BLOW-UP RESULTS FOR A REACTION-DIFFUSION SYSTEM[*]

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Abstract. We consider the Cauchy problem for the reaction-diffusion system with the nonlinear terms $|x|^{\sigma_j} u^{p_j} v^{q_j}$. In this system, the exponents p_1 and q_2 play a crucial role to determine the behavior of the solutions. Using an ODE method, we prove the Fujita-type nonexistence results for $p_1, q_2 < 1$, for $q_2 < 1 < p_1$ or for $p_1, q_2 > 1$. Moreover, we also show the nonexistence results for large initial data.

Key words. blow-up, reaction-diffusion system, Cauchy problem.

AMS subject classifications. Primary 35K57; Secondary 35B33, 35K05, 35K45

1. Introduction. We consider the Cauchy problem for the reaction-diffusion system:

$$(1.1) \qquad u_t - \Delta u = |x|^{\sigma_1} u^{p_1} v^{q_1}, \qquad\qquad x \in \mathbf{R}^N, \ \ t > 0,$$

$$(1.2) \qquad v_t - \Delta v = |x|^{\sigma_2} u^{p_2} v^{q_2}, \qquad\qquad x \in \mathbf{R}^N, \ \ t > 0,$$

$$u(x,0) = u_0(x) \geq 0, \not\equiv 0, \qquad\qquad x \in \mathbf{R}^N,$$

$$v(x,0) = v_0(x) \geq 0, \not\equiv 0, \qquad\qquad x \in \mathbf{R}^N,$$

where $p_j, q_j \geq 0$, $\sigma_j > \max(-2, -N)$ $(j = 1, 2)$, and $p_1, q_2 \neq 1$.

There are some papers on the Cauchy problem for semilinear reaction-diffusion systems. In [2], Escobedo and Herrero proved the existence and nonexistence of global solutions, so-called the Fujita-type result, for $\sigma_1 = \sigma_2 = p_1 = q_2 = 0$, $p_2, q_1 \geq 1$, $p_2 q_1 > 1$. As an extension of [2], Mochizuki and Huang [4] showed the Fujita-type result for $p_1 = q_2 = 0$, $0 \leq \sigma_1 < N(p_2 - 1)$, $0 \leq \sigma_2 < N(q_1 - 1)$, $p_2, q_1 \geq 1$, $p_2 q_1 > 1$. Both of the results show that the interaction between the unknown functions in the nonlinear terms determines the behavior of solutions of the system.

In [3], Escobedo and Levine showed an interesting result for $\sigma_1 = \sigma_2 = 0$, p_1, p_2, q_1, $q_2 \geq 0$. Under the assumption that $p_2 + q_2 \geq p_1 + q_1 > 0$, they showed that if $p_1 > 1$, the solutions of the system behave like a solution of the single equation $u_t - \Delta u = u^{p_1 + q_1}$.

Our aim of this paper is to show the conditions for the nonexistence of global solutions of the system (1.1) and (1.2) in three cases $p_1, q_2 < 1$, $q_2 < 1 < p_1$, or $p_1, q_2 > 1$. The conditions are about the relation between the exponents p_j, q_j, σ_j, and the initial data. See Theorems 2.1-2.3 in the next section. Comparing each part (i) in the theorems with the results in [1], we see that our conditions are optimal because the authors in [1] have proved the following results:
(i) Let $p_1 < 1$, $q_2 < 1$ and $p_2 q_1 - (1 - p_1)(1 - q_2) > 0$. If $\alpha < N/2$ and $\beta < N/2$, then global solutions exist for small initial data.
(ii) Let $p_1 > 1$ and $q_2 < 1$. If $\alpha < N/2$ and $p_1 + q_1 > 1 + (2 + \sigma_1)/N$, then global solutions exist for small initial data.
(iii) Let $p_1 > 1$ and $q_2 > 1$. If $p_1 + q_1 > 1 + (2 + \sigma_1)/N$ and $p_2 + q_2 > 1 + (2 + \sigma_2)/N$, then global solutions exist for small initial data.

(iv) Let $p_1 < 1$, $q_2 < 1$ and $p_2 q_1 - (1 - p_1)(1 - q_2) < 0$. Then all nonnegative solutions are global.

Since this problem includes the sublinear case, p_j or $q_j < 1$, the authors in [1] show the global existence by iteration argument in weighted L^∞ function space.

Moreover, the same result as [3] holds in our problem, that is, if $p_1 > 1$, the solutions of the system behave like a solution of the single equation $u_t - \Delta u = |x|^{\sigma_1} u^{p_1 + q_1}$ under the assumption that $(p_2 + q_2 - 1)/(\sigma_2 + 2) \geq (p_1 + q_1 - 1)/(\sigma_1 + 2)$.

The iteration method of [3] is often used to show blow up for reaction-diffusion systems. However, the method does not seem applicable for our problem because the nonlinear terms have the variable coefficients $|x|^{\sigma_j}$. In this paper, we improve the argument in [4] and apply it to our problem. The argument in [4] is to transform the system of PDEs into the ordinary differential inequalities. In our problem, multiplying the equation by negative power of unknown function makes the transformation possible.

