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EXISTENCE OF SOLUTIONS TO THE THREE DIMENSIONAL BAROTROPIC-VORTICITY EQUATION*

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Abstract. We prove existence of maximizers for a variational problem in $\mathbb{R}^3_+$. Solutions represent steady geophysical flows over a surface of variable height which is bounded from below.

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1. Introduction. In this paper we prove existence of maximizers for a variational problem which describes a geophysical flow over a surface of variable height, bounded from below, such as a seamount in the ocean or a mountain in the atmosphere. The basic equation governing such flows is the three dimensional barotropic vorticity equation given by

$$[\psi, \zeta] = 0,$$

where $[.,.]$ denotes the Jacobian and ψ represents the stream function, $-\zeta$ the potential vorticity given by

$$-\zeta = \Delta\psi + h,$$

where h is the height of the bottom surface.

In [6] and [9] similar problems have been considered in two dimensions. Here, the problem has been formulated in three dimensions which is more realistic. In addition, from a technical point of view, due to drastic differences between the fundamental solutions of $-\Delta$ in two and three dimensions the estimates in [6] and [9] are not applicable. In particular we single out the simple but crucial result stated in Lemma 6 in section 3.

To prove the existence we follow the method proposed by Benjamin [3]. To do this we begin by considering the variational problem over half spheres. In order to prove existence of maximizers in this situation we employ the technology extensively developed by Burton [4,5]. Then using a limiting argument we show that maximizers for *large* half spheres indeed are maximizers for the original problem; the radius of the critical half sphere turns out to be the radius of the smallest two dimensional disc containing the support of the height function h.

2. Definitions and notations. Henceforth we assume $p \in (3, \infty)$. The ball centered at $x \in \mathbb{R}^3$ with radius R is denoted $B_R(x)$; in particular when the center is the origin we write B_R. For $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, we write $\bar{x} = (x_1, x_2, -x_3)$ and we define $\mathbb{R}^3_+ = \{x \in \mathbb{R}^3 : x_3 > 0\}$. For a measurable set $A \subseteq \mathbb{R}^3$, $\mid A \mid$ denotes the three dimensional Lebesgue measure of A. If A is measurable, then $x \in A$ is called a density point of A whenever $\mid B_\varepsilon(x) \bigcap A \mid > 0$, for all positive ε. The set of all density

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points of A is denoted $den(A)$.
For a measurable function $\zeta : \mathbb{R}^3_+ \to \mathbb{R}$, the *strong support* or simply the *support* of ζ denoted $supp(\zeta)$ is defined

$$supp(\zeta) = \{x \in \mathbb{R}^3_+ : \zeta(x) > 0\}.$$

If f and g are non-negative measurable functions that vanish outside sets of finite measure in $\mathbb{R}^3_+$, we say f is a rearrangement of g whenever

$$| \{x \in \mathbb{R}^3_+ : f(x) \geq \alpha\} | = | \{x \in \mathbb{R}^3_+ : g(x) \geq \alpha\} |,$$

for every positive α. Let us fix $\zeta_0 \in L^p(\mathbb{R}^3_+)$ to be a non-negative function vanishing outside a set of measure $\frac{4}{3}\pi a^3$ for some positive a and $\| \zeta_0 \|_p = 1$. The set of all rearrangements of ζ_0 on $\mathbb{R}^3_+$ which vanish outside bounded sets is denoted $\mathcal{F}$. The subset of $\mathcal{F}$ containing functions vanishing outside the ball B_R is denoted $\mathcal{F}(R)$; henceforth we assume $R > a$ in order to ensure $\mathcal{F}(R)$ is non-empty. For a non-negative $\zeta \in L^p(\mathbb{R}^3_+)$ having bounded support, we define the energy functional

$$\Psi(\zeta) = \frac{1}{2} \int_{\mathbb{R}^3_+} \zeta K \zeta + \int_{\mathbb{R}^3_+} \eta \zeta,$$

where

$$K\zeta(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3_+} \left(\frac{1}{| x - y |} + \frac{1}{| x - \bar{y} |} \right) \zeta(y) \, dy$$

and

$$\eta(x) = \frac{1}{2\pi} \int_{\partial \mathbb{R}^3_+} \frac{1}{| x - y |} h(y) \, d\sigma(y).$$

