

DEFORMATION OF SURFACES INDUCED BY MOTIONS OF CURVES IN HIGHER-DIMENSIONAL SIMILARITY GEOMETRIES*

CHANGZHENG QU[†] AND YANYAN LI[‡]

Abstract. In this paper, we study deformation of surfaces induced by adding one and two extra space variables to the motions of space curves in higher-dimensional similarity geometries. It is shown that the 2+1- and 3+1-dimensional nonlinear evolution equations including the 2+1-dimensional mKdV equation and a generalization to the mKdV-Burgers system arise from such motions.

Key words. Similarity geometry, invariant geometric flow, deformation of surface, integrable system.

AMS subject classifications. 37K25, 53C44, 53A55

1. Introduction. Considerable effort has been made by a number of authors to study motions of curves and surfaces in certain geometries, and many interesting results have been obtained. It has been recognized that motions of curves and surfaces in certain geometries are closely related to nonlinear evolution equations. For examples, Mullins's nonlinear diffusion model of groove development [1] describes the curve shortening problem [2]. Hasimoto transformation [3] sets up a one-to-one correspondence between the integrable Schrödinger equation and the binormal motion of a space curve driven by its curvature and torsion. Later, Lamb [4] utilized the Hasimoto transformation to show that certain types of space curve flows can be mapped to integrable equations such as sine-Gordon equation, Schrödinger equation and Hirota equation etc. Langer and Perline [5, 6] further proved that the dynamics of non-stretching vortex filament in \mathbb{R}^3 gives to the NLS hierarchy. To interpret the dynamics of a nonlinear string of fixed length in \mathbb{R}^3 , Lakshmanan [7] considered the motion of an arbitrary rigid body along it, deriving the AKNS spectral problem without spectral parameter. Doliwa and Santini [8] found that the NLS hierarchy and complex mKdV equation arise from motions on $S^3(R)$ where the radius R plays the role of the spectral parameter. Goldstein and Petrich [9] related the mKdV equation and its hierarchies to motions of non-stretching closed curves on the plane in \mathbb{R}^2 . Nakayama, Segur and Wadati [10] obtained the sine-Gordon equation by considering a nonlocal motion of curves in \mathbb{R}^2 . Nakayama [11] also showed that the defocusing nonlinear Schrödinger equation, the Regge-Lund equation, a coupled system of KdV equations and their hyperbolic type arise from motions of curves in hyperboloids in the Minkowski space. In [12], he realized the full AKNS scheme in a hyperboloid in \mathbb{M}^4 . An extended Harry-Dym equation and sine-Gordon equation from binormal motions of curves with constant curvature or torsion were derived by Schief and Rogers [13]. Recently, motions of curves in Klein geometry were investigated systematically by Chou and Qu et al. [14-20], they showed that a couple of 1+1-dimensional integrable equations arise from the curve motions. Motions of curves in Riemannian geometry were studied by Beffa, Sanders and Wang [21]. The relationship between invariant curve flows in conformal, Poisson, Lagrange-Finsler geometries and homoge-

*Received October 2, 2007; accepted for publication April 9, 2008.

[†]Center for Nonlinear Studies, Northwest University, Xi'an, 710069, P. R. China (czqu@nwu.edu.cn).

[‡]Department of Mathematics, Northwest University, Xi'an, 710069, P. R. China.

neous space and Hamiltonian structure was investigated in [21-29] by Meffa and Anco et al. The flows of curves on adjoint orbits were studied by Terng and Thorbergsson [30], which provided a systematic study of the construction of curve flows on adjoint orbits from solutions of AKNS-type soliton equations.

As for the interaction between differential geometry of surfaces and nonlinear differential equations has been studied since the 19th century. In particular, the relationship between deformations of surfaces and integrable systems in 2+1 dimensions were studied by several authors [29-44]. In [35-42], Myrzakulov et al extended the theory of moving space curve formalism in 1+1 dimensions to 2+1 dimensions by adding an extra space variable to the motion of curves in \mathbb{R}^3 . They showed that the 2+1-dimensional Ishimori equation, Myrzakulov I equation, Myrzakulov III equation, 2+1-dimensional isotropic Heisenberg ferromagnet model and the 2+1-dimensional Schrödinger equation arise from such motions. In a similar manner, the 2+1-dimensional integrable shallow water wave equation was obtained by Qu and Zhang [43] by considering the motion of space curves in centro-affine geometry.

In this paper, we will consider motion of space curves in n -dimensional similarity geometries by the approach in [35-43], which induces deformation of surfaces in similarity geometries. The paper is organized as follows: In Subsections 2.1 and 2.2 of Section 2, we investigate respectively motion of space curves in four-dimensional similarity geometry \mathbb{P}^4 by adding one and two extra space variables. Motion of space curves in n -dimensional geometry \mathbb{P}^n ($n > 4$) by adding one and two extra space variables are discussed respectively in Subsections 3.1 and 3.2. Section 4 contains a concluding remarks on this work.

2. Motion of space curves in \mathbb{P}^4 .

2.1. Motion of curves in \mathbb{P}^4 by adding an extra space variable y . Motion of curves in similarity geometry and related topics have been studied by Sapiro, Tannenbaum, Chou and Qu et al in [14,15,45,46]. In this section, we consider motion of space curves in \mathbb{P}^4 by adding an extra space variable y .

We denote by $\kappa_1, \kappa_2, \kappa_3, s$ and $\tau_i, i = 1, 2, 3, 4$ the curvatures, arc-length and unite frame vectors of curves in Euclidean space \mathbb{R}^4 .

In the case of space curves in \mathbb{P}^4 , the Serret-Frenet formula reads as

$$\begin{pmatrix} \mathbf{t}_1 \\ \mathbf{t}_2 \\ \mathbf{t}_3 \\ \mathbf{t}_4 \end{pmatrix}_\theta = \begin{pmatrix} -\alpha_1 & 1 & 0 & 0 \\ -1 & -\alpha_1 & \alpha_2 & 0 \\ 0 & -\alpha_2 & -\alpha_1 & \alpha_3 \\ 0 & 0 & -\alpha_3 & -\alpha_1 \end{pmatrix} \begin{pmatrix} \mathbf{t}_1 \\ \mathbf{t}_2 \\ \mathbf{t}_3 \\ \mathbf{t}_4 \end{pmatrix}, \quad (1)$$

where $\alpha_1, \alpha_2, \alpha_3, \theta$ and $\mathbf{t}_i, i = 1, 2, 3, 4$, are respectively the curvatures, arc-length and frame vectors in \mathbb{P}^4 , they are related to the Euclidean's by

$$\alpha_1 = \kappa_{1,s}/\kappa_1^2, \quad \alpha_2 = \kappa_2/\kappa_1, \quad \alpha_3 = \kappa_3/\kappa_1, \quad d\theta = \kappa_1 ds, \quad \mathbf{t}_i = (1/\kappa_1)\tau_i, \quad i = 1, 2, 3, 4.$$

It is readily to show that they are invariant with respect to the similarity transformation. The invariant geometric motion in \mathbb{P}^4 is governed by

$$\gamma_t = A_1 \mathbf{t}_1 + A_2 \mathbf{t}_2 + A_3 \mathbf{t}_3 + A_4 \mathbf{t}_4, \quad (2)$$

where $A_i, i = 1, 2, 3, 4$, denote velocities along the frame vectors, which depend on curvatures $\alpha_1, \alpha_2, \alpha_3$.

