate the spaces of Dunkl-convolution operators and multiplication operators on $\mathcal{S}'(\mathbb{R})$, and the relation between them via the Dunkl Fourier transform. We organize this paper as follows

In chapter 2, we collect some properties and results about the Dunkl theory on the real line. Then we establish some properties of the Schwartz spaces $\mathcal{S}(\mathbb{R})$ and

*Received November 9, 2007; accepted for publication July 10, 2008.
†Faculty of Sciences of Monastir, Department of Mathematics, 5019 Monastir, Tunisia (slaim.benfarah@fsm.rnu.tn).
‡Faculty of Sciences of Monastir, Department of Mathematics, 5019 Monastir, Tunisia (kamel.mokni@fsm.rnu.tn).

$\mathcal{S}'(\mathbb{R})$, related respectively to the usual derivative of and to the Dunkl operator. In Chapter 3, we prove some equivalent properties of the spaces of multiplication and Dunkl-convolution operators in $\mathcal{S}'(\mathbb{R})$. In particular we show that the Dunkl-Fourier transform, denoted by $\mathcal{F}_d$, is a topological isomorphism between them, and that the space of Dunkl-convolution operators in $\mathcal{S}'(\mathbb{R})$ is equal to the usual space $\mathcal{O}'_C(\mathbb{R})$. In the last chapter we characterize hypoelliptic Dunkl-convolution equations in $\mathcal{S}'(\mathbb{R})$.

2. Basic properties of the Dunkl theory on the real line.

2.1. Dunkl convolution. Let $\mathcal{C}^\infty(\mathbb{R})$ and $\mathcal{D}(\mathbb{R})$ be respectively the spaces of $\mathcal{C}^\infty$-functions and $\mathcal{C}^\infty$-functions with compact support on $\mathbb{R}$. For $n \geq 1$ integer and $f \in \mathcal{C}^\infty(\mathbb{R})$, we denote $(d/dx)^n(f)$ by $f^{(n)}$.
For a fixed real $\alpha \geq -1/2$, we consider the differential-difference operator, defined for a $\mathcal{C}^\infty$-function f on $\mathbb{R}$, by

$$\Lambda f(x) = f'(x) + (2\alpha+1)\frac{f(x)-f(-x)}{2x}.$$

This operator maps the spaces $\mathcal{C}^\infty(\mathbb{R})$, $\mathcal{D}(\mathbb{R})$ and $\mathcal{S}(\mathbb{R})$ into themselves. It is known as the Dunkl operator on $\mathbb{R}$ of index $(\alpha+1/2)$ related to the reflection group $\mathbb{Z}_2$ and the weight function $|x|^{2\alpha+1}$. We are going to collect some results about harmonic analysis associated with this operator. For this, and for the general Dunkl theory, we refer to [5], [6], [10], [13], [14] and [15] and their bibliographies.

For each $z \in \mathbb{C}$, we denote by $E(., iz)$ the unique holomorphic solution of the problem

$$\begin{cases} \Lambda\Psi &= iz\Psi, \\ \Psi(0) &= 1. \end{cases}$$

The function E satisfies $|E(x, iy)| \leq 1$, for all real x, y.
Let $d\mu(x) = |x|^{2\alpha+1}dx$ and $L^1_\mu(\mathbb{R})$ be the space of μ-integrable functions on $\mathbb{R}$. The Dunkl-transform $\mathcal{F}_d$ is then defined for all $f \in L^1_\mu(\mathbb{R})$ and $y \in \mathbb{R}$, by

$$\mathcal{F}_d(f)(y) = \frac{1}{c_\mu}\int_{\mathbb{R}} f(x)E(x, iy)d\mu(x),$$

where

$$c_\mu = \int_{\mathbb{R}} e^{-x^2/2}d\mu(x).$$

It is clear that for a such f, $\mathcal{F}_d(f)$ is a bounded continuous function and

$$||\mathcal{F}_d(f)||_\infty \leq \frac{1}{c_\mu}||f||_{L^1_\mu(\mathbb{R})}.$$

Moreover, $\mathcal{F}_d$ is a topological isomorphism of $\mathcal{S}(\mathbb{R})$ onto itself, and for each $f \in \mathcal{S}(\mathbb{R})$, $x, y \in \mathbb{R}$

$$\mathcal{F}_d^{-1}(f)(x) = \mathcal{F}_d(f)(-x) \qquad \text{and} \qquad \mathcal{F}_d(\Lambda f)(y) = iy\mathcal{F}_d(f)(y).$$

As in the classical analysis, we have a Dunkl-convolution $*_d$, defined for $f, g \in L^1_\mu(\mathbb{R})$ and $x \in \mathbb{R}$ by

$$f *_d g(x) = \int_{\mathbb{R}} f(y)\tau_x g(-y)d\mu(y).$$

Here τ_x is a related translation operator, which has the following explicit formula

$$\begin{aligned} \tau_y f(x) &= \frac{1}{2}\int_{-1}^{1} f(\sqrt{x^2+y^2-2xyt})(1+\frac{x-y}{\sqrt{x^2+y^2-2xyt}})\Phi(t)dt \\ &+ \frac{1}{2}\int_{-1}^{1} f(-\sqrt{x^2+y^2-2xyt})(1-\frac{x-y}{\sqrt{x^2+y^2-2xyt}})\Phi(t)dt, \end{aligned} \tag{2.1}$$

where

$$\Phi(t) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+1/2)}(1+t)(1-t^2)^{\alpha-1/2}. \tag{2.2}$$

τ_x, $*_d$, $\mathcal{F}_d$ and Λ satisfy the following properties and relations between them.

(1) $\big(L^1_\mu(\mathbb{R}), +, *_d\big)$ is a commutative Banach algebra.
(2) $\mathcal{S}(\mathbb{R}) *_d \mathcal{S}(\mathbb{R}) \subset \mathcal{S}(\mathbb{R})$.
(3) τ_x maps the spaces $\mathcal{C}^\infty(\mathbb{R})$, $\mathcal{D}(\mathbb{R})$, $L^1_\mu(\mathbb{R})$ and $\mathcal{S}(\mathbb{R})$ into themselves.
(4) $\tau_x f(-y) = \tau_y f(-x)$.
(5) $||\tau_x f||_\infty \le ||f||_\infty$.
(6) $||\tau_x f||_{L^1_\mu(\mathbb{R})} \le ||f||_{L^1_\mu(\mathbb{R})}$.
(7) $\Lambda(\tau_x f) = \tau_x(\Lambda f)$.
(8) $\mathcal{F}_d\big[f *_d g\big] = \mathcal{F}_d(f).\mathcal{F}_d(g)$.
(9) $\Lambda\big[f *_d g\big] = (\Lambda f) *_d g$.

Now we give two technical useful lemmas for the next.

LEMMA 2.1. *There is a positive constant C such that, for all $f \in \mathcal{S}(\mathbb{R})$, $k \in \mathbb{N}$ and $x \in \mathbb{R}$,*

$$||(1+y^2)^k \tau_x f||_\infty \le C(1+x^2)^k ||(1+y^2)^k f||_\infty.$$

Proof. Let $f \in \mathcal{S}(\mathbb{R})$ and $x \in \mathbb{R}$. One has $\tau_x f = \tau_x f_e + \tau_x f_o$, where $f_e = (f+f^\vee)/2$, $f_o = (f-f^\vee)/2$ and $f^\vee(x) = f(-x)$. The fact that f_e is even and f_o is odd implies that

$$(1+y^2)^k \tau_x f_e(y) = (1+y^2)^k \int_{-1}^{1} f_e(\sqrt{x^2+y^2-2xyt})\Phi(t)dt,$$

$$(1+y^2)^k \tau_x f_o(y) = (1+y^2)^k \int_{-1}^{1} f_o(\sqrt{x^2+y^2-2xyt})\frac{y-x}{\sqrt{x^2+y^2-2xyt}}\Phi(t)dt.$$

We conclude by using the well known fact saying that there is some constant C_1 independent of x and y, such that $(1+y^2)^k \le C_1(1+x^2)^k(1+(|x|-|y|)^2)^k$. □

LEMMA 2.2. *Let φ be a positive even function in $\mathcal{D}(\mathbb{R})$, with support in $[-1,1]$ and $\varphi \ge 1$ on $[-1/2, 1/2]$. Then*
i) there is a constant $C > 0$ such that, $\tau_x(\varphi)(x) \ge C/|x|^{2\alpha+1}$ when $|x| \ge 1$,
ii) $\tau_x(\varphi)(y) = 0$ whenever $||x|-|y|| \ge 1$.

Proof. φ is even, so $\tau_x(\varphi)(y) = \int_{-1}^{1} \varphi(\sqrt{x^2+y^2-2xyt})\Phi(t)dt$.
i) Let x be a real number such that $|x| \ge 1$, then

$$\tau_x(\varphi)(x) = \int_{-1}^{1} \varphi(\sqrt{2}|x|\sqrt{1-t})\Phi(t)dt.$$

The change of variable $u = \sqrt{2}|x|\sqrt{1-t}$ and the expression (2.2) of Φ implies that

$$\tau_x(\varphi)(x) = \frac{C}{|x|^{2\alpha+1}} \int_0^{2|x|} \varphi(u)(2 - \frac{u^2}{2x^2})^{\alpha+1/2} u^{2\alpha} du,$$

where $C = \dfrac{\Gamma(\alpha+1)}{2^{\alpha-1/2}\sqrt{\pi}\Gamma(\alpha+1/2)}$.