Here $h \in L^p(\partial \mathbb{R}^3_+)$ is a non-negative function with compact support.
Let B_{r_h} be the smallest ball containing $supp(h)$; we assume that

$$r_h > \max\{a, r_*\} \tag{1}$$

where

$$r_* \ln \frac{r_*}{2\sqrt{e}} = 2,$$

(such r_* is unique and $1.81e < r_* < 1.82e$) and

$$h(x_1, x_2) \geq c \, \ln | x_1 x_2 |, \tag{2}$$

almost everywhere in $supp(h)$, where c is some constant given in Lemma 1.

Let us now introduce the following variational problem(P):

$$\sup_{\zeta \in \mathcal{F}} \Psi(\zeta) \, .$$

The solution set for (P) is denoted Σ. Now we can state the main result of this paper is the following

Theorem. *The variational problem (P) is solvable; that is, Σ is not empty. Moreover, if $\hat{\zeta} \in \Sigma$ and we set $\hat{\psi} = K\hat{\zeta} + \eta$, then $\hat{\psi}$ satisfies the following partial differential equation*

$$-\Delta \hat{\psi} = \phi \circ \hat{\psi} + h, \tag{3}$$

almost everywhere in $\mathbb{R}^3_+$, for some increasing function ϕ unknown a priori.

3. Preliminaries. In this section we present some lemmas which will be used in the proof of the Theorem.

LEMMA 1. *Suppose $\zeta \in L^p(\mathbb{R}^3_+)$ is a non-negative function with compact support. Then*

$$K\zeta(x) \leq c \parallel \zeta \parallel_p \ , \ \forall x \in \mathbb{R}^3_+. \tag{4}$$

Proof. We have

$$K\zeta(x) \leq \frac{1}{2\pi} \int_{\mathbb{R}^3_+} \frac{\zeta(y)}{\mid x-y \mid} \ dy \leq \frac{1}{2\pi} \int_{B_{r^*}(x)} \frac{\tilde{\zeta}(y)}{\mid x-y \mid} \ dy \ ,$$

where $\tilde{\zeta}$ is the Schwarz rearrangement of ζ with respect to x and $r^* = (\frac{3|supp(\zeta)|}{4\pi})^{\frac{1}{3}}$. The second inequality is a consequence of Hardy-Littlewood inequality [8]. Now by Hölder's inequality, we get (4), where

$$c = \frac{1}{2\pi} \left(\int_{B_{r^*}(x)} \frac{1}{\mid x-y \mid^{p\prime}} \ dy \right)^{\frac{1}{p\prime}} = \frac{2(3 \mid supp(\zeta) \mid)^{\frac{1}{p\prime}-\frac{1}{3}}}{(4\pi)^{\frac{2}{3}}(3-p\prime)^{\frac{1}{p\prime}}} \ , \tag{5}$$

and $p\prime$ is the conjugate exponent of p. □

Remark. The constant c evaluated in (5) is the constant used in (2).

LEMMA 2. *Let $q \geq 1$ and let U be a bounded open subset of $\mathbb{R}^3_+$. Then $K : L^p(U) \to L^q(U)$ is a compact linear operator. Moreover, if $\zeta \in L^p(\mathbb{R}^3_+)$ vanishes outside U, then $K\zeta \in W^{2,p}_{loc}(\mathbb{R}^3_+)$ and verifies*

$$-\Delta u = \zeta \ a.e \ in \ \mathbb{R}^3_+$$

and

$$\frac{\partial u}{\partial x_3} = 0 \ on \ \partial\mathbb{R}^3_+.$$

Proof. From Lemma 1, it readily follows that the map K from $L^p(U)$ into $L^q(U)$ is well defined. Notice that functions in $L^p(U)$ are interpreted as functions in $L^p(\mathbb{R}^3_+)$ which vanish outside U. Now consider $\zeta \in L^p(\mathbb{R}^3_+)$, which vanishes outside U. Then there exists a sequence $\{\zeta_n\}$ in $C^\infty_0(\mathbb{R}^3_+)$ such that $supp(\zeta_n) \subseteq U$, and $\zeta_n \to \zeta$ in $L^p(\mathbb{R}^3_+)$, as $n \to \infty$. We deduce from (4) that

$$\mid K(\zeta_n - \zeta)(x) \mid \leq c \parallel \zeta_n - \zeta \parallel_p .$$