It follows from (1) and (2) that the time evolution of the frame vectors in \mathbb{P}^4 is given by

$$\begin{pmatrix} \mathbf{t}_1 \\ \mathbf{t}_2 \\ \mathbf{t}_3 \\ \mathbf{t}_4 \end{pmatrix}_t = \begin{pmatrix} E & F & G & H \\ -F & E & F_1 & G_1 \\ -G & -F_1 & E & H_1 \\ -H & -G_1 & -H_1 & E \end{pmatrix} \begin{pmatrix} \mathbf{t}_1 \\ \mathbf{t}_2 \\ \mathbf{t}_3 \\ \mathbf{t}_4 \end{pmatrix}, \tag{3}$$

where $E, F, G, H, F_1, G_1, H_1$ depend on the curvatures $\alpha_1, \alpha_2, \alpha_3$ too.

The compatibility condition of equations (1) and (3), i.e.

$$\begin{pmatrix} \mathbf{t}_1 \\ \mathbf{t}_2 \\ \mathbf{t}_3 \\ \mathbf{t}_4 \end{pmatrix}_{t\theta} = \begin{pmatrix} \mathbf{t}_1 \\ \mathbf{t}_2 \\ \mathbf{t}_3 \\ \mathbf{t}_4 \end{pmatrix}_{\theta t}$$

gives rise to the following system

$$\begin{aligned} \alpha_{1,t} &= -E_\theta, \\ \alpha_{2,t} &= G + F_{1,\theta} - \alpha_3 G_1, \\ \alpha_{3,t} &= H_{1,\theta} + \alpha_2 G_1, \\ F_\theta &= \alpha_2 G, \\ F_1 &= \alpha_2 F + G_\theta - \alpha_3 H, \\ G_1 &= \alpha_3 G + H_\theta, \\ H &= -\alpha_3 F_1 - G_{1,\theta} + \alpha_2 H_1. \end{aligned} \tag{4}$$

Let's extend the theory of moving space curve formalism in 1+1 dimension to the case in 2+1 dimension by endowing an extra space variable y to the motion of curves in \mathbb{P}^n . Let's assume that the y -evolution of the frame vectors is determined by

$$\begin{pmatrix} \mathbf{t}_1 \\ \mathbf{t}_2 \\ \mathbf{t}_3 \\ \mathbf{t}_4 \end{pmatrix}_y = \begin{pmatrix} e & f & g & h \\ -f & e & f_1 & g_1 \\ -g & -f_1 & e & h_1 \\ -h & -g_1 & -h_1 & e \end{pmatrix} \begin{pmatrix} \mathbf{t}_1 \\ \mathbf{t}_2 \\ \mathbf{t}_3 \\ \mathbf{t}_4 \end{pmatrix}, \tag{5}$$

where $e, f, g, h, f_1, g_1, h_1$ are functions of curvatures α_1, α_2 and α_3 , to be determined.

The compatibility condition between equations (1) and (5) yields

$$\begin{aligned} \alpha_{1,y} &= -e_\theta, \\ \alpha_{2,y} &= g + f_{1,\theta} - \alpha_3 g_1, \\ \alpha_{3,y} &= h_{1,\theta} + \alpha_2 g_1, \\ f_\theta &= \alpha_2 g, \\ f_1 &= \alpha_2 f + g_\theta - \alpha_3 h, \\ g_1 &= \alpha_3 g + h_\theta, \\ h &= \alpha_2 h_1 - \alpha_3 f_1 - g_{1,\theta}. \end{aligned} \tag{6}$$

Similarly, the compatibility condition

$$\begin{pmatrix} \mathbf{t}_1 \\ \mathbf{t}_2 \\ \mathbf{t}_3 \\ \mathbf{t}_4 \end{pmatrix}_{ty} = \begin{pmatrix} \mathbf{t}_1 \\ \mathbf{t}_2 \\ \mathbf{t}_3 \\ \mathbf{t}_4 \end{pmatrix}_{yt}$$

leads to the following equations

$$\begin{aligned}
e_t &= E_y, \\
f_t &= F_y + gF_1 + hG_1 - f_1G - g_1H, \\
g_t &= G_y + f_1F - fF_1 + hH_1 - h_1H, \\
h_t &= H_y + g_1F + h_1G - fG_1 - gH_1, \\
f_{1,t} &= F_{1,y} + fG + g_1H_1 - gF - h_1G_1, \\
g_{1,t} &= G_{1,y} + fH + h_1F_1 - f_1H_1 - hF, \\
h_{1,t} &= H_{1,y} + gH + f_1G_1 - hG - g_1F_1.
\end{aligned} \tag{7}$$

From (4) and (6), one obtains

$$\begin{aligned}
F &= \partial_\theta^{-1}(\alpha_2 G), \\
F_1 &= \alpha_2 \partial_\theta^{-1}(\alpha_2 G) + G_\theta - \alpha_3 H, \\
G_1 &= \alpha_3 G + H_\theta, \\
H_1 &= \alpha_2^{-1}[H_{\theta\theta} + (1 - \alpha_3^2)H + 2\alpha_3 G_\theta + \alpha_{3,\theta}G + \alpha_2 \alpha_3 \partial_\theta^{-1}(\alpha_2 G)], \\
f &= \partial_\theta^{-1}(\alpha_2 g), \\
f_1 &= \alpha_2 \partial_\theta^{-1}(\alpha_2 g) + g_\theta - \alpha_3 h, \\
g_1 &= \alpha_3 g + h_\theta, \\
h_1 &= \alpha_2^{-1}[h_{\theta\theta} + (1 - \alpha_3^2)h + 2\alpha_3 g_\theta + \alpha_{3,\theta}g + \alpha_2 \alpha_3 \partial_\theta^{-1}(\alpha_2 g)],
\end{aligned} \tag{8}$$

and that the curvatures fulfill the following equations

$$\begin{aligned}
\alpha_{1,t} &= -E_\theta, \\
\alpha_{2,t} &= (\partial_\theta^2 + \alpha_2^2 - \alpha_3^2 + 1 + \alpha_{2,\theta} \partial_\theta^{-1} \alpha_2)G - 2\alpha_3 H_\theta - \alpha_{3,\theta} H, \\
\alpha_{3,t} &= 2\alpha_2 \alpha_3 G + \alpha_{3,\theta} \partial_\theta^{-1}(\alpha_2 G) + \alpha_2 H_\theta \\
&\quad + [\alpha_2^{-1}(H_{\theta\theta} - \alpha_3^2 H + H + 2\alpha_3 G_\theta + \alpha_{3,\theta} G)]_\theta, \\
\alpha_{1,y} &= -e_\theta, \\
\alpha_{2,y} &= (\partial_\theta^2 + \alpha_2^2 - \alpha_3^2 + 1 + \alpha_{2,\theta} \partial_\theta^{-1} \alpha_2)g - 2\alpha_3 h_\theta - \alpha_{3,\theta} h, \\
\alpha_{3,y} &= 2\alpha_2 \alpha_3 g + \alpha_{3,\theta} \partial_\theta^{-1}(\alpha_2 g) + \alpha_2 h_\theta + [\alpha_2^{-1}(h_{\theta\theta} - \alpha_3^2 h + h + 2\alpha_3 g_\theta + \alpha_{3,\theta} g)]_\theta.
\end{aligned} \tag{9}$$

Substituting equations (8) into (7), we obtain

$$\begin{aligned}
g_t &= G_y + g_\theta \partial_\theta^{-1}(\alpha_2 G) - G_\theta \partial_\theta^{-1}(\alpha_2 g) + \alpha_2^{-1} h(H_{\theta\theta} + 2\alpha_3 G_\theta + \alpha_{3,\theta} G) \\
&\quad - \alpha_2^{-1} H(h_{\theta\theta} + 2\alpha_3 g_\theta + \alpha_{3,\theta} g), \\
h_t &= H_y + h_\theta \partial_\theta^{-1}(\alpha_2 G) - H_\theta \partial_\theta^{-1}(\alpha_2 g) + \alpha_2^{-1} G[h_{\theta\theta} + (1 - \alpha_3^2)h + 2\alpha_3 g_\theta] \\
&\quad - \alpha_2^{-1} g[H_{\theta\theta} + (1 - \alpha_3^2)H + 2\alpha_3 G_\theta].
\end{aligned} \tag{10}$$