If we use that φ is positive and that, $\varphi(u) \geq 1$ and $(2 - \frac{u^2}{2x^2}) \geq 1$ for $u \in [0, 1/2]$, we obtain

$$\tau_x(\varphi)(x) \geq \frac{C}{(2\alpha+1)2^{2\alpha+1}} \frac{1}{|x|^{2\alpha+1}}.$$

ii) This is quite trivial if we note that $supp(\varphi) \subset [-1, 1]$ and

$$x^2 + y^2 - 2xyt = (|x| - |y|)^2 + 2|x||y|(1 - \operatorname{sgn}[xy]t) \geq 1.$$

□

We need also, the classical Taylor's formula with integral remainder for a $\mathcal{C}^\infty$-function f on $\mathbb{R}$, which asserts that, for any $n \in \mathbb{N}$, there is a polynomial Q_n of degree n such that

$$f(x) = Q_n(x) + \frac{1}{n!} \int_0^x (x-t)^n f^{(n+1)}(t) dt. \tag{2.3}$$

In [12], an extension of this formula to the Dunkl operator Λ proves the existence, for any $n \in \mathbb{N}$, of a polynomial P_n of degree n and a function W_n defined on $\mathbb{R}^2$, such that

$$f(x) = P_n(x) + \int_{-|x|}^{|x|} W_n(x,t) \Lambda^{n+1} f(t) |t|^{2\alpha+1} dt, \tag{2.4}$$

Next we shall refer to this formula as Dunkl-Taylor's formula. In fact the functions W_n are defined by induction, and we retain the following properties when $(0 < |y| < |x|)$

(1) $W_0(x,y) = \operatorname{sgn}(x)/2|x|^{2\alpha+1} + \operatorname{sgn}(y)/2|y|^{2\alpha+1}$.
(2) $\Lambda_x W_{n+1}(x,y) = W_n(x,y)$.
(3) $W_n(x,x) = W_n(-x,x) = 0$.
(4) $|W_i(x,y| \leq C|x|^i/|y|^{2\alpha+1}$.

In order to lighten the mean proofs of this paper, we conclude this section by the lemma and remark below.

Lemma 2.3. *Let $f \in \mathcal{C}^\infty(\mathbb{R})$, then for all integer $n \geq 1$,*

$$\Lambda^n f'(x) = \sum_{i=0}^n \frac{1}{|x|^{i+1}} \int_{-|x|}^{|x|} \left[A_{i,n} W_i(x,t) + B_{i,n} W_i(-x,t)\right] \Lambda^{n+1} f(t) |t|^{2\alpha+1} dt.$$

where $A_{i,n}$ and $B_{i,n}$ are constants.

Proof. The result can be obtained by induction on n, if we use the Dunkl-Taylor's formula

$$f(x) = P_n(x) + \int_{-|x|}^{|x|} W_n(x,t) \Lambda^{n+1} f(t) |t|^{2\alpha+1} dt,$$

and the fact that

$$\frac{d}{dx}W_n(x,t) = \Lambda_x W_n(x,t) - (2\alpha+1)\frac{W_n(x,t) - W_n(-x,t)}{2x},$$

with $\Lambda_x W_n(x,t) = W_{n-1}(x,t)$. □

REMARK 2.1. *Let f be a $\mathcal{C}^\infty$-function and define $f^\vee$ by $f^\vee(x) = f(-x)$, then for all $m, k \in \mathbb{N}$,*

i) $\Lambda^m[(1+x^2)^k f]$ is a linear combination of terms of the form

$$x^\varepsilon (1+x^2)^s \Lambda^j f^{\vee^\eta},$$

where ε, η, s, j are integers such that $0 \le \varepsilon, \eta \le 1$, $0 \le s \le k_0$ and $0 \le j \le m$. (Here $f^{\vee^0} = f$ and $f^{\vee^1} = f^\vee$).

ii) $\Lambda^m\big[\frac{f}{(1+x^2)^k}\big]$ is a linear combination of terms of the form

$$\frac{x^\varepsilon \Lambda^j f^{\vee^\eta}(x)}{(1+x^2)^{k+s}},$$

where ε, η, s, j are integers such that $0 \le \varepsilon, \eta \le 1$, $0 \le j, s \le m$.

This remark can be obtained by induction on m, if we use the following formulas

$$\Lambda(f^\vee) = -(\Lambda f)^\vee \qquad ; \qquad (f^\vee)' = -(f')^\vee$$

$$\Lambda((1+x^2)^p g) = 2px(1+x^2)^{p-1}g + (1+x^2)^p \Lambda g,$$

$$\Lambda(x(1+x^2)^p g) = (1+2p)(1+x^2)^p g - 2p(1+x^2)^{p-1}g + x(1+x^2)^p \Lambda g + (2\alpha+1)(1+x^2)^p g^\vee,$$

where p is any integer in $\mathbb{Z}$. □

2.2. Definitions and properties of some functional spaces. The topology of $\mathcal{S}(\mathbb{R})$ is defined by the family of semi-norms, given for all $k, n \in \mathbb{N}$ by

$$p_{k,n}(f) = \max_{0 \le m \le n} ||(1+x^2)^k f^{(m)}||_\infty.$$

We consider the semi-norms associated to the operator Λ, defined on $\mathcal{S}(\mathbb{R})$ for all $k, n \in \mathbb{N}$

$$q_{k,n}(f) = \max_{0 \le m \le n} ||(1+x^2)^k \Lambda^m f||_\infty.$$

It is known that the $p_{k,n}$ and the $q_{k,n}$ generate the same topology on $\mathcal{S}(\mathbb{R})$ (see [2]). In fact it can be proved that, for any integers k and n, there are positive constants C, D such that

$$C.p_{k,n} \le q_{k,n} \le D.p_{k,n}.$$

In particular this implies that

$$\mathcal{S}(\mathbb{R}) = \mathcal{S}_b(\mathbb{R}) := \{f \in \mathcal{C}^\infty(\mathbb{R}) \ ; \ \forall n, k \in \mathbb{N}, \ ||(1+x^2)^k \Lambda^n f||_\infty < \infty\}.$$

We introduce now two classes of functions, which are a natural generalization of the usual spaces denoted by $\mathcal{B}(\mathbb{R})$ and $\mathcal{D}_{(\mathbb{R})}(\mathbb{R})$ in [16].

1) $\mathcal{B}(\mathbb{R})$ is the space of $\mathcal{C}^\infty$-functions f on $\mathbb{R}$, such that $f^{(n)}$ is bounded on $\mathbb{R}$, for all $n \geq 0$. Its topology is defined by the family of semi-norms $||f^{(n)}||_\infty$, $n \in \mathbb{N}$.

1') $\mathcal{B}_d(\mathbb{R})$ will be the space of $\mathcal{C}^\infty$-functions f on $\mathbb{R}$, such that $\Lambda^n f$ is bounded on $\mathbb{R}$ for all $n \geq 0$. The semi-norms for this space are $||\Lambda^n f||_\infty$, $n \in \mathbb{N}$.

2) $\mathcal{D}_{L^1(\mathbb{R})}(\mathbb{R})$ is the space of $\mathcal{C}^\infty$-functions f on $\mathbb{R}$, such that $f^{(n)}$ is $L^1(\mathbb{R})$ for all $n \geq 0$, equipped with the family of semi-norms $||f^{(n)}||_{L^1(\mathbb{R})}$, $n \in \mathbb{N}$.

2') $\mathcal{D}_{L^1_\mu(\mathbb{R})}(\mathbb{R})$ will be the space of $\mathcal{C}^\infty$-functions f on $\mathbb{R}$, such that $\Lambda^n f$ is $L^1_\mu(\mathbb{R})$ for all $n \geq 0$, and the semi-norms are $||f^{(n)}||_{L^1_\mu(\mathbb{R})}$, $n \in \mathbb{N}$.

We are going to prove that $\mathcal{B}_d(\mathbb{R}) = \mathcal{B}(\mathbb{R})$ as topological spaces, this is a consequence of the following proposition which is useful for the remainder.

Proposition 2.1. *For all $k \geq 0$, $n \geq 0$ there exist C_1 and C_2 positive real numbers such that, for all $f \in \mathcal{B}(\mathbb{R})$,*

$$C_1 || \frac{\Lambda^n f}{(1+x^2)^k} ||_\infty \leq || \frac{f^{(n)}}{(1+x^2)^k} ||_\infty \leq C_2 || \frac{\Lambda^n f}{(1+x^2)^k} ||_\infty .$$

In particular, $\mathcal{B}(\mathbb{R}) = \mathcal{B}_d(\mathbb{R})$ as topological spaces.