Therefore, $K\zeta_n \to K\zeta$, uniformly in $\mathbb{R}^3_+$. Whence

$$\int (K\zeta_n)\phi \to \int (K\zeta)\phi \qquad \forall \phi \in C^\infty_0(\mathbb{R}^3_+). \tag{6}$$

On the other hand by [7, lemmas 4.1, 4.2], we have

$$-\Delta(K\zeta_n) = \zeta_n \ .$$

Thus by applying the Lebesgue dominated convergence theorem, we infer

$$\int -\Delta(K\zeta_n)\phi \to \int \zeta\phi. \tag{7}$$

Note that,

$$\int -\Delta(K\zeta_n)\phi = \int (K\zeta_n)(-\Delta\phi) \ .$$

From (6) we now deduce

$$\int -\Delta(K\zeta_n)\phi \to \int (K\zeta)(-\Delta\phi). \tag{8}$$

Hence, from (7) and (8), we find

$$\int \zeta\phi = \int (K\zeta)(-\Delta\phi) \ ,$$

so

$$-\Delta(K\zeta) = \zeta \qquad in \quad \mathcal{D}'(\mathbb{R}^3_+). \tag{9}$$

Now, by Agmon's regularity theory [2, theorem 6.1] we infer that $K\zeta \in W^{2,p}_{loc}(\mathbb{R}^3_+)$. Therefore equation (9) holds almost everywhere in $\mathbb{R}^3_+$. According to Sobolev embedding theorem [1] in order to show compactness of K it suffices to prove the boundedness of K as a map from $L^p(U)$ into $W^{1,3}(U)$. To do this, we first show

$$|\nabla K\zeta(x)| \leq M \parallel \zeta \parallel_p \qquad \forall x \in \mathbb{R}^3_+, \tag{10}$$

where M is a constant independent of x.
We begin with

$$\nabla K\zeta(x) = -\frac{1}{4\pi}\int_{\mathbb{R}^3_+}\left(\frac{x-y}{|\ x-y\ |^3} + \frac{x-\bar{y}}{|\ x-\bar{y}\ |^3}\right)\zeta(y)\ dy \ .$$

Therefore

$$|\nabla K\zeta(x)| \leq \frac{1}{4\pi}\int_{\mathbb{R}^3_+}\left(\frac{1}{|\ x-y\ |^2} + \frac{1}{|\ x-\bar{y}\ |^2}\right)\zeta(y)\ dy$$

$$\leq \frac{1}{2\pi}\int_{\mathbb{R}^3_+}\frac{1}{|\ x-y\ |^2}\zeta(y)\ dy$$

$$\leq \frac{1}{2\pi}\int_{B_{r^*}(x)}\frac{1}{|\ x-y\ |^2}\tilde{\zeta}(y)\ dy \ ,$$

where $\tilde{\zeta}$ and r^* are the same as in the proof of Lemma 1.
Now, by Hölder's inequality, we obtain (10), where

$$M = \frac{2(3\,|\,supp(\zeta)\,|)^{\frac{1}{p\prime}-\frac{2}{3}}}{(4\pi)^{\frac{1}{3}}(3-2p\prime)^{\frac{1}{p\prime}}}\ .$$

This implies that

$$\|\ \nabla K\zeta\ \|_{L^3(U)} \leq M\ \|\ \zeta\ \|_p\ |\,U\,|^{\frac{1}{3}}\ .$$

Also, from (4), we have

$$\|\ K\zeta\ \|_{L^3(U)} \leq c\ \|\ \zeta\ \|_p\ |\,U\,|^{\frac{1}{3}}\ .$$

Therefore

$$\|\ K\zeta\ \|_{W^{1,3}(U)} \leq \mathcal{C}\ \|\ \zeta\ \|_p\ .$$

So K is bounded as desired. Finally, by calculation, we have

$$\frac{\partial K\zeta}{\partial x_3} = 0 \qquad on \quad \partial\mathbb{R}^3_+,$$

as desired. □

The next lemma has been proved in [4].