Setting

$$G = -\alpha_{2,y},$$

and assuming that H satisfies $2\alpha_3 H_\theta - \alpha_3^2 \alpha_{2,y} + \alpha_{3,\theta} H = 0$, i.e.

$$H = (\alpha_3 \partial_\theta + \partial_\theta \alpha_3)^{-1} \alpha_3^2 \alpha_{2,y},$$

one deduces from (9) that

$$\begin{aligned} \alpha_{2,t} + \alpha_{2,\theta\theta y} + \alpha_{2,\theta}\partial_\theta^{-1}(\alpha_2\alpha_{2,y}) + \alpha_2^2\alpha_{2,y} + \alpha_{2,y} &= 0, \\ \alpha_{3,t} &= -2\alpha_2\alpha_3\alpha_{2,y} - \alpha_{3,\theta}\partial_\theta^{-1}(\alpha_2\alpha_{2,y}) - [\alpha_2^{-1}(\alpha_3\partial_\theta + \partial_\theta\alpha_3)\alpha_{2,y}]_\theta \\ &+ [\alpha_2^{-1}(\partial_\theta^2 + \alpha_2^2 - \alpha_2\partial_\theta^{-1}\alpha_{2,\theta} - \alpha_3^2 + 1)(\alpha_3\partial_\theta + \partial_\theta\alpha_3)^{-1}\alpha_3^2\alpha_{2,y}]_\theta. \end{aligned} \tag{11}$$

Using the compatibility condition $\gamma_{t\theta} = \gamma_{\theta t}$, we arrive at the following system

$$\begin{aligned} E &= A_{1,\theta} - \alpha_1 A_1 - A_2, \\ F &= A_1 + A_{2,\theta} - \alpha_1 A_2 - \alpha_2 A_3, \\ G &= \alpha_2 A_2 + A_{3,\theta} - \alpha_1 A_3 - \alpha_3 A_4, \\ H &= \alpha_3 A_3 + A_{4,\theta} - \alpha_1 A_4. \end{aligned}$$

From $F = \partial_\theta(\alpha_2 G)$, we also have

$$\begin{aligned} A_1 &= -A_{2,\theta} + \alpha_1 A_2 + \alpha_2 A_3 + \partial_\theta^{-1}[\alpha_2(A_{3,\theta} + \alpha_2 A_2 - \alpha_1 A_3 - \alpha_3 A_4)], \\ \alpha_{1,t} &= [(\partial_\theta - \alpha_1 - \alpha_{1,\theta}\partial_\theta^{-1})(\partial_\theta^2 - \partial_\theta\alpha_1 - \alpha_2^2) + \partial_\theta]A_2 - (\partial_\theta - \alpha_1 - \alpha_{1,\theta}\partial_\theta^{-1}) \\ &* (\partial_\theta\alpha_2 + \alpha_2\partial_\theta - \alpha_1\alpha_2)A_3 + (\partial_\theta - \alpha_1 - \alpha_{1,\theta}\partial_\theta^{-1})\alpha_2\alpha_3 A_4. \end{aligned} \tag{12}$$

Letting $A_2 = -\alpha_1$ in (12), one obtains the following equation for the curvatures α_1

$$\begin{aligned} \alpha_{1,t} &= -\alpha_{1,\theta\theta\theta} + 3\alpha_{1,\theta}^2 + 3\alpha_1\alpha_{1,\theta\theta} + 3\alpha_{2,\theta}\alpha_{2,y} + 2\alpha_2\alpha_{2,\theta y} - 3\alpha_1^2\alpha_{1,\theta} - \alpha_{1,\theta}\partial_\theta^{-1}(\alpha_2\alpha_{2,y}) \\ &- \alpha_1\alpha_2\alpha_{2,y} - \alpha_{1,\theta} - \alpha_{1,\theta}\alpha_2^2 - 3\alpha_1\alpha_2\alpha_{2,\theta} - \alpha_2\alpha_3(\partial_\theta\alpha_3 + \alpha_3\partial_\theta)^{-1}\alpha_3^2\alpha_{2,y} \\ &- (\alpha_{2,\theta\theta} + \alpha_1\alpha_{2,\theta} - \alpha_2\alpha_3^2)[(\partial_\theta - \alpha_1)^{-1}(\alpha_1\alpha_2 - \alpha_{2,y}) + (\partial_\theta - \alpha_1)^{-1}[\alpha_3(\partial_\theta - \alpha_1 \\ &+ \alpha_3(\partial_\theta - \alpha_1)^{-1}\alpha_3)^{-1}[(\alpha_3\partial_\theta + \partial_\theta\alpha_3)^{-1}\alpha_3^2\alpha_{2,y} - \alpha_3(\partial_\theta - \alpha_1)^{-1}(\alpha_1\alpha_2 - \alpha_{2,y})]] \\ &- (2\alpha_3\alpha_{2,\theta} + \alpha_2\alpha_{3,\theta} + \alpha_1\alpha_2\alpha_3)[\partial_\theta - \alpha_1 + \alpha_3(\partial_\theta - \alpha_1)^{-1}\alpha_3]^{-1} \\ &* [(\alpha_3\partial_\theta + \partial_\theta\alpha_3)^{-1}\alpha_3^2\alpha_{2,y} - \alpha_3(\partial_\theta - \alpha_1)^{-1}(\alpha_1\alpha_2 - \alpha_{2,y})]. \end{aligned} \tag{13}$$

It is noted that the first equation in (11) is the 2+1-dimensional mKdV equation, which is integrable [47]. The equations (11) and (13) can be regarded as a generalization of the mKdV-Burgers system [15].

2.2. Motion of curves in \mathbb{P}^4 by adding two extra space variables y and z .

On the basis of Section 2.1, we introduce another one more space variable z . Assume that the z -evolution is governed by

$$\begin{pmatrix} \mathbf{t}_1 \\ \mathbf{t}_2 \\ \mathbf{t}_3 \\ \mathbf{t}_4 \end{pmatrix}_z = \begin{pmatrix} \tilde{e} & \tilde{f} & \tilde{g} & \tilde{h} \\ -\tilde{f} & \tilde{e} & \tilde{f}_1 & \tilde{g}_1 \\ -\tilde{g} & -\tilde{f}_1 & \tilde{e} & \tilde{h}_1 \\ -\tilde{h} & -\tilde{g}_1 & -\tilde{h}_1 & \tilde{e} \end{pmatrix} \begin{pmatrix} \mathbf{t}_1 \\ \mathbf{t}_2 \\ \mathbf{t}_3 \\ \mathbf{t}_4 \end{pmatrix}, \tag{14}$$

where $\tilde{e}, \tilde{f}, \tilde{g}, \tilde{h}, \tilde{f}_1, \tilde{g}_1, \tilde{h}_1$ are functions of curvatures α_1, α_2 and α_3 , to be determined. The compatibility condition

$$\begin{pmatrix} \mathbf{t}_1 \\ \mathbf{t}_2 \\ \mathbf{t}_3 \\ \mathbf{t}_4 \end{pmatrix}_{z\theta} = \begin{pmatrix} \mathbf{t}_1 \\ \mathbf{t}_2 \\ \mathbf{t}_3 \\ \mathbf{t}_4 \end{pmatrix}_{\theta z}$$

leads to the following system

$$\begin{aligned}
\alpha_{1,z} &= -\tilde{e}_\theta, \\
\alpha_{2,z} &= \tilde{g} + \tilde{f}_{1,\theta} - \alpha_3\tilde{g}_1, \\
\alpha_{3,z} &= \tilde{h}_{1,\theta} + \alpha_2\tilde{g}_1, \\
\tilde{f}_1 &= \alpha_2\tilde{f} + \tilde{g}_\theta - \alpha_3\tilde{h}, \\
\tilde{g}_1 &= \alpha_3\tilde{g} + \tilde{h}_\theta, \quad \tilde{f}_\theta = \alpha_2\tilde{g}, \\
\tilde{h} &= -\alpha_3\tilde{f}_1 - \tilde{g}_{1,\theta} + \alpha_2\tilde{h}_1.
\end{aligned} \tag{15}$$