Proof. The inequalities will be proved by induction on n. They are trivial for $n = 0$, suppose that

$$|| \frac{\Lambda^n f}{(1+x^2)^k} ||_\infty \leq C || \frac{f^{(n)}}{(1+x^2)^k} ||_\infty$$

for some positive constant C, then

$$|| \frac{\Lambda^{n+1} f}{(1+x^2)^k} ||_\infty \leq C || \frac{(\Lambda f)^{(n)}}{(1+x^2)^k} ||_\infty ,$$

but $\Lambda f(x) = f'(x) + (2\alpha+1) \dfrac{f(x) - f(-x)}{2x}$, so

$$|| \frac{\Lambda^{n+1} f}{(1+x^2)^k} ||_\infty \leq C (|| \frac{f^{(n+1)}}{(1+x^2)^k} ||_\infty + (\alpha + 1/2) || \frac{1}{(1+x^2)^k} (\frac{f - f^\vee}{x})^{(n)} ||_\infty),$$

using the Taylor formula:

$$f(x) = Q_n(x) + \frac{1}{n!} \int_0^x (x-t)^n f^{(n+1)}(t) dt,$$

we obtain

$$\frac{f(x) - f(-x)}{x} = R_{n-1}(x) + \frac{1}{n!} \frac{1}{x} \int_0^x (x-t)^n \big[f^{(n+1)}(t) + (-1)^n f^{(n+1)}(-t) \big] dt.$$

So, by Leibnitz Formula, there are some constants $(A_j)_{0 \leq j \leq n}$ such that

$$(\frac{f - f^\vee}{x})^{(n)}(x) = \sum_{j=0}^n A_j \frac{1}{x^{j+1}} \int_0^x (x-t)^j \big[f^{(n+1)}(t) + (-1)^n f^{(n+1)}(-t) \big] dt. \quad (2.5)$$

Observe that, for $|t| \leq |x|$

$$\frac{|f^{(n+1)}(t) + (-1)^n f^{(n+1)}(-t)|}{(1+x^2)^k} \leq \frac{|f^{(n+1)}(t) + (-1)^n f^{(n+1)}(-t)|}{(1+t^2)^k}$$
$$\leq \text{const.}||\frac{f^{(n+1)}}{(1+x^2)^k}||_\infty,$$

so for any $0 \leq j \leq n$

$$|\frac{1}{(1+x^2)^k}\frac{1}{x^{j+1}}\int_0^x (x-t)^j\big[f^{(n+1)}(t) + (-1)^n f^{(n+1)}(-t)\big]dt| \leq \text{const.}||\frac{f^{(n+1)}}{(1+x^2)^k}||_\infty.$$

This gives the first inequality.
For the second inequality, we suppose that

$$||\frac{f^{(n)}}{(1+x^2)^k}||_\infty \leq C||\frac{\Lambda^n f}{(1+x^2)^k}||_\infty$$

for some positive constant C, then

$$||\frac{f^{(n+1)}}{(1+x^2)^k}||_\infty \leq C||\frac{\Lambda^n f'}{(1+x^2)^k}||_\infty.$$

In the other hand, by Lemma 2.3 one has

$$\Lambda^n f'(x) = \sum_{i=0}^{n} \frac{1}{|x|^{i+1}}\int_{-|x|}^{|x|}\big[A_{i,n}W_i(x,t) + B_{i,n}W_i(-x,t)\big]\Lambda^{n+1}f(t)|t|^{2\alpha+1}dt,$$

$$\frac{|\Lambda^{n+1}f(t)|}{(1+x^2)^k} \leq \frac{|\Lambda^{n+1}f(t)|}{(1+t^2)^k} \leq \text{const.}||\frac{\Lambda^{(n+1)}f}{(1+x^2)^k}||_\infty,$$

and

$$|W_i(x,t| \leq C\frac{|x|^i}{|t|^{2\alpha+1}},$$

combining all this as before we get the result. □

3. Dunkl-Convolution operators in $\mathcal{S}'(\mathbb{R})$. The aim of this section is to define and characterize, as in [16], the spaces $\mathcal{O}_{M,d}(\mathbb{R})$ and $\mathcal{O}'_{C,d}(\mathbb{R})$ of Dunkl-multiplication and convolution operators on $\mathcal{S}'(\mathbb{R})$.

3.1. Multiplication operators. Let us recall some properties of $\mathcal{O}_M(\mathbb{R})$, this space is identified with the space of $\mathcal{C}^\infty$-functions slowly increasing together with all their derivatives. In other words, for a $\mathcal{C}^\infty$-function f one has

$$f \in \mathcal{O}_M(\mathbb{R}) \quad \Leftrightarrow \quad \forall p \in \mathbb{N},\ \forall \varphi \in \mathcal{S}(\mathbb{R});\quad ||\varphi.f^{(p)}||_\infty < +\infty$$
$$\Leftrightarrow \quad \forall p \in \mathbb{N},\ \exists k_p \in \mathbb{N};\quad ||\frac{f^{(p)}}{(1+x^2)^{k_p}}||_\infty < +\infty.$$

For the topology, a sequence $(f_n)_n$ converges to 0 in $\mathcal{O}_M(\mathbb{R})$ if and only if

$$\forall p \in \mathbb{N},\ \forall \varphi \in \mathcal{S}(\mathbb{R});\quad ||\varphi.f_n^{(p)}||_\infty \to 0 \quad \text{as} \quad n \to \infty,$$

the convergence is uniform on φ when it belongs to a bounded subset of $\mathcal{S}(\mathbb{R})$. This is also equivalent to

$$\forall p \in \mathbb{N},\ \exists k_p \in \mathbb{N};\quad ||\frac{f_n^{(p)}}{(1+x^2)^{k_p}}||_\infty \to 0 \quad \text{as} \quad n \to \infty,$$

where k_p depends only on p and not on n.

In fact the spaces $\mathcal{O}_{M,d}(\mathbb{R})$ and $\mathcal{O}_M(\mathbb{R})$ are the same space, since we have the same Schwartz space $\mathcal{S}'(\mathbb{R})$, we shall then use the notation $\mathcal{O}_M(\mathbb{R})$ for them. Moreover the above definitions are compatible with the Dunkl-operator, indeed, Proposition 2.1 allows us to assert trivially the following

Proposition 3.1.
i) If f is a $\mathcal{C}^\infty$-function then

$$\begin{aligned} f \in \mathcal{O}_M(\mathbb{R}) \quad &\Leftrightarrow \quad \forall p \in \mathbb{N},\ \forall \varphi \in \mathcal{S}(\mathbb{R});\quad ||\varphi.\Lambda^p f||_\infty < +\infty \\ &\Leftrightarrow \quad \forall p \in \mathbb{N},\ \exists k_p \in \mathbb{N};\quad ||\frac{\Lambda^p f}{(1+x^2)^{k_p}}||_\infty < +\infty. \end{aligned}$$

ii) A sequence $(f_n)_n$ in $\mathcal{O}_M(\mathbb{R})$ converges to 0 if and only if

$$\forall p \in \mathbb{N},\ \forall \varphi \in \mathcal{S}(\mathbb{R});\quad ||\varphi.\Lambda^p f_n||_\infty \to 0 \quad \textit{as} \quad n \to \infty,$$

the convergence is uniform on φ when it belongs to a bounded subset of $\mathcal{S}(\mathbb{R})$. This is also equivalent to

$$\forall p \in \mathbb{N},\ \exists k_p \in \mathbb{N};\quad ||\frac{\Lambda^p f_n}{(1+x^2)^{k_p}}||_\infty \to 0 \quad \textit{as} \quad n \to \infty,$$

where k_p depends only on p and not on n.

3.2. Dunkl-bounded distributions . Now to define $\mathcal{O}'_{C,d}(\mathbb{R})$ we need the topological dual space of $\mathcal{D}_{L^1_\mu(\mathbb{R})}(\mathbb{R})$ which we denote by $\mathcal{B}'_d(\mathbb{R})$. For its topology, a sequence $(T_n)_n$ converges to 0 in $\mathcal{B}'_d(\mathbb{R})$ if, for any bounded subset A of $\mathcal{D}_{L^1_\mu(\mathbb{R})}(\mathbb{R})$, $\langle T_n, \varphi\rangle$ tends to 0 uniformly on $\varphi \in A$. $\mathcal{D}(\mathbb{R})$ is dense in $\mathcal{D}_{L^1_\mu(\mathbb{R})}(\mathbb{R})$, so $\mathcal{B}'_d(\mathbb{R})$ is a space of distributions and we have

Theorem 3.1. *The following assertions are equivalent, for a distribution T.*
i) $T \in \mathcal{B}'_d(\mathbb{R})$.
ii) $\forall \varphi \in \mathcal{D}(\mathbb{R})$; $T *_d \varphi$ is bounded.
iii) There are bounded functions $f_1, \ldots, f_q$ such that $T = \sum_{j=1}^q \Lambda^{l_j} f_j d\mu_\alpha$.

Proof. i) $\Rightarrow$ ii): Suppose that $T \in \mathcal{B}'_d(\mathbb{R})$ and let $\varphi \in \mathcal{D}(\mathbb{R})$. For all real x, one has $|T *_d \varphi(x)| = |\langle T, \tau_x\varphi\rangle|$. But $T \in \mathcal{B}'_d(\mathbb{R})$, so there are a positive constant C and a positive integer k, such that $|\langle T, \tau_x\varphi\rangle| \le C||\Lambda^k(\tau_x\varphi)||_{L^1_\mu(\mathbb{R})}$, this implies that $|T *_d \varphi(x)| \le C||\Lambda^k\varphi||_{L^1_\mu(\mathbb{R})}$, and then

$$||T *_d \varphi||_\infty \le C||\Lambda^k\varphi||_{L^1_\mu(\mathbb{R})}.$$

ii) $\Rightarrow$ iii): Let $\varphi \in \mathcal{D}(\mathbb{R})$, by hypothesis $T *_d \varphi$ is bounded. Then, as function of $\psi \in \mathcal{D}(\mathbb{R}) \cap L^1_\mu(\mathbb{R})$ such that $||\psi||_{L^1_\mu(\mathbb{R})} \le 1$, the quantity $I := |\int_\mathbb{R} T *_d \varphi(x)\psi(x)d\mu(x)|$ is uniformly bounded. But $I = |\int_\mathbb{R} T *_d \psi(x)\varphi(x)d\mu(x)|$, so $\{T *_d \psi d\mu;\ \psi \in \mathcal{D}(\mathbb{R}) \cap L^1_\mu(\mathbb{R})$ and $||\psi||_{L^1_\mu(\mathbb{R})} \le 1\}$ is bounded in $\mathcal{D}'(\mathbb{R})$. This means that for any compact set