Lemma 3. *If $\overline{\mathcal{F}(R)^\omega}$ denotes the weak closure of $\mathcal{F}(R)$ in $L^p(B_R)$, then $\overline{\mathcal{F}(R)^\omega}$ is convex and weakly sequentially compact.*

In order to prove the existence part of the Theorem, we first consider the following truncated variational problem (P_R):

$$\sup_{\zeta\in\mathcal{F}(R)} \ \Psi(\zeta).$$

We denote the solution set of (P_R) by Σ_R. We show that (P_R) is solvable. To do this we need the following result, which is a simple variation of [5, Lemma 2.15].

Lemma 4 *Let $g \in L^{p\prime}(B_R)$ and denote by $L_\alpha(g)$ the level set of g at height α; that is,*

$$L_\alpha(g) = \{x \in B_R : g(x) = \alpha\}.$$

Let $\mathcal{I} : L^p(B_R) \to \mathbb{R}$ be the linear functional defined by

$$\mathcal{I}(\zeta) = \int_{B_R} \zeta\, g\ .$$

If $\hat{\zeta}$ is a maximizer of $\mathcal{I}$ relative to $\overline{\mathcal{F}(R)^\omega}$ and if

$$|\,L_\alpha(g) \cap supp(\hat{\zeta})\,| = 0\ ,$$

for every $\alpha \in \mathbb{R}$, then $\hat{\zeta} \in \mathcal{F}(R)$ and

$$\hat{\zeta} = \phi_R \circ g\ ,$$

almost everywhere in B_R, for some increasing function ϕ_R.

Remark. In Lemma 4, by redefining $\hat{\zeta}$ on a set of zero measure on B_R, if necessary, we can make the conclusion to hold everywhere in B_R.

LEMMA 5. *The variational problem (P_R) is solvable. Moreover if $\hat{\zeta}_R \in \Sigma_R$, then*

$$\hat{\zeta}_R = \phi_R \circ (K\hat{\zeta}_R + \eta) ,$$

almost everywhere in B_R for some increasing function ϕ_R .

Proof. By Lemma 2 we have $-\Delta\eta = h$; hence using elliptic regularity theory it follows that $\eta \in W^{2,p}_{loc}(\mathbb{R}^3_+)$, thus $\eta \in C(\mathbb{R}^3_+)$, by the Sobolev embedding theorem. Note that Ψ is the summation of a quadratic and a linear functional; that is, $\Psi = Q + \mathcal{L}$. By Lemma 2, Q is weakly sequentially continuous. Also since η is continuous, it follows that $\mathcal{L}$ is also weakly sequentially continuous. This proves that Ψ is weakly sequentially continuous on $L^p(B_R)$. Since $\overline{\mathcal{F}(R)^\omega}$ is weakly sequentially compact, by Lemma 3, it follows that Ψ has a maximizer relative to $\overline{\mathcal{F}(R)^\omega}$, say $\tilde{\zeta}$. Fix $\zeta \in \overline{\mathcal{F}(R)^\omega}$, by convexity of $\overline{\mathcal{F}(R)^\omega}$, see Lemma 3, it follows that for any $t \in [0,1]$, $\tilde{\zeta} + t(\zeta - \tilde{\zeta}) \in \overline{\mathcal{F}(R)^\omega}$. Next using the first variation of Ψ at $\tilde{\zeta}$ we get

$$\Psi(\tilde{\zeta} + t(\zeta - \tilde{\zeta}) - \Psi(\tilde{\zeta}) = t < \Psi'(\tilde{\zeta}), \zeta - \tilde{\zeta} > + o(t) ,$$

as $t \to 0^+$; here $<\ ,\ >$ stands for the pairing between $L^p(B_R)$ and its dual, and $\Psi'(.)$ stands for the derivative. Since $\tilde{\zeta}$ is a maximizer we infer