On the other hand, the compatibility condition

$$\begin{pmatrix} \mathbf{t}_1 \\ \mathbf{t}_2 \\ \mathbf{t}_3 \\ \mathbf{t}_4 \end{pmatrix}_{zt} = \begin{pmatrix} \mathbf{t}_1 \\ \mathbf{t}_2 \\ \mathbf{t}_3 \\ \mathbf{t}_4 \end{pmatrix}_{tz}$$

leads to the following equations

$$\begin{aligned}
\tilde{e}_t &= E_z, \\
\tilde{f}_t &= F_z + \tilde{g}F_1 + \tilde{h}G_1 - \tilde{f}_1G - \tilde{g}_1H, \\
\tilde{g}_t &= G_z + \tilde{f}_1F - \tilde{f}F_1 + \tilde{h}H_1 - \tilde{h}_1H, \\
\tilde{h}_t &= H_z + \tilde{g}_1F + \tilde{h}_1G - \tilde{f}G_1 - \tilde{g}H_1, \\
\tilde{f}_{1,t} &= F_{1,z} + \tilde{f}G + \tilde{g}_1H_1 - \tilde{g}F - \tilde{h}_1G_1, \\
\tilde{g}_{1,t} &= G_{1,z} + \tilde{f}H + \tilde{h}_1F_1 - \tilde{f}_1H_1 - \tilde{h}F, \\
\tilde{h}_{1,t} &= H_{1,z} + \tilde{g}H + \tilde{f}_1G_1 - \tilde{h}G - \tilde{g}_1F_1.
\end{aligned} \tag{16}$$

One can easily verify that the compatibility condition $\mathbf{t}_{i,zy} = \mathbf{t}_{i,yz}$, $i = 1, 2, 3, 4$, can be obtained by the other compatibility conditions.

Thus from the compatibility conditions, one obtains the following equations

$$\begin{aligned}
A_1 &= -A_{2,\theta} + \alpha_1A_2 + \alpha_2A_3 + \partial_\theta^{-1}[\alpha_2(A_{3,\theta} + \alpha_2A_2 - \alpha_1A_3 - \alpha_3A_4)], \\
E &= A_{1,\theta} - \alpha_1A_1 - A_2, \\
F &= A_{2,\theta} + A_1 - \alpha_1A_2 - \alpha_2A_3, \\
G &= A_{3,\theta} + \alpha_2A_2 - \alpha_1A_3 - \alpha_3A_4, \\
H &= A_{4,\theta} + \alpha_3A_3 - \alpha_1A_4, \\
F_1 &= A_{3,\theta\theta\theta} + 2\alpha_2A_{2,\theta} - \alpha_1A_{3,\theta} - 2\alpha_3A_{4,\theta} + \alpha_2A_1 + (\alpha_{2,\theta} - \alpha_1\alpha_2)A_2 \\
&\quad - (\alpha_{1,\theta} + \alpha_2^2 + \alpha_3^2)A_3 - (\alpha_{3,\theta} - \alpha_1\alpha_3)A_4, \\
G_1 &= A_{4,\theta\theta} + 2\alpha_3A_{3,\theta} - \alpha_1A_{4,\theta} + \alpha_2\alpha_3A_2 + (\alpha_{3,\theta} - \alpha_1\alpha_3)A_3 - (\alpha_{1,\theta} + \alpha_3^2)A_4, \\
H_1 &= \alpha_2^{-1}[\alpha_3A_{3,\theta\theta\theta} + A_{4,\theta\theta\theta} + 2\alpha_3A_{3,\theta\theta} - \alpha_1A_{4,\theta\theta} + 3\alpha_2\alpha_3A_{2,\theta} \\
&\quad + (3\alpha_{3,\theta} - 2\alpha_1\alpha_3)A_{3,\theta} - (2\alpha_{1,\theta} + 3\alpha_3^2 - 1)A_{4,\theta} \\
&\quad + \alpha_2\alpha_3A_1 + (2\alpha_{2,\theta}\alpha_3 - \alpha_1\alpha_3^2 + \alpha_2\alpha_3,\theta)A_2 \\
&\quad + [\alpha_{3,\theta\theta} - \alpha_1\alpha_{3,\theta} - \alpha_3(2\alpha_{1,\theta} + \alpha_2^2 + \alpha_3^2 - 1)]A_3 \\
&\quad - (\alpha_{1,\theta\theta} + 3\alpha_3\alpha_{3,\theta} + \alpha_1 - \alpha_1\alpha_2\alpha_3)A_4],
\end{aligned} \tag{17}$$

$$\begin{aligned}
f &= \partial_\theta^{-1}(\alpha_2 g), \quad \tilde{f} = \partial_\theta^{-1}(\alpha_2 \tilde{g}), \\
f_1 &= \alpha_2 \partial_\theta^{-1}(\alpha_2 g) + g_\theta - \alpha_3 h, \\
g_1 &= \alpha_3 g + h_\theta, \quad \tilde{g}_1 = \alpha_3 \tilde{g} + \tilde{h}_\theta, \\
h_1 &= \alpha_2^{-1}[h_{\theta\theta} + (1 - \alpha_3^2)h + 2\alpha_3 g_\theta + \alpha_{3,\theta}g + \alpha_2 \alpha_3 \partial_\theta^{-1}(\alpha_2 g)], \\
\tilde{f}_1 &= \alpha_2 \partial_\theta^{-1}(\alpha_2 \tilde{g}) + \tilde{g}_\theta - \alpha_3 \tilde{h}, \\
\tilde{h}_1 &= \alpha_2^{-1}[\tilde{h}_{\theta\theta} + (1 - \alpha_3^2)\tilde{h} + 2\alpha_3 \tilde{g}_\theta + \alpha_{3,\theta}\tilde{g} + \alpha_2 \alpha_3 \partial_\theta^{-1}(\alpha_2 \tilde{g})],
\end{aligned} \tag{18}$$