K of $\mathbb{R}$, there are a positive constant C and a positive integer m such that, for all $\varphi \in \mathcal{D}_K(\mathbb{R})$ and $\psi \in \mathcal{D}(\mathbb{R}) \cap L^1_\mu(\mathbb{R})$ such that $||\psi||_{L^1_\mu(\mathbb{R})} \leq 1$

$$|\int_{\mathbb{R}} T *_d \psi(x)\varphi(x)d\mu(x)| \leq C \max_{0\leq l \leq m} ||\Lambda^l \varphi||_\infty.$$

We remark that this inequality can be extended to $\varphi \in \mathcal{D}^m_K(\mathbb{R})$, to affirm that $T *_d \varphi$ stays bounded for those φ.
Let δ_0 be the Dirac measure at the origin and E a solution of the equation $\Lambda^p E = \delta_0$, where p is chosen such that E is of class $\mathcal{C}^m(\mathbb{R})$. Take $\gamma \in \mathcal{D}(\mathbb{R})$ equal to 1 in a neighborhood of 0 with support K, then $\Lambda^p(\gamma E) = \delta_0 + \zeta$, where $\zeta \in \mathcal{D}(\mathbb{R})$. To conclude we observe that

$$T = T *_d \delta_0 = \Lambda^p(T *_d \gamma E) - T *_d \zeta,$$

this is the desired because $\zeta \in \mathcal{D}(\mathbb{R})$ and $\gamma E \in \mathcal{D}^m_K(\mathbb{R})$, so $T *_d \zeta$ and $T *_d \gamma E$ are bounded.
iii) $\Rightarrow$ i): If $T = \sum_{j=1}^q \Lambda^{l_j} f_j d\mu$, with $f_1, \dots, f_q$ bounded, then for all $\varphi \in \mathcal{D}(\mathbb{R}) \cap \mathcal{D}_{L^1_\mu(\mathbb{R})}(\mathbb{R})$

$$|\langle T, \varphi \rangle| \leq \sum_{j=1}^q \int_{\mathbb{R}} |f_j(x)|.|\Lambda^{l_j}\varphi(x)|d\mu(x) \leq \sum_{j=1}^q ||f_j||_\infty.||\Lambda^{l_j}\varphi||_{L^1_\mu(\mathbb{R})},$$

this proves that $T \in \mathcal{B}'_d(\mathbb{R})$. □

3.3. Dunkl-Convolution operators. Now, we can define the topological space of Dunkl-convolution operators on $\mathcal{S}'(\mathbb{R})$ by

$$\mathcal{O}'_{C,d}(\mathbb{R}) = \{T \in \mathcal{S}'(\mathbb{R});\ \forall k \in \mathbb{N},\ (1+x^2)^k T \in \mathcal{B}'_d(\mathbb{R})\},$$

and we say that a sequence $(T_n)_n$ converges to 0 in $\mathcal{O}'_{C,d}(\mathbb{R})$ if, for all $k \in \mathbb{N}$, the sequence $((1+x^2)^k T_n)_n$ converges to 0 in $\mathcal{B}'_d(\mathbb{R})$.
The space $\mathcal{O}'_{C,d}(\mathbb{R})$ is characterized as in ([16] p. 244) by

THEOREM 3.2. *The following assertions are equivalent, for a distribution T.*
i) $T \in \mathcal{O}'_{C,d}(\mathbb{R})$.
ii) $\forall \varphi \in \mathcal{D}(\mathbb{R})$; $T *_d \varphi \in \mathcal{S}(\mathbb{R})$.
iii) $\forall k \in \mathbb{N}$, $\{(1+x^2)^k \tau_x T;\ x \in \mathbb{R}\}$ *is bounded in* $\mathcal{D}'(\mathbb{R})$.
iv) $\forall k \in \mathbb{N}$, *there are functions* $f_1, \dots, f_q$ *such that* $T = \sum_{j=1}^q \Lambda^{l_j} f_j d\mu$ *with* $(1+x^2)^k f_j$ *bounded for* $1 \leq j \leq q$.

Proof. Note that ii) and iii) are trivially equivalent.
i) $\Rightarrow$ ii): Let $\varphi \in \mathcal{D}(\mathbb{R})$, to prove that $T *_d \varphi \in \mathcal{S}(\mathbb{R})$ we have to show that

$$\forall k, m \in \mathbb{N};\quad (1+x^2)^k(T *_d \Lambda^m \varphi) \text{ is bounded.}$$

$(1+x^2)^k(T *_d \Lambda^m\varphi)$ is continuous, so it is sufficient to prove that, for all $f \in \mathcal{D}(\mathbb{R})$;

$$I := |\int_{\mathbb{R}} (1+x^2)^k f(x)(T *_d \Lambda^m\varphi)(x)d\mu(x)| \leq \text{const.}||f||_{L^1_\mu(\mathbb{R})}.$$

Let $p \in \mathbb{N}$ such that $1/(1+y^2)^{p-1} \in L^1_\mu(\mathbb{R})$ and note that

$$\begin{aligned} I &= |\langle T_y, \left[(1+x^2)^k f *_d \Lambda^m\varphi\right](y)\rangle| \\ &= |\langle (1+y^2)^{k+p} T_y, \frac{1}{(1+y^2)^{k+p}}\left[(1+x^2)^k f *_d \Lambda^m\varphi\right](y)\rangle|. \end{aligned}$$

The fact that $(1+y^2)^{k+p}T \in \mathcal{B}'_d(\mathbb{R})$ implies that there are $C>0$ and $l \in \mathbb{N}$ such that

$$I \leq C.||\Lambda_y^l\big[\frac{1}{(1+y^2)^{k+p}}((1+x^2)^k f *_d \Lambda^m\varphi)\big](y)||_{L^1_\mu(\mathbb{R})},$$

which means that

$$I \leq C.\int_{\mathbb{R}} |\Lambda_y^l \int_{\mathbb{R}} \frac{\tau_y(\Lambda^m\varphi)(x)}{(1+y^2)^{k+p}}(1+x^2)^k f(x)d\mu(x)|d\mu(y).$$

By remark 2.1, $\Lambda_y^l\big[\frac{\tau_y(\Lambda^m\varphi)(x)}{(1+y^2)^{k+p}}\big]$ is a linear combination of terms of the form

$$\frac{y^\varepsilon \tau_y(\Lambda^{m+j}\varphi^{\vee^\eta})(x)}{(1+y^2)^{k+p+s}},$$

where ε, η, j, s are nonnegative integers such that $0 \leq \varepsilon, \eta \leq 1$ and $0 \leq j, s \leq l$. For any one of those terms let

$$J := \int_{\mathbb{R}} |\int_{\mathbb{R}} \frac{y^\varepsilon \tau_y(\Lambda^{m+j}\varphi^{\vee^\eta})(x)}{(1+y^2)^{k+p+s}}(1+x^2)^k f(x)d\mu(x)|d\mu(y),$$

we suppose that $supp(\varphi) \subset [-a,a]$, then $supp(\tau_y(\Lambda^{m+j}\varphi^{\vee^\eta})) \subset [-a-|y|, a+|y|]$ and so

$$J \leq \int_{\mathbb{R}} \frac{1}{(1+y^2)^{p-1}} \int_{-a-|y|}^{a+|y|} |\tau_y(\Lambda^{m+j}\varphi^{\vee^\eta})(x)| \frac{(1+x^2)^k}{(1+y^2)^k}|f(x)|d\mu(x)d\mu(y).$$

Now if we use the inequality

$$|\tau_y(\Lambda^{m+j}\varphi^{\vee^\eta})(x)| \leq \sup_{0\leq q\leq m+l} ||\Lambda^{m+q}\varphi||_\infty,$$

and the fact that $\frac{(1+x^2)^k}{(1+y^2)^k}$ is bounded for all $(x,y) \in [-a-|y|, a+|y|] \times \mathbb{R}$ by a constant depending only on $supp(\varphi)$, we obtain

$$J \leq \text{ const.} \sup_{0\leq q\leq m+l} ||\Lambda^{m+q}\varphi||_\infty.||\frac{1}{(1+y^2)^{p-1}}||_{L^1_\mu(\mathbb{R})}.||f||_{L^1_\mu(\mathbb{R})},$$

this implies the desired estimation for I.
i) $\Rightarrow$ ii): Let $k \in \mathbb{N}$, we have to prove that T can be written as $T = \sum_{j=1}^q \Lambda^{l_j} f_j d\mu$, where $f_1, \ldots, f_q$ are functions such that $(1+x^2)^{k+p} f_j$ is bounded for $1 \leq j \leq q$. By hypothesis, $T *_d \varphi$ is in $\mathcal{S}(\mathbb{R})$ so $(1+x^2)^k T *_d \varphi$ is bounded for all $\varphi \in \mathcal{D}(\mathbb{R})$, by the same methods of Theorem 3.1, if we fix a compact set K of $\mathbb{R}$ and a positive integer m, we can extend the above to say that $(1+x^2)^k T *_d \varphi$ is bounded for all $\varphi \in \mathcal{D}_K^m(\mathbb{R})$. After this we take a solution E of the equation $\Lambda^p E = \delta_0$, where p is chosen such that E is of class $\mathcal{C}^m(\mathbb{R})$. Let $\gamma \in \mathcal{D}(\mathbb{R})$ equal to 1 in a neighborhood of 0 and denote by K its support, then $\Lambda^p(\gamma E) = \delta_0 + \zeta$, where $\zeta \in \mathcal{D}(\mathbb{R})$. As before

$$T = T *_d \delta_0 = \Lambda^p(T *_d \gamma E) - T *_d \zeta,$$

which is the desired.