$$< \Psi'(\tilde{\zeta}), \zeta - \tilde{\zeta} > \leq 0 .$$

Therefore $\tilde{\zeta}$ is a maximizer for the linear functional $< \Psi'(\tilde{\zeta}), . >$, relative to $\overline{\mathcal{F}(R)^\omega}$. Since $\Psi'(\tilde{\zeta})$ can be identified with $K\tilde{\zeta} + \eta \in L^{p'}(B_R)$, it follows that $\tilde{\zeta}$ is a maximizer of $\int_{B_R} \zeta(K\tilde{\zeta} + \eta)$ relative to $\zeta \in \overline{\mathcal{F}(R)^\omega}$. From Lemma 2 we obtain

$$-\Delta(K\tilde{\zeta} + \eta) = \tilde{\zeta} + h .$$

Thus the level sets of $K\tilde{\zeta} + \eta$ on $\mathrm{supp}(\tilde{\zeta})$ are negligible , by [7, Lemma 7.7]. Whence we can apply Lemma 4 to deduce that $\tilde{\zeta} \in \mathcal{F}(R)$ and

$$\tilde{\zeta} = \phi \circ (K\tilde{\zeta} + \eta) ,$$

almost everywhere in B_R for some increasing function ϕ; in particular $\tilde{\zeta} \in \Sigma_R$. Now consider $\hat{\zeta}_R \in \Sigma_R$. Since Ψ is weakly sequentially continuous, it follows that $\hat{\zeta}_R$ maximizes Ψ relative to $\overline{\mathcal{F}(R)^\omega}$. Next by applying the first variation argument above we can similarly prove existence of an increasing function ϕ_R such that

$$\hat{\zeta}_R = \phi_R \circ (K\hat{\zeta}_R + \eta) ,$$

almost everywhere in B_R. □

LEMMA 6. *Let $\gamma = \int_{\partial\mathbb{R}^3_+} h$. Then $\gamma > 4\pi c\, r_h$.*

Proof. By the hypotheses on r_h and h, that's (1) and (2), we have

$$\gamma = \int\int_{x_1^2+x_2^2 \leq r_h^2} h(x_1, x_2)\ dx_1 dx_2 \geq c \int\int_{x_1^2+x_2^2 \leq r_h^2} \ln \mid x_1 x_2 \mid\ \ dx_1 dx_2$$

$$= c\int_0^{2\pi}\int_0^{r_h} r\ln r^2 \mid \sin\theta\cos\theta \mid drd\theta$$

$$= c\int_0^{2\pi} r_h^2\ln r_h - \frac{1}{2}r_h^2 + \frac{1}{2}r_h^2(-\ln 2 + \ln\mid\sin 2\theta\mid)d\theta$$

$$= 2c\pi r_h^2(\ln r_h - \frac{1}{2}) - c\pi r_h^2\ln 2 + \frac{c}{2}r_h^2\int_0^{2\pi}\ln\mid\sin 2\theta\mid d\theta$$

$$= 2c\pi r_h^2\ln\frac{r_h}{\sqrt{2e}} + \frac{c}{2}r_h^2\int_0^{2\pi}\ln\mid\sin\theta\mid d\theta = 2c\pi r_h^2\ln\frac{r_h}{\sqrt{2e}}$$

$$+\frac{c}{2}r_h^2\left(\int_0^{\frac{\pi}{2}}\ln\sin\theta d\theta + \int_{\frac{\pi}{2}}^{\pi}\ln\sin\theta d\theta + \int_{\pi}^{\frac{3\pi}{2}}\ln(-\sin\theta)d\theta + \int_{\frac{3\pi}{2}}^{2\pi}\ln(-\sin\theta)d\theta\right)$$

$$= 2c\pi r_h^2\ln\frac{r_h}{\sqrt{2e}} + \frac{c}{2}r_h^2\left(4\int_0^{\frac{\pi}{2}}\ln\sin\theta d\theta\right) = 2c\pi r_h^2\ln\frac{r_h}{\sqrt{2e}}$$

$$+\frac{c}{2}r_h^2(-2\pi\ln 2) = 2c\pi r_h^2\ln\frac{r_h}{2\sqrt{e}} > 4\pi c\, r_h.$$

□

Lemma 7. *Let $R > r_h$ and $\hat{\zeta}_R \in \Sigma_R$, then $supp(\hat{\zeta}_R) \subseteq B_{r_h}$, modulo a set of zero measure.*