and the equations for α_1 , α_2 and α_3

$$\begin{aligned}
\alpha_{1,t} &= [(\partial_\theta - \alpha_1 - \alpha_{1,\theta} \partial_\theta^{-1})(\partial_\theta^2 - \partial_\theta \alpha_1 - \alpha_2^2) + \partial_\theta] A_2 - (\partial_\theta - \alpha_1 - \alpha_{1,\theta} \partial_\theta^{-1}) \\
&\quad * (\partial_\theta \alpha_2 + \alpha_2 \partial_\theta - \alpha_1 \alpha_2) A_3 + (\partial_\theta - \alpha_1 - \alpha_{1,\theta} \partial_\theta^{-1}) \alpha_2 \alpha_3 A_4, \\
\alpha_{2,t} &= (\partial_\theta^2 + \alpha_{2,\theta} \partial_\theta^{-1} \alpha_2 + \alpha_2^2 - \alpha_3^2 + 1) \alpha_2 A_2 + [(\partial_\theta^2 + \alpha_{2,\theta} \partial_\theta^{-1} \alpha_2 + \alpha_2^2 - \alpha_3^2 + 1) \\
&\quad * (\partial_\theta - \alpha_1) - (\partial_\theta \alpha_3 + \alpha_3 \partial_\theta) \alpha_3] A_3 - [(\partial_\theta^2 + \alpha_{2,\theta} \partial_\theta^{-1} \alpha_2 + \alpha_2^2 - \alpha_3^2 + 1) \alpha_3 \\
&\quad + (\partial_\theta \alpha_3 + \alpha_3 \partial_\theta)(\partial_\theta - \alpha_1)] A_4, \\
\alpha_{3,t} &= [\partial_\theta (\alpha_2^{-1} \partial_\theta (\alpha_2 \alpha_3) + \alpha_2^{-1} \alpha_3 \partial_\theta \alpha_2 + \alpha_3 \partial_\theta^{-1} \alpha_2^2) + \alpha_2^2 \alpha_3] A_2 \\
&\quad + [\partial_\theta [\alpha_2^{-1} (\partial_\theta^2 - \alpha_3^2 + 1) \alpha_3 + \alpha_2^{-1} (\partial_\theta \alpha_3 + \alpha_3 \partial_\theta) (\partial_\theta - \alpha_1) + \alpha_3 \partial_\theta^{-1} \alpha_2 (\partial_\theta - \alpha_1)] \\
&\quad + \alpha_2 (\partial_\theta \alpha_3 + \alpha_3 \partial_\theta - \alpha_1 \alpha_3)] A_3 + [\partial_\theta [\alpha_2^{-1} (\partial_\theta^2 - \alpha_3^2 + 1) (\partial_\theta - \alpha_1) \\
&\quad - \alpha_2^{-1} (\partial_\theta \alpha_3 + \alpha_3 \partial_\theta) \alpha_3 - \alpha_3 \partial_\theta^{-1} \alpha_2 \alpha_3] + \alpha_2 (\partial_\theta (\partial_\theta - \alpha_1) - \alpha_3^2)] A_4,
\end{aligned} \tag{19}$$

$$\begin{aligned}
\alpha_{1,y} &= -e_\theta, \\
\alpha_{2,y} &= (\partial_\theta^2 + \alpha_{2,\theta} \partial_\theta^{-1} \alpha_2 + \alpha_2^2 - \alpha_3^2 + 1) g - 2\alpha_3 h_\theta - \alpha_{3,\theta} h, \\
\alpha_{3,y} &= 2\alpha_2 \alpha_3 g + \alpha_{3,\theta} \partial_\theta^{-1}(\alpha_2 g) + \alpha_2 h_\theta + [\alpha_2^{-1} (h_{\theta\theta} - \alpha_3^2 h + h + 2\alpha_3 g_\theta + \alpha_{3,\theta} g)]_\theta, \\
\alpha_{1,z} &= -\tilde{e}_\theta, \\
\alpha_{2,z} &= (\partial_\theta^2 + \alpha_{2,\theta} \partial_\theta^{-1} \alpha_2 + \alpha_2^2 - \alpha_3^2 + 1) \tilde{g} - 2\alpha_3 \tilde{h}_\theta - \alpha_{3,\theta} \tilde{h}, \\
\alpha_{3,z} &= 2\alpha_2 \alpha_3 \tilde{g} + \alpha_{3,\theta} \partial_\theta^{-1}(\alpha_2 \tilde{g}) + \alpha_2 \tilde{h}_\theta + [\alpha_2^{-1} (\tilde{h}_{\theta\theta} - \alpha_3^2 \tilde{h} + \tilde{h} + 2\alpha_3 \tilde{g}_\theta + \alpha_{3,\theta} \tilde{g})]_\theta,
\end{aligned} \tag{20}$$

as well as the following equations

$$\begin{aligned}
 g_t &= (A_{3,\theta} + \alpha_2 A_2 - \alpha_1 A_3 - \alpha_3 A_4)_y + g_\theta \partial_\theta^{-1} [\alpha_2 (A_{3,\theta} + \alpha_2 A_2 - \alpha_1 A_3 - \alpha_3 A_4)] \\
 &\quad - (A_{3,\theta} + \alpha_2 A_2 - \alpha_1 A_3 - \alpha_3 A_4)_\theta \partial_\theta^{-1} (\alpha_2 g) + \alpha_2^{-1} h [(A_{4,\theta} + \alpha_3 A_3 - \alpha_1 A_4)_{\theta\theta}] \\
 &\quad + 2\alpha_3 (A_{3,\theta} + \alpha_2 A_2 - \alpha_1 A_3 - \alpha_3 A_4)_\theta + \alpha_{3,\theta} (A_{3,\theta} + \alpha_2 A_2 - \alpha_1 A_3 - \alpha_3 A_4)] \\
 &\quad - \alpha_2^{-1} (A_{4,\theta} + \alpha_3 A_3 - \alpha_1 A_4) (h_{\theta\theta} + 2\alpha_3 g_\theta + \alpha_{3,\theta} g), \\
 h_t &= (A_{4,\theta} + \alpha_3 A_3 - \alpha_1 A_4)_y + h_\theta \partial_\theta^{-1} [\alpha_2 (A_{3,\theta} + \alpha_2 A_2 - \alpha_1 A_3 - \alpha_3 A_4)] - (A_{4,\theta} \\
 &\quad + \alpha_3 A_3 - \alpha_1 A_4)_\theta \partial_\theta^{-1} (\alpha_2 g) + \alpha_2^{-1} (A_{3,\theta} + \alpha_2 A_2 - \alpha_1 A_3 - \alpha_3 A_4) [h_{\theta\theta} \\
 &\quad + (1 - \alpha_3^2) h + 2\alpha_3 g_\theta] - \alpha_2^{-1} g [(A_{4,\theta} + \alpha_3 A_3 - \alpha_1 A_4)_{\theta\theta} + (1 - \alpha_3^2) (A_{4,\theta} + \alpha_3 A_3 \\
 &\quad - \alpha_1 A_4) + 2\alpha_3 (A_{3,\theta} + \alpha_2 A_2 - \alpha_1 A_3 - \alpha_3 A_4)_\theta], \\
 \tilde{g}_t &= (A_{3,\theta} + \alpha_2 A_2 - \alpha_1 A_3 - \alpha_3 A_4)_z + \tilde{g}_\theta \partial_\theta^{-1} [\alpha_2 (A_{3,\theta} + \alpha_2 A_2 - \alpha_1 A_3 - \alpha_3 A_4)] \\
 &\quad - (A_{3,\theta} + \alpha_2 A_2 - \alpha_1 A_3 - \alpha_3 A_4)_\theta \partial_\theta^{-1} (\alpha_2 \tilde{g}) + \alpha_2^{-1} h [(A_{4,\theta} + \alpha_3 A_3 - \alpha_1 A_4)_{\theta\theta}] \\
 &\quad + 2\alpha_3 (A_{3,\theta} + \alpha_2 A_2 - \alpha_1 A_3 - \alpha_3 A_4)_\theta + \alpha_{3,\theta} (A_{3,\theta} + \alpha_2 A_2 - \alpha_1 A_3 - \alpha_3 A_4)] \\
 &\quad - \alpha_2^{-1} (A_{4,\theta} + \alpha_3 A_3 - \alpha_1 A_4) (\tilde{h}_{\theta\theta} + 2\alpha_3 \tilde{g}_\theta + \alpha_{3,\theta} \tilde{g}), \\
 \tilde{h}_t &= (A_{4,\theta} + \alpha_3 A_3 - \alpha_1 A_4)_z + \tilde{h}_\theta \partial_\theta^{-1} [\alpha_2 (A_{3,\theta} + \alpha_2 A_2 - \alpha_1 A_3 - \alpha_3 A_4)] - (A_{4,\theta} \\
 &\quad + \alpha_3 A_3 - \alpha_1 A_4)_\theta \partial_\theta^{-1} (\alpha_2 \tilde{g}) + \alpha_2^{-1} (A_{3,\theta} + \alpha_2 A_2 - \alpha_1 A_3 - \alpha_3 A_4) [\tilde{h}_{\theta\theta} + (1 - \alpha_3^2) \tilde{h} \\
 &\quad + 2\alpha_3 \tilde{g}_\theta] - \alpha_2^{-1} \tilde{g} [(A_{4,\theta} + \alpha_3 A_3 - \alpha_1 A_4)_{\theta\theta} + (1 - \alpha_3^2) (A_{4,\theta} + \alpha_3 A_3 - \alpha_1 A_4) \\
 &\quad + 2\alpha_3 (A_{3,\theta} + \alpha_2 A_2 - \alpha_1 A_3 - \alpha_3 A_4)_\theta].
 \end{aligned} \tag{21}$$

Choosing A_2 , A_3 and A_4 properly so that the equations (19), (20) and (21) are compatible, we can obtain a system of 3+1-dimensional evolution equations for curvatures α_1 , α_2 and α_3 .