iv) $\Rightarrow$ i): Let $k \in \mathbb{N}$, we shall prove that $(1+x^2)^k T \in \mathcal{B}'_d(\mathbb{R})$. We choose $p \in \mathbb{N}$ such that $1/(1+x^2)^{p-1} \in L^1_\mu(\mathbb{R})$. By hypothesis, for $k+p$, there are $f_1, \dots, f_q$ such that $(1+x^2)^{k+p} f_j$ bounded for $1 \le j \le q$ and $T = \sum_{j=1}^q \Lambda^{l_j} f_j d\mu$, in those conditions $(1+x^2)^k f_j \in L^1_\mu(\mathbb{R})$. To have the result and according to Theorem 3.1, it is sufficient to show that for all $\varphi \in \mathcal{D}(\mathbb{R})$, the function $(1+x^2)^k T *_d \varphi$ is bounded. Let $\varphi \in \mathcal{D}(\mathbb{R})$, so

$$\big[(1+x^2)^k T *_d \varphi\big](y) = \sum_{j=1}^q \int_{\mathbb{R}} f_j(x) \Lambda_x^{l_j} \big[(1+x^2)^k \tau_y \varphi\big](x) d\mu(x).$$

For $1 \le j \le q$, $\Lambda_x^{l_j}\big[(1+x^2)^k \tau_y \varphi\big](x)$ is a finite linear combination of terms of the form $x^\varepsilon (1+x^2)^s \tau_y(\Lambda^m \varphi^{\vee^\eta})(x)$,where ε, η, s, m are integers such that $0 \le \varepsilon, \eta \le 1$, $0 \le s \le k$ and $0 \le m \le l_j$. On the other hand

$$|\int_{\mathbb{R}} x^\varepsilon (1+x^2)^s f_j(x) \tau_y(\Lambda^m \varphi^{\vee^\eta})(x) d\mu(x)| \le \text{ const.} \sup_{0 \le n \le l_j} ||\Lambda^n \varphi||_\infty . ||(1+x^2)^k f_j||_{L^1_\mu(\mathbb{R})},$$

this estimation is independent of y, so $(1+x^2)^k T *_d \varphi$ is a bounded function. □

COROLLARY 3.1. $\mathcal{O}'_{C,b}(\mathbb{R}) = \mathcal{O}'_C(\mathbb{R})$.

Proof. Let $T \in \mathcal{O}'_C(\mathbb{R})$. To prove that T belongs to $\mathcal{O}'_{C,b}(\mathbb{R})$ it is sufficient to show that $T *_b \varphi \in \mathcal{S}(\mathbb{R})$ for all $\varphi \in \mathcal{D}(\mathbb{R})$ (see Theorem 3.2 (ii)). For this we fix $\varphi \in \mathcal{D}(\mathbb{R})$ with support in $[-a, a]$, $k, n \in \mathbb{N}$ and show that $(1+x^2)^k T *_b \Lambda^m \varphi(x)$ is bounded. But $(1+x^2)^k (T *_b \Lambda^m \varphi)$ is continuous, so it is sufficient to prove that, for all $\psi \in \mathcal{D}(\mathbb{R})$,

$$|\int_{\mathbb{R}} (1+x^2)^k \psi(x) (T *_b \Lambda^m \varphi)(x) d\mu(x)| \le \text{const.} ||\psi||_{L^1_\mu(\mathbb{R})}.$$

We have

$$\begin{aligned}(1+x^2)^k T *_b \Lambda^m \varphi(x) &= (1+x^2)^k \langle T_y, \tau_x \Lambda^m \varphi(y)\rangle \\ &= (1+x^2)^k \langle (1+y^2)^{k+p} T_y, \frac{\tau_x \Lambda^m \varphi(y)}{(1+y^2)^{k+p}}\rangle,\end{aligned}$$

where p is such that $1/(1+y^2)^p \in L^1_\mu(\mathbb{R})$. In the other hand $(1+y^2)^{k+p} T$ is in $\mathcal{B}'(\mathbb{R})$, so there are bounded even functions $f_1, \dots, f_q$ such that $(1+y^2)^{k+p} T = \sum_{j=1}^q \frac{d^{l_j}}{dy^{l_j}}(f_j)$, and then

$$(1+x^2)^k T *_b \Lambda^m \varphi(x) = (1+x^2)^k \sum_{j=1}^q \int_{\mathbb{R}} f_j(y) \frac{d^{l_j}}{dy^{l_j}} \big[\frac{\tau_x \Lambda^m \varphi}{(1+y^2)^{k+p}}\big](y) dy.$$

By Leibnitz Formula $(1+x^2)^k T *_b \Lambda^m \varphi(x)$ is then a linear combination of terms of the form

$$H(x) := (1+x^2)^k \int_{\mathbb{R}} f_j(y) \frac{d^s}{dy^s} \big[\frac{1}{(1+y^2)^{k+p}}\big] \frac{d^r}{dy^r} \big[\tau_x \Lambda^m \varphi\big](y) dy,$$

where $1 \le j \le q$ and $0 \le s, r \le l_j$. Now to get the desired we have just to prove that

$$|\int_{\mathbb{R}} H(x) \psi(x) dx| \le \text{const.} ||\psi||_{L^1_\mu(\mathbb{R})}.$$

Taking account the support of φ we assert that

$$|\int_{\mathbb{R}} H(x)\psi(x)dx| \leq \int_{\mathbb{R}} \frac{|f_j(y)|}{(1+y^2)^p} \int_{-a-|y|}^{a+|y|} \frac{(1+x^2)^k}{(1+y^2)^k} |\frac{d^r}{dy^r}\big[\tau_x \Lambda^m \varphi\big](y)||\psi(x)|dxd\mu(y)$$

Now, if we use the inequality

$$\frac{d^r}{dy^r}\big[\tau_y(\Lambda^m\varphi)(x)\big]| \leq \text{ const.} \max_{0\leq s\leq l_j} ||\Lambda^{m+s}\varphi||_\infty,$$

and the fact that $\dfrac{(1+x^2)^k}{(1+y^2)^k}$ is bounded for all $(x,y) \in [-a-|y|, a+|y|] \times \mathbb{R}$, we obtain

$$|\int_{\mathbb{R}} H(x)\psi(x)dx| \leq \text{const.} \max_{0\leq s\leq l_j} ||\Lambda^{m+s}\varphi||_\infty . ||\frac{1}{(1+y^2)^p}||_{L^1_\mu(\mathbb{R})} . ||f_j||_\infty . ||\psi||_{L^1(\mathbb{R})},$$

this implies the estimation.
For the converse we use the same technique with a few modification. □
We note that the result of Corollary 3.1 is obtained by Betancor in [2] by other methods.

Theorem 3.3. *The mapping* $(S,T) \longmapsto S *_d T$, *from* $\mathcal{O}'_C(\mathbb{R}) \times \mathcal{S}'(\mathbb{R})$ *to* $\mathcal{S}'(\mathbb{R})$, *is hypocontinuous.*

Proof. Let $(S,T,\varphi) \in \mathcal{O}'_C(\mathbb{R}) \times \mathcal{S}'(\mathbb{R}) \times \mathcal{D}(\mathbb{R})$, we have to show that if two of those elements stay bounded and the third tends to 0, then the quantity $\langle S_x \times T_y, \tau_x\varphi(y)\rangle$ tends to 0. For example, if S tends to 0 in $\mathcal{O}'_C(\mathbb{R})$ then for all $l \in \mathbb{N}$, $(1+x^2)^l S_x$ tends to 0 in $\mathcal{B}'_d(\mathbb{R})$. If T stays bounded in $\mathcal{S}'(\mathbb{R})$ then there is $k \in \mathbb{N}$ such that $T_y/(1+y^2)^k$ belongs to $\mathcal{B}'_d(\mathbb{R})$, and by Theorem 3.1, $T_y = (1+y^2)^k \sum_{j=1}^q \Lambda^{l_j} f_j d\mu$ where $f_1,\ldots,f_q$ stay bounded on $\mathbb{R}$. We choose $p \in \mathbb{N}$ such that $1/(1+x^2)^{p-1} \in L^1_\mu(\mathbb{R})$ and $l = k+2p$, in those conditions

$$\langle S_x \times T_y, \tau_x\varphi(y)\rangle = \langle S_x, T *_d \varphi(x)\rangle = \langle (1+x^2)^{k+2p} S_x, \frac{T *_d \varphi(x)}{(1+x^2)^{k+2p}}\rangle.$$

To conclude we must prove that $\dfrac{T *_d \varphi(x)}{(1+x^2)^{k+2p}}$ is bounded in $\mathcal{D}_{L^1_\mu(\mathbb{R})}(\mathbb{R})$ uniformly in T and φ when they stay respectively bounded in $\mathcal{S}'(\mathbb{R})$ and $\mathcal{S}(\mathbb{R})$. This is equivalent to fined, for all integer m, a positive constant M independent of T and φ, such that

$$||\Lambda^m\big[\frac{T *_d \varphi}{(1+x^2)^{k+2p}}\big]||_{L^1_\mu(\mathbb{R})} \leq M.$$