Proof. Suppose the assertion is false. Then there exist sequences $\{R_n\}$, $\{x_n\}$ and $\{\hat{\zeta}_{R_n}\} := \{\hat{\zeta}_n\}$ such that
(1) $R_n \to \infty$
(2) $\hat{\zeta}_n \in \Sigma_{R_n}$
(3) $x_n \in den(supp(\hat{\zeta}_n))$ and $\parallel x_n \parallel_{\mathbb{R}^3_+} \to \infty$, where $\parallel . \parallel_{\mathbb{R}^3_+}$ denote the usual Euclidean norm in $\mathbb{R}^3_+$. Without loss of generality we may assume that $\parallel x_n \parallel_{\mathbb{R}^3_+} = R_n$ and $\{R_n\}$ is increasing; moreover we may assume that $R_n > r_h$. Let us set $\psi_n := K\hat{\zeta}_n + \eta$, and estimate $K\hat{\zeta}_n(x_n)$:

$$K\hat{\zeta}_n(x_n) \le \frac{1}{2\pi}\int_{\mathbb{R}^3_+}\frac{1}{\mid x_n - y\mid}\hat{\zeta}_n(y)\; dy = \frac{1}{2\pi}\int_{B_{r_h}}\frac{1}{\mid x_n - y\mid}\hat{\zeta}_n(y)\; dy +$$

$$\frac{1}{2\pi}\int_{\mathbb{R}^3_+ - B_{r_h}}\frac{1}{\mid x_n - y\mid}\hat{\zeta}_n(y)\; dy \le \frac{1}{2\pi}\parallel\hat{\zeta}_n\parallel_{1,B_{r_h}}\frac{1}{R_n - r_h} + c\,. \tag{11}$$

Now we estimate $\eta(x_n)$:

$$\eta(x_n) = \frac{1}{2\pi}\int_{\partial\mathbb{R}^3_+}\frac{1}{\mid x_n - y\mid}h(y)\; d\sigma(y) \le \frac{\gamma}{2\pi}\frac{1}{R_n - r_h}\,. \tag{12}$$

From (11) and (12) we obtain

$$\psi_n(x_n) \le c + \frac{1}{2\pi} \parallel \hat{\zeta}_n \parallel_{1,B_{r_h}} \frac{1}{R_n - r_h} + \frac{\gamma}{2\pi} \frac{1}{R_n - r_h} \,. \tag{13}$$

Notice that since $r_h > a$, we can find a sequence $\{y_n\}$ in B_{r_h} such that $y_n \notin den(supp(\hat{\zeta}_n))$. Now, we estimate $\psi_n(y_n)$ from below.

$$K\hat{\zeta}_n(y_n) \ge \frac{1}{4\pi} \int_{\mathbb{R}^3_+} \frac{1}{|y_n - y|} \hat{\zeta}_n(y)\, dy = \frac{1}{4\pi} \int_{B_{r_h}} \frac{1}{|y_n - y|} \hat{\zeta}_n(y)\, dy +$$

$$\frac{1}{4\pi} \int_{\mathbb{R}^3_+ - B_{r_h}} \frac{1}{|y_n - y|} \hat{\zeta}_n(y)\, dy \ge \frac{1}{4\pi} \parallel \hat{\zeta}_n \parallel_{1,B_{r_h}} \frac{1}{2r_h} +$$

$$\frac{1}{4\pi} (\parallel \zeta_0 \parallel_1 - \parallel \hat{\zeta}_n \parallel_{1,B_{r_h}}) \frac{1}{R_n + r_h} \,. \tag{14}$$

Also,

$$\eta(y_n) = \frac{1}{2\pi} \int_{\partial\mathbb{R}^3_+} \frac{1}{|y_n - y|} h(y)\, d\sigma(y) \ge \frac{\gamma}{2\pi} \frac{1}{2r_h} \,. \tag{15}$$