3. Motion of space curves in \mathbb{P}^n .

3.1. Motion of curves in \mathbb{P}^n by adding an extra space variable y . Denote by $\kappa_1, \kappa_2, \dots, \kappa_{n-1}$, s and τ_i , $i = 1, 2, \dots, n$ respectively the Euclidean curvatures, Euclidean arc-length and Euclidean frame vectors of a curve. One can readily verify that $\alpha_1 = \kappa_{1,s}/\kappa_1^2$, $\alpha_j = \kappa_j/\kappa_1$, $\mathbf{t}_i = (1/\kappa_1)\tau_i$ and $d\theta = \kappa_1 ds$, $i = 1, 2, \dots, n$, $j = 2, 3, \dots, n-1$ are invariant with respect to the isometry group of the similarity geometry \mathbb{P}^n . We define them to be the curvatures, frame vectors and arc-length element of the curve in \mathbb{P}^n .

The Serret-Frenet formulas in \mathbb{P}^n read as [15]

$$\begin{pmatrix} \mathbf{t}_1 \\ \mathbf{t}_2 \\ \vdots \\ \mathbf{t}_n \end{pmatrix}_\theta = \begin{pmatrix} -\alpha_1 & 1 & 0 & & & \\ -1 & -\alpha_1 & \alpha_2 & \ddots & & \\ 0 & -\alpha_2 & -\alpha_1 & \ddots & 0 & \\ & \ddots & \ddots & \ddots & \alpha_{n-1} & \\ & & 0 & -\alpha_{n-1} & -\alpha_1 & \end{pmatrix} \begin{pmatrix} \mathbf{t}_1 \\ \mathbf{t}_2 \\ \vdots \\ \mathbf{t}_n \end{pmatrix}, \tag{22}$$

The geometric motion flow in \mathbb{P}^n is specified by

$$\gamma_t = A_1 \mathbf{t}_1 + A_2 \mathbf{t}_2 + \dots + A_n \mathbf{t}_n, \tag{23}$$

where A_i , $i = 1, 2, \dots, n$, are velocities depending on the curvatures α_j , $j = 1, 2, 3, \dots, n-1$.

From $\gamma_{\theta t} = \gamma_{t\theta}$, we deduce that the time evolution for the frame vectors \mathbf{t}_i is given by

$$\mathbf{t}_{i,t} = (A_{1,\theta} - A_2 - \alpha_1 A_1)\mathbf{t}_i + \sum_{j=1}^n M_{ij}\mathbf{t}_j, i = 1, 2, \dots, n, \quad (24)$$

where the $n \times n$ matrix M_{ij} are determined recursively by

$$\begin{aligned} M_{ij} &= -M_{ji}, \\ M_{1j} &= A_{j,\theta} - \alpha_1 A_j + \hat{\alpha}_{j-1} A_{j-1} - \alpha_j A_{j+1}, 2 \leq j \leq n-1, \\ M_{1n} &= A_{n,\theta} - \alpha_1 A_n + \alpha_{n-1} A_{n-1}, \\ M_{i+1,j} &= 1/\alpha_i (M_{ij,\theta} + \hat{\alpha}_{i-1} M_{i-1,j} - \alpha_j M_{i,j+1} + \hat{\alpha}_{j-1} M_{i,j-1}), j > i+1, \end{aligned}$$

with

$$\hat{\alpha}_j = \begin{cases} 1, & j = 1 \\ \alpha_j, & j \neq 1. \end{cases}$$

The compatibility condition of equations (22) and (24) i.e.

$$\begin{pmatrix} \mathbf{t}_1 \\ \mathbf{t}_2 \\ \vdots \\ \mathbf{t}_n \end{pmatrix}_{t\theta} = \begin{pmatrix} \mathbf{t}_1 \\ \mathbf{t}_2 \\ \vdots \\ \mathbf{t}_n \end{pmatrix}_{\theta t},$$

leads to the equations

$$\begin{aligned} (A_{2,\theta} - \alpha_1 A_2 + A_1 - \alpha_2 A_3)\theta &= \alpha_2 (A_{3,\theta} - \alpha_1 A_3 + \alpha_2 A_2 - \alpha_3 A_4), \\ \alpha_{1,t} &= -(A_{1,\theta} - \alpha_1 A_1 - A_2)\theta, \\ \alpha_{i,t} &= M_{i,i+1,\theta} - \alpha_{i+1} M_{i,i+2} + \hat{\alpha}_{i-1} M_{i-1,i+1}, i = 2, 3, \dots, n-1. \end{aligned} \quad (25)$$

We now introduce the extra space variable y to the curve motion flow. It follows from (24) that the y -evolution for the frame vectors \mathbf{t}_i is

$$\mathbf{t}_{i,y} = (a_{1,\theta} - a_2 - \alpha_1 a_1)\mathbf{t}_i + \sum_{j=1}^n m_{ij}\mathbf{t}_j, i = 1, 2, \dots, n, \quad (26)$$

where a_i , $i = 1, 2, \dots, n$, are velocities depending on the curvatures α_j , $j = 1, 2, 3, \dots, n-1$, and $n \times n$ matrix m_{ij} are determined recursively by

$$\begin{aligned} m_{ij} &= -m_{ji}, \\ m_{1j} &= a_{j,\theta} - \alpha_1 a_j + \hat{\alpha}_{j-1} a_{j-1} - \alpha_j a_{j+1}, 2 \leq j \leq n-1, \\ m_{1n} &= a_{n,\theta} - \alpha_1 a_n + \alpha_{n-1} a_{n-1}, \\ m_{i+1,j} &= 1/\alpha_i (m_{ij,\theta} + \hat{\alpha}_{i-1} m_{i-1,j} - \alpha_j m_{i,j+1} + \hat{\alpha}_{j-1} m_{i,j-1}), j > i+1. \end{aligned}$$

The compatibility condition between equations (22) and (26) gives rise to the following equations

$$\begin{aligned} (a_{2,\theta} - \alpha_1 a_2 + a_1 - \alpha_2 a_3)\theta &= \alpha_2 (a_{3,\theta} - \alpha_1 a_3 + \alpha_2 a_2 - \alpha_3 a_4), \\ \alpha_{1,y} &= -(a_{1,\theta} - \alpha_1 a_1 - a_2)\theta, \\ \alpha_{i,y} &= m_{i,i+1,\theta} - \alpha_{i+1} m_{i,i+2} + \hat{\alpha}_{i-1} m_{i-1,i+1}, i = 2, 3, \dots, n-1. \end{aligned} \quad (27)$$

On the other hand, the compatibility condition

$$\begin{pmatrix} \mathbf{t}_1 \\ \mathbf{t}_2 \\ \vdots \\ \mathbf{t}_n \end{pmatrix}_{ty} = \begin{pmatrix} \mathbf{t}_1 \\ \mathbf{t}_2 \\ \vdots \\ \mathbf{t}_n \end{pmatrix}_{yt},$$

leads to the following equations for $2 \leq j \leq n$

$$m_{1j,t} = M_{1j,y} + \sum_{l=2}^{j-1} (m_{lj}M_{1l} - m_{1l}M_{lj}) + \sum_{k=1}^{n-j} (m_{1,j+k}M_{j,j+k} - m_{j,j+k}M_{1,j+k}). \tag{28}$$

We note that the equations satisfied by $m_{11,t}$, $m_{ij,t}$, $i = 2, 3, \dots, n$, $j = 1, 2, \dots, n$, can be represented by (25), (27) and (28).