As before $\Lambda^m\big[\dfrac{T *_d \varphi}{(1+x^2)^{k+2p}}\big](x)$, is a linear combination of terms of the form

$$\frac{x^\varepsilon (T *_d \Lambda^j\varphi)^{\vee^\eta}(x)}{(1+x^2)^{k+2p+s}},$$

where ε, η, j, s are nonnegative integers such that $0 \leq \varepsilon, \eta \leq 1$ and $0 \leq j \leq m$. For such a term

$$||\frac{x^\varepsilon (T *_d \Lambda^j\varphi)^{\vee^\eta}}{(1+x^2)^{k+2p+s}}||_{L^1_\mu(\mathbb{R})} \leq \int_{\mathbb{R}} \frac{1}{(1+x^2)^{p-1}} ||\frac{T *_d \Lambda^j\varphi}{(1+x^2)^{k+p}}||_\infty d\mu(x),$$

so it is sufficient to estimate $||\frac{T *_d \Lambda^j\varphi}{(1+x^2)^{k+p}}||_\infty$ uniformly on T and φ. For this we use the fact that $T_y = (1+y^2)^k \sum_{j=1}^q \Lambda^{l_j} f_j d\mu_\alpha$ to obtain

$$\frac{|T *_d \Lambda^j\varphi(x)|}{(1+x^2)^{k+p}} \leq \frac{1}{(1+x^2)^{k+p}} \sum_{j=1}^q \int_{\mathbb{R}} |f_j(y)|.|\Lambda_y^{l_j}\left[(1+y^2)^k \tau_x \Lambda^j\varphi\right](y)|d\mu(y).$$

On the other hand $\Lambda_y^{l_j}\left[(1+y^2)^k \tau_x \Lambda^j\varphi\right](y)$ is a finite linear combination of terms of the form $y^{\sigma_1}(1+y^2)^r \tau_x(\Lambda^{h+j}\varphi^{\vee\sigma_2})(y)$,where σ_1, σ_2, r, h are integers such that $0 \leq \sigma_1, \sigma_2 \leq 1$, $0 \leq r \leq k$ and $0 \leq h \leq l_j$. If we denote

$$J := \frac{1}{(1+x^2)^{k+p}} \int_{\mathbb{R}} |f_j(y)|.|y^{\sigma_1}(1+y^2)^r \tau_x(\Lambda^{h+j}\varphi^{\vee\sigma_2})(y)|d\mu(y),$$

then

$$J \leq \int_{\mathbb{R}} \frac{|f_j(y)|}{(1+y^2)^{p-1}} ||\frac{(1+y^2)^{k+p}}{(1+x^2)^{k+p}} \tau_x(\Lambda^{h+j}\varphi^{\vee\sigma_2})(y)||_{\infty,y} d\mu(y).$$

By Lemma 2.1 there is a positive constant C such that

$$||\frac{(1+y^2)^{k+p}}{(1+x^2)^{k+p}} \tau_x(\Lambda^{h+j}\varphi^{\vee\sigma_2})(y)||_{\infty,y} \leq C||(1+y^2)^{k+p}\Lambda^{h+j}\varphi||_\infty,$$

thus

$$J \leq \text{ const.} \sup_{0\leq n\leq m+l_j} ||(1+x^2)^{k+p}\Lambda^n\varphi||_\infty.||f_j||_\infty.||\frac{1}{(1+x^2)^{p-1}}||_{L^1_\mu(\mathbb{R})},$$

and then

$$\begin{aligned} &||\frac{T *_d \Lambda^j\varphi}{(1+x^2)^{k+p}}||_\infty \\ \leq\ & \text{const.}||\frac{1}{(1+x^2)^{p-1}}||_{L^1_\mu(\mathbb{R})} \sum_{j=1}^q \sup_{0\leq n\leq m+l_j} ||(1+x^2)^{k+p}\Lambda^n\varphi||_\infty.||f_j||_\infty, \end{aligned}$$

this is uniformly bounded in T, φ when they stay respectively bounded in $\mathcal{S}'(\mathbb{R})$ and $\mathcal{S}(\mathbb{R})$. □

As a consequence of Theorem 3.3 and the density of $\mathcal{S}(\mathbb{R})$ in $\mathcal{O}'_C(\mathbb{R})$ and in $\mathcal{S}'(\mathbb{R})$ we have

Corollary 3.2.
i) $\mathcal{O}'_C(\mathbb{R}) *_d \mathcal{S}(\mathbb{R}) \subset \mathcal{S}(\mathbb{R})$,
ii) $\mathcal{O}'_C(\mathbb{R}) *_d \mathcal{O}'_C(\mathbb{R}) \subset \mathcal{O}'_C(\mathbb{R})$.
iii) The Dunkl-convolution product of a finite number of distributions of $\mathcal{S}'(\mathbb{R})$*, all of which but one at most are in* $\mathcal{O}'_C(\mathbb{R})$*, is associative and "commutative".*

As in the classical case, $\mathcal{O}'_C(\mathbb{R})$ and $\mathcal{O}_M(\mathbb{R})$ are related via $\mathcal{F}_d$ in the following result.

Theorem 3.4. *$\mathcal{F}_d$ is an isomorphism from $\mathcal{O}'_C(\mathbb{R})$ onto $\mathcal{O}_M(\mathbb{R})$, moreover*
i) $\mathcal{F}_d$ and $\mathcal{F}_d^{-1}$ are sequentially continuous.

ii) $\mathcal{F}_d(S *_d T) = \mathcal{F}_d(S).\mathcal{F}_d(T)$ *for all* S, T *in* $\mathcal{O}'_C(\mathbb{R}) \times \mathcal{S}'(\mathbb{R})$.
iii) $\mathcal{F}_d(f.T) = \mathcal{F}_d(f) *_d \mathcal{F}_d(T)$ *for all* f, T *in* $\mathcal{O}_M(\mathbb{R}) \times \mathcal{S}'(\mathbb{R})$.

Proof. Let $T \in \mathcal{O}'_C(\mathbb{R})$. For any positive integer p, we have to fined a polynomial P and a bounded function g such that $\Lambda^p\mathcal{F}_d(T) = Pg$, this to prove that $\mathcal{F}_d(T) \in \mathcal{O}_M(\mathbb{R})$. $\Lambda^p\mathcal{F}_d(T) = (i)^p\mathcal{F}_d(x^pT)$, and $x^pT \in \mathcal{B}'_d(\mathbb{R})$ so, by Theorem 3.1, $x^pT = \sum_{j=1}^q \Lambda^{l_j} f_j d\mu$ where $f_1, \ldots, f_q$ can be taken in L^1_μ. Thus

$$\Lambda^p\mathcal{F}_d(T) = (i)^p \sum_{j=1}^{q} \mathcal{F}_d(\Lambda^{l_j} f_j) = \sum_{j=1}^{q} (i)^{p+l_j} x^{l_j} \mathcal{F}_d(f_j).$$

If $k = \max_{1\le j\le q} l_j$, then

$$\Lambda^p\mathcal{F}_d(T) = (1+x^2)^k \sum_{j=1}^{q} (i)^{p+l_j} \frac{x^{l_j}}{(1+x^2)^k} \mathcal{F}_d(f_j),$$

and $\sum_{j=1}^{q} (i)^{p+l_j} \frac{x^{l_j}}{(1+x^2)^k} \mathcal{F}_d(f_j)$ is a bounded function since $f_1, \ldots, f_q$ are in L^1_μ.

Let $f \in \mathcal{O}_M(\mathbb{R})$ and consider the distribution $\mathcal{F}_d(f)$. We have to show that for any positive integer k, $(1+x^2)^k\mathcal{F}_d(f) \in \mathcal{B}'_d(\mathbb{R})$. Since $(1+x^2)^k\mathcal{F}_d(f) = \mathcal{F}_d((1-\Lambda^2)^k f)$ and $f \in \mathcal{O}_M(\mathbb{R})$, then there are $(k+1)$ polynomials $P_0, \ldots, P_k$ and $(k+1)$ functions $g_0, \ldots, g_k$ in L^1_μ, such that $(1-\Lambda^2)^k f = \sum_{j=0}^k P_j.g_j$, this implies that

$$(1+x^2)^k\mathcal{F}_d(f) = \sum_{j=0}^{k} P_j(i\Lambda)\mathcal{F}_d(g_j),$$

to conclude we note that $\mathcal{F}_d(g_j)$ are bounded because $g_j \in L^1_\mu$.
For the rest of the proof it is sufficient to show i), because the formulas in ii) and iii) are true for f, S, T in $\mathcal{S}(\mathbb{R})$ which is dense respectively in $\mathcal{O}'_C(\mathbb{R})$, $\mathcal{S}'(\mathbb{R})$ and $\mathcal{O}_M(\mathbb{R})$. So, let $(f_n)_n$ be a sequence in $\mathcal{O}_M(\mathbb{R})$ converging to 0. To prove that $(\mathcal{F}_d(f_n))_n$ tends to 0 in $\mathcal{O}'_C(\mathbb{R})$, we must show that for all $k \in \mathbb{N}$ and for all $\varphi \in \mathcal{D}(\mathbb{R})$ staying bounded in $\mathcal{D}_{L^1_\mu(\mathbb{R})}(\mathbb{R})$, the sequence $\langle (1+x^2)^k\mathcal{F}_d(f_n), \varphi\rangle$ tends to 0 uniformly on φ. We have