Therefore, from (14) and (15), we have

$$\psi_n(y_n) \ge \frac{1}{4\pi} \parallel \hat{\zeta}_n \parallel_{1,B_{r_h}} \frac{1}{2r_h} +$$

$$\frac{1}{4\pi} (\parallel \zeta_0 \parallel_1 - \parallel \hat{\zeta}_n \parallel_{1,B_{r_h}}) \frac{1}{R_n + r_h} + \frac{\gamma}{2\pi} \frac{1}{2r_h} \,. \tag{16}$$

Therefore, from (13)and (16) we drive

$$\psi_n(x_n) - \psi_n(y_n) \le c + \frac{1}{2\pi} \parallel \hat{\zeta}_n \parallel_{1,B_{r_h}} \frac{1}{R_n - r_h} + \frac{\gamma}{2\pi} \frac{1}{R_n - r_h}$$

$$-\frac{1}{4\pi} \parallel \hat{\zeta}_n \parallel_{1,B_{r_h}} \frac{1}{2r_h} - \frac{1}{4\pi} (\parallel \zeta_0 \parallel_1 - \parallel \hat{\zeta}_n \parallel_{1,B_{r_h}}) \frac{1}{R_n + r_h} - \frac{\gamma}{2\pi} \frac{1}{2r_h} \,.$$

Thus

$$\limsup_{n\to\infty} (\psi_n(x_n) - \psi_n(y_n)) \le c - \frac{\gamma}{4\pi r_h} < 0 \,. \tag{17}$$

Where in the last inequality we have used Lemma 6. From (17)we infer existence of $n_0 \in \mathbb{N}$ for which

$$\psi_{n_0}(x_{n_0}) - \psi_{n_0}(y_{n_0}) < 0 \,. \tag{18}$$

However, from Lemma 4, and the Remark following it, there exists ϕ_{n_0} , an increasing function, such that

$$\hat{\zeta}_{n_0} = \phi_{n_0} \circ \psi_{n_0} \,,$$

everywhere in $B_{R_{n_0}}$. Therefore, ψ_{n_0} attains its largest values over $den(supp(\hat{\zeta}_{n_0}))$, so (18) is false. Hence we are done. □

Remark. From Lemma 7 it readily follows that $\Sigma = \Sigma_{r_h}$.

4. Proof of Theorem. The existence part of the Theorem follows from Lemma 7 and the remark following it. Now consider $\hat{\zeta} \in \Sigma$. Since $\hat{\zeta} \in \Sigma_{r_h}$, it follows that

$$\hat{\zeta} = \phi_{r_h} \circ \psi, \tag{19}$$

almost everywhere in B_{r_h}, where $\psi = K\hat{\zeta} + \eta$, thanks to Lemma 5. Note that to derive (3) we only need to modify ϕ_{r_h} in order to have a similar functional equation as (19) to hold throughout $\mathbb{R}^3_+$. Since ϕ_{r_h} is an increasing function, we obtain

$$supp(\hat{\zeta}) = \{x \in B_{r_h} :\ \psi \geq \lambda\} , \tag{20}$$

modulo a set of zero measure, where λ is a positive constant. On the other hand for $|x| \geq 2r_h$, we derive the following estimate

$$\psi(x) \leq \frac{\|\hat{\zeta}\|_1 + \gamma}{\pi \mid x \mid} .$$

Thus, there exists $R' > r_h$ such that

$$\psi(z) < \frac{\lambda}{2} , \tag{21}$$

provided $z \in \mathbb{R}^3_+ - B_{R'}$. Finally, since $\hat{\zeta} \in \Sigma_{R'}$ we can apply Lemma 5 once again to deduce the existence of another increasing function, say ϕ', such that

$$\hat{\zeta} = \phi' \circ \psi , \tag{22}$$

almost everywhere in $B_{R'}$. We now define

$$\phi(t) = \begin{cases} \phi'(t) & \text{if } t \geq \lambda \\ 0 & \text{if } t < \lambda \end{cases}$$

Therefore by applying (20), (21) and (22) we obtain $\hat{\zeta} = \phi \circ \psi$, almost everywhere in $\mathbb{R}^3_+$, as desired. Now using Lemma 2 and the fact that

$$-\Delta\eta = h ,$$

almost everywhere in $\mathbb{R}^3_+$, we derive (3). This completes the proof of the Theorem. □

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