Choosing A_j , $j = 2, 3, \dots, n$, properly so that equations (25), (27) and (28) are compatible, one can obtain 2+1-dimensional evolution equations for curvatures α_i $i = 1, 2, \dots, n - 1$.

3.2. Motion of curves in \mathbb{P}^n by adding two extra space variables y and z . We assume that the z -evolution is governed by

$$\mathbf{t}_{i,z} = (\tilde{a}_{1,\theta} - \tilde{a}_2 - \alpha_1 \tilde{a}_1) \mathbf{t}_i + \sum_{j=1}^n \tilde{m}_{ij} \mathbf{t}_j, \quad i = 1, 2, \dots, n,$$

where $\tilde{\alpha}_i$, $i = 1, 2, \dots, n$, are some functions to be determined, and $n \times n$ matrix \tilde{m}_{ij} are governed recursively by

$$\begin{aligned} \tilde{m}_{ij} &= -\tilde{m}_{ji}, \\ \tilde{m}_{1j} &= \tilde{a}_{j,\theta} - \alpha_1 \tilde{a}_j + \hat{\alpha}_{j-1} \tilde{a}_{j-1} - \alpha_j \tilde{a}_{j+1}, \quad 2 \leq j \leq n - 1, \\ \tilde{m}_{1n} &= \tilde{a}_{n,\theta} - \alpha_1 \tilde{a}_n + \alpha_{n-1} \tilde{a}_{n-1}, \\ \tilde{m}_{i+1,j} &= 1/\alpha_i (\tilde{m}_{ij,\theta} + \hat{\alpha}_{i-1} \tilde{m}_{i-1,j} - \alpha_j \tilde{m}_{i,j+1} + \hat{\alpha}_{j-1} \tilde{m}_{i,j-1}), \quad j > i + 1. \end{aligned}$$

From

$$\begin{pmatrix} \mathbf{t}_1 \\ \mathbf{t}_2 \\ \vdots \\ \mathbf{t}_n \end{pmatrix}_{z\theta} = \begin{pmatrix} \mathbf{t}_1 \\ \mathbf{t}_2 \\ \vdots \\ \mathbf{t}_n \end{pmatrix}_{\theta z},$$

we have

$$\begin{aligned} (\tilde{a}_{2,\theta} - \alpha_1 \tilde{a}_2 + \tilde{a}_1 - \alpha_2 \tilde{a}_3) \theta &= \alpha_2 (\tilde{a}_{3,\theta} - \alpha_1 \tilde{a}_3 + \alpha_2 \tilde{a}_2 - \alpha_3 \tilde{a}_4), \\ \alpha_{1,z} &= -(\tilde{a}_{1,\theta} - \alpha_1 \tilde{a}_1 - \tilde{a}_2) \theta, \\ \alpha_{i,z} &= m_{i,i+1,\theta} - \alpha_{i+1} m_{i,i+2} + \hat{\alpha}_{i-1} m_{i-1,i+1}, \quad i = 2, 3, \dots, n - 1. \end{aligned} \tag{29}$$

The compatibility condition

$$\begin{pmatrix} \mathbf{t}_1 \\ \mathbf{t}_2 \\ \vdots \\ \mathbf{t}_n \end{pmatrix}_{zt} = \begin{pmatrix} \mathbf{t}_1 \\ \mathbf{t}_2 \\ \vdots \\ \mathbf{t}_n \end{pmatrix}_{tz}$$

gives the following system $2 \leq j \leq n$,

$$\tilde{m}_{1j,t} = M_{1j,z} + \sum_{l=2}^{j-1} (\tilde{m}_{lj} M_{1l} - \tilde{m}_{1l} M_{lj}) + \sum_{k=1}^{n-j} (\tilde{m}_{1,j+k} M_{j,j+k} - \tilde{m}_{j,j+k} M_{1,j+k}). \quad (30)$$

One can readily verify that $\mathbf{t}_{i,yz} = \mathbf{t}_{i,zy}$, $i = 1, 2, \dots, n$, can be represented by the other compatibility conditions.

Thus we can obtain 3+1-dimensional evolution equations for curvatures α_i , $i = 1, 2, \dots, n-1$, by choosing properly A_j , $j = 2, 3, \dots, n$, subject to that the equations (25), (27), (28), (29) and (30) are compatible.

4. Concluding remarks. In this paper, we have discussed integrable deformation of surfaces induced by adding one and two extra space variables to the motions of space curves in higher-dimensional similarity geometries. The 2+1- and 3+1-dimensional evolution equations including the 2+1-dimensional mKdV equation and a generalization of the mKdV-Burgers system [15] were obtained. Similarly, we can discuss motions of curves by endowing one and two additional space variables in other Klein geometries, and we believe that there will be some new integrable equations associated with such motions.

Acknowledgements. This work was supported by the National NSF (Grant No. 10671156) of China, Program for New Century Excellent Talents in University (NCET-04-0968).

REFERENCES

- [1] W.W. MULLINS, *Theory of thermal grooving*, J. Appl. Phys., 28 (1957), pp. 333–339.
- [2] K.S. CHOU AND X.P. ZHU, *The Curve Shortening Problem*, Chapman & Hall/CRC Press, Boca Raton, FL, 2001.
- [3] H. HASIMOTO, *A soliton on a vortex filament*, J. Fluid Mech., 51 (1972), pp. 477–485.
- [4] G.L. LAMB, *Solitons on moving space curves*, J. Math. Phys., 18 (1977), pp. 1654–1661.
- [5] J.S. LANGER AND R. PERLINE, *Poisson geometry of the filament equation*, J. Nonlinear Sci., 1 (1991), pp. 71–93.
- [6] J.S. LANGER AND R. PERLINE, *Curve motion inducing modified Korteweg-de Vries systems*, Phys. Lett. A, 239 (1998), pp. 36–40.
- [7] M. LAKSHMANAN, *Rigid body motions, space curves, prolongation structures, fiber bundles, and solitons*, J. Math. Phys., 20 (1979), pp. 1667–1672.
- [8] A. DOLIWA AND P.M. SANTINI, *An elementary geometric characterization of the integrable motions of a curve*, Phys. Lett. A, 85 (1994), pp. 373–384.
- [9] R.E. GOLDSTEIN AND D. M. PETRICH, *The Korteweg-de Vries hierarchy as dynamics of closed curves in the plane*, Phys. Rev. Lett., 67 (1991), pp. 3203–3206.
- [10] K. NAKAYAMA, H. SEGUR AND M. WADATI, *Integrability and the motion of curves*, Phys. Rev. Lett., 69 (1992), pp. 2603–2606.
- [11] K. NAKAYAMA, *Motion of curves in hyperboloids in the Minkowski space I*, J. Phys. Soc. Jpn., 67 (1998), pp. 3031–3037.
- [12] K. NAKAYAMA, *Motion of curves in hyperboloids in the Minkowski space II*, J. Phys. Soc. Jpn., 68 (1999), pp. 3214–3218.
- [13] W.K. SCHIEF AND C. ROGERS, *Binormal motion of curves of constant curvature and torsion, Generation of soliton surfaces*, Proc. Roy. Soc. Lond. A, 455 (1999), pp. 3163–3188.
- [14] K.S. CHOU AND C.Z. QU, *Integrable equations arising from motions of plane curves*, Physica D, 162 (2002), pp. 9–33.
- [15] K.S. CHOU AND C.Z. QU, *Motions of curves in similarity geometries and Burgers-mKdV hierarchies*, Chaos, Solitons and Fractals, 19 (2004), pp. 47–53.
- [16] K.S. CHOU AND C.Z. QU, *The KdV equation and motion of plane curves*, J. Phys. Soc. Jpn., 70 (2001), pp. 1912–1916.
- [17] K.S. CHOU AND C.Z. QU, *Integrable equations arising from motions of plane curves II*, J. Nonlin. Sci., 13 (2003), pp. 487–517.