$$\langle (1+x^2)^k\mathcal{F}_d(f_n), \varphi\rangle = \langle \mathcal{F}_d((1-\Lambda^2)^k f_n), \varphi\rangle = \sum_{j=0}^{k} a_j \langle \Lambda^{2j} f_n, \overline{\mathcal{F}}_d(\varphi)\rangle,$$

here the a_j's appears in the development of $(1-\Lambda^2)^k$. Now by the definition of the convergence in $\mathcal{O}_M(\mathbb{R})$, see Proposition 3.1, there is a positive integer m_k such that $\|\frac{\Lambda^{2j} f_n}{(1+x^2)^{m_k}}\|_\infty$ tends to 0 for $0 \le j \le k$. We choose $p \in \mathbb{N}$ such that $1/(1+x^2)^p$ belongs to $L^1_\mu(\mathbb{R})$, then

$$|\langle (1+x^2)^k\mathcal{F}_d(f_n), \varphi\rangle| \le \sum_{j=0}^{k} |a_j|.|\langle \frac{\Lambda^{2j} f_n}{(1+x^2)^{m_k+p}}, (1+x^2)^{m_k+p}\overline{\mathcal{F}}_d(\varphi)\rangle|,$$

and finally

$$|\langle (1+x^2)^k\mathcal{F}_d(f_n), \varphi\rangle| \le \|\frac{1}{(1+x^2)^p}\|_{L^1_\mu} \sum_{j=0}^{k} |a_j|.\|\frac{\Lambda^{2j} f_n}{(1+x^2)^{m_k}}\|_\infty.\|(1-\Lambda^2)^{m_k+p}\varphi\|_{L^1_\mu}.$$

Conversely, let $(T_n)_n$ in $\mathcal{O}'_C(\mathbb{R})$ converging to 0. We have to show that for all $k \in \mathbb{N}$ and all f staying bounded in $\mathcal{S}(\mathbb{R})$, the quantity $||f.\Lambda^k \mathcal{F}_d(T_n)||_\infty$ tends to 0 uniformly on those f. Let $g = \mathcal{F}_d^{-1} f$ then g stays bounded in $\mathcal{S}(\mathbb{R})$ as f, and we have

$$||f.\Lambda^k \mathcal{F}_d(T_n)||_\infty = ||f.\mathcal{F}_d((ix)^k T_n)||_\infty = ||\mathcal{F}_d\big[(ix)^k T_n *_d g\big]||_\infty \leq ||x^k T_n *_d g||_{L^1_\mu}.$$

In the other hand, if $p \in \mathbb{N}$ is chosen such that $D = ||1/(1+y^2)^p||_{L^1_\mu}$ is finite, then

$$||x^k T_n *_d g||_{L^1_\mu} \leq D.||(1+y^2)^p \langle x^k T_n, \tau_y g\rangle||_\infty \leq D.||\langle (1+x^2)^{2p} x^k T_n, \frac{(1+y^2)^p}{(1+x^2)^{2p}} \tau_y g\rangle||_\infty.$$

Now $(1+x^2)^{2p} x^k T_n$ tends to 0 in $\mathcal{B}'_d(\mathbb{R})$, so it is sufficient to show that the family

$$\{\frac{(1+y^2)^p}{(1+x^2)^{2p}} \tau_y g;\ y \in \mathbb{R}\},$$

stays bounded in $\mathcal{D}_{L^1_\mu(\mathbb{R})}(\mathbb{R})$ when f stays bounded in $\mathcal{S}(\mathbb{R})$. For this we just apply Remark 2.1 and Lemma 2.1 with the above methods. □

4. Hypoelliptic Dunkl-convolution equations. In this section we are going to study Hypoelliptic Dunkl-convolution equations. To expose the problem we need the space $\mathcal{E}(\mathcal{S}'(\mathbb{R}))$ of functions $f \in \mathcal{C}^\infty(\mathbb{R}) \cap \mathcal{S}'(\mathbb{R})$ such that, for all $S \in \mathcal{O}'_C(\mathbb{R})$, $S * f \in \mathcal{C}^\infty(\mathbb{R})$ and the mapping $S \longmapsto S * f \in \mathcal{C}^\infty(\mathbb{R})$ from $\mathcal{O}'_C(\mathbb{R})$ to $\mathcal{C}^\infty(\mathbb{R})$ is continuous. In [8], $\mathcal{E}(\mathcal{S}'(\mathbb{R}))$ is identified to $\mathcal{O}_C(\mathbb{R})$ the space of very slowly increasing $\mathcal{C}^\infty$-functions, which is the dual of $\mathcal{O}'_C(\mathbb{R})$. In other words $f \in \mathcal{E}(\mathcal{S}'(\mathbb{R}))$ if and only if there is a constant ρ such that for all $n \in \mathbb{N}$, $f^{(n)} = O(|x|^\rho)$, as $|x| \to \infty$. By Proposition 2.1, this is equivalent to $\Lambda^n f = O(|x|^\rho)$, as $|x| \to \infty$. Moreover we have

Proposition 4.1.
i) For all $f \in \mathcal{E}(\mathcal{S}'(\mathbb{R}))$ and all $a \in \mathbb{R}$, $\tau_a f$ stays in $\mathcal{E}(\mathcal{S}'(\mathbb{R}))$.
ii) $f \in \mathcal{E}(\mathcal{S}'(\mathbb{R}))$ if and only if, for all $S \in \mathcal{O}'_C(\mathbb{R})$, $S *_d f \in \mathcal{C}^\infty(\mathbb{R})$, and the mapping $S \longmapsto S *_d f$ from $\mathcal{O}'_C(\mathbb{R})$ to $\mathcal{C}^\infty(\mathbb{R})$ is continuous.
iii) For all $f \in \mathcal{E}(\mathcal{S}'(\mathbb{R}))$ and $S \in \mathcal{O}'_C(\mathbb{R})$, $S *_d f$ belongs to $\mathcal{E}(\mathcal{S}'(\mathbb{R}))$.

Proof. To prove this result, we note that
i) is immediate from Lemma 2.1 and Proposition 2.1.
ii) is a consequence of i) and the fact that $S *_d f(x) = \langle S, \tau_x f\rangle = S * \tau_x f(0)$.
iii) Let $f \in \mathcal{E}(\mathcal{S}'(\mathbb{R}))$ and $S \in \mathcal{O}'_C(\mathbb{R})$, then $S *_d f \in \mathcal{C}^\infty(\mathbb{R}) \cap \mathcal{S}'(\mathbb{R})$. Moreover if $T \in \mathcal{O}'_C(\mathbb{R})$ then, by Corollary 3.2, $T *_d (S *_d f) = (T *_d S) *_d f$ and $(T *_d S) \in \mathcal{O}'_C(\mathbb{R})$, so $T \longmapsto T *_d (S *_d f)$ is continuous and then $S *_d f \in \mathcal{E}(\mathcal{S}'(\mathbb{R}))$. □

We consider now the Dunkl-convolution equation

$$S *_d U = F, \tag{4.1}$$

where $S \in \mathcal{O}'_C(\mathbb{R})$ and $U, F \in \mathcal{S}'(\mathbb{R})$. This equation is well defined for fixed S, F and we say

Definition 4.1. *The Dunkl-convolution equation 4.1 is said to be hypoelliptic in $\mathcal{S}'(\mathbb{R})$, if all solutions $U \in \mathcal{S}'(\mathbb{R})$ are in $\mathcal{E}_d(\mathcal{S}'(\mathbb{R}))$, when F is in $\mathcal{E}_d(\mathcal{S}'(\mathbb{R}))$. In this case, we say also that S is hypoelliptic.*

The mean result here is to characterize hypoelliptic distributions $S \in \mathcal{O}'_C(\mathbb{R})$ as was made by Zieleźny in [20] in the case of the classical convolution on $\mathbb{R}^d$. For this we need the following lemma

LEMMA 4.1. *Let* $(\xi_n)_{n\geq 1}$ *be a real sequence such that*

$$|\xi_{n+1}| > 2|\xi_n| > 2^{n+1},$$

and $(a_n)_{n\geq 1}$ *a complex sequence satisfying*

$$a_n = O(|\xi_n|^\rho),$$

for some real ρ. *The series* $\sum_{n\geq 1} a_n \delta_{\xi_n}$ *converges in* $\mathcal{S}'(\mathbb{R})$ *and can be written as* $\mathcal{F}_d(T)$. *Then* T *belongs to* $\mathcal{E}_d(\mathcal{S}'(\mathbb{R}))$ *if and only if*

$$a_n = o(|\xi_n|^k),$$

for every positive integer k.

Proof. The series $\sum_{n\geq 1} a_n \delta_{\xi_n}$ converges trivially in $\mathcal{S}'(\mathbb{R})$ and we have

$$T_x = \sum_{n\geq 1} a_n E(x, i\xi_n).$$

Suppose that $a_n = o(|\xi_n|^k)$, then $\sum_{n\geq 1} a_n E(x, i\xi_n)$ converges uniformly on $\mathbb{R}$ together with all its term-by-term derivatives, so T is a $\mathcal{C}^\infty$ function. Moreover, $T \in \mathcal{B}(\mathbb{R})$ because $|E(x, i\xi_n)| \leq 1$ and $a_n = o(|\xi_n|^k)$, consequently $T \in \mathcal{E}_d(\mathcal{S}'(\mathbb{R}))$.
Conversely, if T belongs to $\mathcal{E}_d(\mathcal{S}'(\mathbb{R}))$ then $\Lambda^p(T).\varphi \in \mathcal{S}(\mathbb{R})$ for all $\varphi \in \mathcal{S}(\mathbb{R})$ and $p \in \mathbb{N}$ and so $\mathcal{F}_d\big[\Lambda^p(T).\varphi\big] \in \mathcal{S}(\mathbb{R})$, which implies that

$$\big[x^p \mathcal{F}_d(T) *_d \mathcal{F}_d(\varphi)\big](t) \longrightarrow 0 \quad \text{as} \quad |t| \longrightarrow +\infty.$$

This means that

$$\sum_{n\geq 1} a_n \xi_n^p \tau_t \big[\mathcal{F}_d(\varphi)\big](\xi_n) \longrightarrow 0 \quad \text{as} \quad |t| \longrightarrow +\infty. \tag{4.2}$$

Now suppose that for all positive integer k, the condition $a_n = o(|\xi_n|^k)$ is not satisfied, then one can found $q \in \mathbb{N}$, $r > 0$ and a subsequence of $(a_n)_n$ (which can be taken the whole $(a_n)_n$ with a few modification), such that $|\xi_n^q a_n| \geq r$ for all n.
Let φ such that $\mathcal{F}_d(\varphi)$ is a positive even function in $\mathcal{D}(\mathbb{R})$, with support in $[-1, 1]$ and $\mathcal{F}_d(\varphi) \geq 1$ on $[-1/2, 1/2]$. If we note that $||\xi_n| - |\xi_m|| > 1$ for $m \neq n$, then by Lemma 2.2 there is a constant $C > 0$ such that,

$$\tau_{\xi_m}(\varphi)(\xi_m) \geq C/|\xi_m|^{2\alpha+1} \quad \text{and} \quad \tau_{\xi_m}(\varphi)(\xi_n) = 0 \quad \text{for } n \neq m.$$

And for $t = \xi_m$ and $p \geq q + (2\alpha + 1)$ we have

$$|\sum_{n\geq 1} a_n \xi_n^p \tau_{\xi_m} \big[\mathcal{F}_d(\varphi)\big](\xi_n)| = |a_m|.|\xi_m^p|.|\tau_{\xi_m}(\varphi)(\xi_m)| \geq rC,$$

which cannot tend to 0 as m tends to $+\infty$ and then contradicts (4.2). □

THEOREM 4.1. *A distribution* $S \in \mathcal{O}'_C(\mathbb{R})$ *is hypoelliptic if and only if there are constants* ρ *and* A *such that* $\mathcal{F}_d(S)$ *satisfies*

$$|\mathcal{F}_d(S)(\xi)| \geq |\xi|^\rho \quad \textit{for} \quad \xi \in \mathbb{R},\ |\xi| \geq A. \tag{4.3}$$

Proof. If condition (4.3) of the theorem is not satisfied then there is a sequence $(\xi_n)_{n\geq 1}$, as in Lemma 4.1 such that

$$|\mathcal{F}_d(S)(\xi_n)| \leq |\xi_n|^{-(n+1)}, \qquad \text{for} \qquad n \geq 1. \tag{4.4}$$

By Lemma 4.1 the series $\sum_{n\geq 1} \delta_{\xi_n}$ converges in $\mathcal{S}'(\mathbb{R})$ to a distribution denoted by $\mathcal{F}_d(U)$ such that $U \in \mathcal{S}'(\mathbb{R})$ and is not in $\mathcal{E}_d(\mathcal{S}'(\mathbb{R}))$. Now

$$\mathcal{F}_d(S *_d U) = \mathcal{F}_d(S).\mathcal{F}_d(U) = \sum_{n\geq 1} \mathcal{F}_d(S)(\xi_n)\delta_{\xi_n},$$

so, by inequality (4.4) and Lemma 4.1, $S *_d U$ belongs to $\mathcal{E}_d(\mathcal{S}'(\mathbb{R}))$ and contradicts the hypoellipticity of S.
For the converse we need the following construction. Let A be the constant in the hypothesis and take $\psi \in \mathcal{D}(\mathbb{R})$ such that, $\psi \equiv 1$ on $[-A, A]$ and $\text{supp}(\psi) \subset [-A-1, A+1]$. Define the distribution H by

$$\mathcal{F}_d(H)(\xi) = \begin{cases} 0 & \text{for} \quad |\xi| \leq A, \\ \dfrac{1-\psi(\xi)}{\mathcal{F}_d(S)(\xi)} & \text{for} \quad |\xi| > A, \end{cases}$$

the facts that $\mathcal{F}_d(S) \in \mathcal{O}_M(\mathbb{R})$ and verifies the condition (4.3) imply trivially that $\mathcal{F}_d(H) \in \mathcal{O}_M(\mathbb{R})$ and then H is in $\mathcal{O}'_C(\mathbb{R})$, moreover

$$\mathcal{F}_d(S *_d H)(\xi) = \mathcal{F}_d(S)(\xi)\mathcal{F}_d(H)(\xi) = 1 - \psi(\xi).$$

The inverse Dunkl transform gives

$$S *_d H = \delta_0 - \mathcal{F}_d^{-1}(\psi),$$

so if $U \in \mathcal{S}'(\mathbb{R})$ is a solution of the equation

$$S *_d U = F,$$

where $F \in \mathcal{E}_d(\mathcal{S}'(\mathbb{R}))$, then

$$U = U *_d \delta_0 = U *_d (S *_d H) + U *_d \mathcal{F}_d^{-1}(\psi) = F *_d H + U *_d \mathcal{F}_d^{-1}(\psi).$$

To conclude we use Proposition 4.1 to say that $F *_d H$ is in $\mathcal{E}(\mathcal{S}'(\mathbb{R}))$. On the other hand $U *_d \mathcal{F}_d(\psi)$ can be written as $\mathcal{F}_d(\mathcal{F}_d(U).\psi)$ i.e the Fourier-Dunkl transform of a distribution of compact support, so by Paley-Weiner theorem (see [17]), $U *_d \mathcal{F}_d(\psi)$ is an entire function slowly increasing of exponential type and then belongs to $\mathcal{E}(\mathcal{S}'(\mathbb{R}))$. Thus $U \in \mathcal{E}(\mathcal{S}'(\mathbb{R}))$ and S is hypoelliptic. □

REFERENCES

[1] M. Belhadji and J. J. Betancor, *Hypoellipticity of Hankel convolution equations on distributions*, Arch. Math., 79 (2002), pp. 188–196.

[2] J. J. Betancor, *Distributional Dunkl Transform and Dunkl Convolution Operators*, Bollettino U.M.I, 8 (2006), pp. 221–245.

[3] J. J. Betancor, J. D. Betancor and J. M. R. Méndez, *Hypoelliptic Jacobi convolution operators on Schwartz distributions*, Positivity, 8 (2004), pp. 407–422.

[4] J. J. Betancor and I. Marrero, *Structure and convergence in certain spaces of distributions and the generalized Hankel convolution*, Math. Japonica, 38 (1993), pp. 1141–1155.

[5] C. F. Dunkl, *Differential-difference operators associated to reflection groups*, Trans. Amer. Math. Soc., 311 (1989), pp. 167–183.
[6] C. F. Dunkl, *Integral kernels with reflection group invariance*, Canad. J. Math., 43 (1991), pp. 1213–1227.
[7] L. Ehrenpreis, *Solutions of some problems of division IV. Invertible and elliptic operators*, Amer. J. Math., 82 (1960), pp. 522–588.
[8] A. Grothendieck, *Produits tensoriels topologiques et espaces nucléaires*, Mem. Amer. Math. Soc., 16 (1955).
[9] L. Hörmander, *Hypoelliptic convolution equations*, Math. Scand., 9 (1961), pp. 178–181.
[10] M. F. E. de Jeu, *The Dunkl transform*, Invent. Math., 113 (1993), pp. 147–162.
[11] M. Mili and K. Trimèche, *Hypoelliptic Jacobi-Dunkl Convolution of Distributions* , Mediterr. J. math., 4 (2007), pp. 263–276.
[12] M. A. Mourou, *Taylor series associated with a differential-difference operator on the real line*, Journal of Computational and Applied Mathematics, 153 (2003), pp. 343–354.
[13] M. Rösler, *Bessel-type signed hypergroups on $\mathbb{R}$ in: H. Heyer, A. Mukherjea (eds), Probability measures on groups and related structures XI*, Proc Oberwolfach 1994, World Scintific, Singapore (1995), pp. 292–304.
[14] M. Rösler, *Positivity of Dunkl's intertwining operator*, Duke Math. J., 98 (1999), pp. 445–463.
[15] M. Rösler, *A positive radial product formula for the Dunkl kernel*, Trans. Amer. Math. Soc., 355 (2003), pp. 2413–2438.
[16] L. Schwartz, *Théorie des distributions*, Hermann, Paris (1998).
[17] K. Trimèche, *Paley-Wiener theorems for the Dunkl transform and Dunkl translation operators*, Integral Transforms and Special Functions, 13 (2002), pp. 17–38.
[18] K. Trimèche, *Hypoelliptic Dunkl convolution equations in the space of distributions on $\mathbb{R}^d$*, J. Fourier Anal. Appl., 12 (2006), pp. 517–542.
[19] K. Trimèche, *Hypoelliptic distributions on Chebli-Trimèche hypergroups*, Glob. J. Pure Appl. Math., 1 (2005), pp. 251–271.
[20] Z. Zieleźny, *Hypoelliptic and entire elliptic convolution equations in subspaces of the space of distributions (I)*, Studia Mathematica, 28 (1967), pp. 317–332.