- [18] C.Z. QU, S.L. ZHANG AND Q.J. ZHANG, *Integrability of models arising from motions of plane curves*, Z. Naturforsch A, 58 (2003), pp. 75–83.
- [19] K.S. CHOU AND C.Z. QU, *Integrable motions of space curves in affine geometry*, Chaos, Solitons and Fractals, 14 (2002), pp. 29–44.
- [20] W.F. WO AND C.Z. QU, *Integrable motions of curves in $S^1 \times \mathbb{R}$* , J. Geom. Phys., 57 (2007), pp. 1733–1755.
- [21] G.M. BEFFA, J.A. SANDERS AND J.P. WANG, *Integrable systems in three-dimensional Riemannian geometry*, J. Nonlinear Sci., 12 (2002), pp. 143–167.
- [22] G.M. BEFFA, *Poisson brackets associated to the conformal geometry of curves*, Trans. Amer. Math. Soc., 357 (2004), pp. 2799–2827.
- [23] G.M. BEFFA, *Invariant evolutions of curves and surfaces and completely integrable Hamiltonian systems*, Contemp. Math., 285 (2001), pp. 29–38.
- [24] G.M. BEFFA, *Poisson geometry of differential invariants of curves in some nonsemisimple homogeneous space*, Proc. Amer. Math. Soc., 134 (2005), pp. 779–791.
- [25] T.A. IVEY, *Integrable geometric evolution equations for curves*, Contemp. Math., 285 (2001), pp. 71–84.
- [26] S.C. ANCO, *Hamiltonian flows of curves in $G/SO(N)$ and hyperbolic/evolutionary vector soliton equations*, SIGMA, 2 (044) (2006), pp. 1–17.
- [27] S.C. ANCO, *Bi-Hamiltonian operators, integrable flows of curves using moving frames and geometric map equations*, J. Phys. A: Math. Gen., 39 (2006), pp. 2043–2072.
- [28] S.C. ANCO, *Group-invariant soliton equations and Bi-Hamiltonian geometric curve flows in Riemannian symmetric spaces*, J. Geom. Phys., 58 (2008), pp. 1–37. SI/0703041, 2007.
- [29] S.C. ANCO AND S.I. VACARU, *Curve flows and solitonic hierarchies generated by Riemannian metrics*, preprint arXiv: math-ph/0609070, 2006.
- [30] C.L. TERNG AND G. THORBERGSSON, *Completely integrable curve flows on adjoint orbits (dedicated to Shiing-Shen Chern on his 90th birthday)*, Results Math., 40 (2001), pp. 286–309.
- [31] B.G. KONOPELCHENKO, *Nets in \mathbf{R}^3 , their integrable evolutions and the DS hierarchy*, Phys. Lett. A, 183 (1993), pp. 153–159.
- [32] I.A. TAIMANOV, *Modified Novikov-Veselov equation and differential geometry of surfaces, Solitons, geometry, and topology: on the crossroad*, Amer. Math. Soc. Transl. Ser. 2., 179 (1997), pp. 133–151.
- [33] V.E. ZAKHAROV, *Description of the n -orthogonal curvilinear coordinate systems and Hamiltonian integrable systems of hydrodynamic type. I. Integration of the Lamé equations*, Duke Math. J., 94 (1998), pp. 103–139.
- [34] W.K. SCHIEF, *On the geometry of an integrable $(2+1)$ -dimensional sine-Gordon system*, Proc. Roy. Soc. Lond. A, 453 (1997), pp. 1671–1688.
- [35] R. MYRZAKULOV, S. VIJAYALAKSHMI, G.N. NUGMANOVA AND M. LAKSHMANAN, *A $(2+1)$ dimensional integrable spin model: Geometrical and gauge equivalent counterpart, solitons and localized coherent structures*, Preprint solv-int/9704005, 1997.
- [36] R. MYRZAKULOV, G.N. NUGMANOVA AND R.N. SYZDYKOVA, *Gauge equivalence between $(2+1)$ -dimensional continuous Heisenberg ferromagnetic models and nonlinear Schrödinger-type equations*, J. Phys. A: Math. Gen., 31 (1998), pp. 9535–9545.
- [37] R. MYRZAKULOV, S. VIJAYALAKSHMI, R.N. SYZDYKOVA AND M. LAKSHMANAN, *On the simplest $(2+1)$ -dimensional integrable spin systems and their equivalent nonlinear Schrödinger equations*, J. Math. Phys., 39 (1998), pp. 2122–2140.
- [38] L. MARTINA, KUR. MYRZAKUL, R. MYRZAKULOV AND G. SOLIANI, *Deformation of surfaces, integrable systems, and Chern-Simons theory*, J. Math. Phys., 42 (2001), pp. 1397–1417.
- [39] R. MYRZAKULOV, S. VIJAYALAKSHMI, G.N. NUGMANOVA AND M. LAKSHMANAN, *A $(2+1)$ -dimensional integrable spin model: geometrical and gauge equivalent counterpart, solitons and localized coherent structures*, Phys. Lett. A, 233 (1997), pp. 391–396.
- [40] M. LAKSHMANAN, R. MYRZAKULOV, S. VIJAYALAKSHMI AND A.K. DANLYBAEVA, *Motion of curves and surfaces and nonlinear evolution equations in $(2+1)$ dimensions*, J. Math. Phys., 39 (1998), pp. 3765–3771.
- [41] R. MYRZAKULOV, *Solitons, Surfaces, Curves, and the Spin Description of Nonlinear Evolution Equations*, preprint solv-int/9802017, 1998.
- [42] R. MYRZAKULOV, *Soliton equations in $2+1$ dimensions and Differential geometry of curves/surfaces*, preprint math.DG/9908155, 1999.
- [43] C.Z. QU AND S.L. ZHANG, *Motion of curves and surfaces in affine geometry*, Chaos, Solitons and Fractals, 20 (2004), pp. 1013–1019.
- [44] K.S. CHOU AND C.Z. QU, *Geometric Motions of Surfaces and $2+1$ -dimensional Integrable Equations*, J. Phys. Soc. Jpn., 71 (2002), pp. 1039–1043.
- [45] G. SPAIRO AND A. TANNENBAUM, *Area and length preserving geometric invariant scale-spaces*,

- IEEE Trans PAMI., 17 (1995), pp. 67–72.
- [46] A. FUJIOKA AND J. INOBUCHI, *Deformations of surfaces preserving conformal or similarity invariants*, From geometry to quantum mechanics, pp. 53–67, Progr. Math. 252, Birkhäuser, Boston, MA, 2007.
- [47] I.A.B. STRACHAN, *Some integrable hierarchies in $(2+1)$ dimensions and their twistor description*, J. Math. Phys., 34 (1993), pp. 243–259.