Our plan of this paper is as follows. In Section 2, we state main theorem and some notation. In Section 3, we prepare several pointwise estimates for solutions. These estimates are obtained from the system of integral equations associated to (1.1) and (1.2). In Sections 4 and 5, we prove the nonexistence results in the case $\max(p_1, q_2) < 1$ and the case $\max(p_1, q_2) > 1$, respectively. Because the interaction between u and v in the former case is stronger than that in the latter, we employ a system of ordinary differential inequalities. On the other hand, because self-growth of the solution in the latter case is stronger than that in the former, we employ a single ordinary differential inequality. In both of the cases, we first show that functions used in the differential inequality have upper bounds under the assumption that the global solutions exist. Next, we show lower bounds from the estimates in Section 3. This contradicts the upper bounds. Hence, the nonexistence of global solutions is shown. In Appendix, we introduce a comparison principle used in the proofs and a local existence result for the associated system of integral equations.

Remark 1.1. In [2], [3], [4] and [5], the authors show that the solution blows up in critical case. This critical blow-up also occurs in our system (1.1)-(1.2), which is shown in Sections 4 and 5.

2. Main results. For simplicity, let

$$
(2.1) \quad
\begin{cases}
\alpha = \dfrac{q_1(\sigma_2 + 2) + (1 - q_2)(\sigma_1 + 2)}{2\{p_2 q_1 - (1 - p_1)(1 - q_2)\}}, \\[2ex]
\beta = \dfrac{p_2(\sigma_1 + 2) + (1 - p_1)(\sigma_2 + 2)}{2\{p_2 q_1 - (1 - p_1)(1 - q_2)\}},
\end{cases}
$$

$$
(2.2) \quad
\begin{cases}
\delta_1 = \dfrac{q_1 \sigma_2 + (1 - q_2)\sigma_1}{p_2 q_1 - (1 - p_1)(1 - q_2)}, \\[2ex]
\delta_2 = \dfrac{p_2 \sigma_1 + (1 - p_1)\sigma_2}{p_2 q_1 - (1 - p_1)(1 - q_2)}.
\end{cases}
$$

For $a \in \mathbf{R}$, we define the function spaces:

$$
I^a = \{w \in C(\mathbf{R}^N); w(x) \geq 0, \ \limsup_{|x| \to \infty} |x|^a w(x) < \infty\},
$$

$$
I_a = \{w \in C(\mathbf{R}^N); w(x) \geq 0, \ \liminf_{|x| \to \infty} |x|^a w(x) > 0\},
$$

and

$$L_a^\infty = \{w \text{ is measurable function on } \boldsymbol{R}^N;$$
$$w(x) \geq 0, \ \|w\|_{\infty,a} \equiv \sup_{x \in \boldsymbol{R}^N} \langle x \rangle^a \, w(x) < \infty\},$$

where $\langle x \rangle = \left(1 + |x|^2\right)^{1/2}$. We also define

$$E_T = \{(u,v); \ [0,T] \to L_{\delta_1}^\infty \times L_{\delta_2}^\infty, \ \|(u,v)\|_{E_T} < \infty\},$$

where

$$\|(u,v)\|_{E_T} = \sup_{t \in [0,T]} (\|u(t)\|_{\infty,\delta_1} + \|v(t)\|_{\infty,\delta_2}).$$

Now, we state our main results. Throughout this paper, we assume that the initial data $(u_0, v_0) \in I^{\delta_1} \times I^{\delta_2}$.

THEOREM 2.1. *Let* $p_1 < 1$, $q_2 < 1$ *and* $p_2 q_1 - (1 - p_1)(1 - q_2) > 0$.
(i) *If* $\max(\alpha, \beta) \geq N/2$, *then no nontrivial global solutions exist.*
(ii) *If* $u_0 \in I_a$ $(a < 2\alpha)$ *or* $v_0 \in I_b$ $(b < 2\beta)$, *then no global solutions exist.*
(iii) *For any* $\nu > 0$, *there exists large* $C > 0$ *such that no global solutions with* $u_0(x) \geq C \exp(-\nu|x|^2)$ *exist.*

THEOREM 2.2. *Let* $p_1 > 1$, $q_2 < 1$.
(i) *If* $\alpha \geq N/2$ *or* $p_1 + q_1 \leq 1 + (2 + \sigma_1)/N$, *then no nontrivial global solutions exist.*
(ii) *If* $u_0 \in I_a$ $(a < \max((\sigma_1 + 2 - Nq_1)/(p_1 - 1),$
$-\{q_1(\sigma_2 + 2) + (1 - q_2)(\sigma_1 + 2) - p_2 q_1 N\}/\{(1 - p_1)(1 - q_2)\})$, *then no global solutions exist.*
(iii) *For any* $\nu > 0$, *there exists large* $C > 0$ *such that no global solutions with* $u_0(x) \geq C \exp(-\nu|x|^2)$ *exist.*

THEOREM 2.3. *Let* $p_1 > 1$, $q_2 > 1$.
(i) *If* $p_1 + q_1 \leq 1 + (2 + \sigma_1)/N$ *or* $p_2 + q_2 \leq 1 + (2 + \sigma_2)/N$, *then no nontrivial global solutions exist.*
(ii) *If* $u_0 \in I_a$ $(a < (\sigma_1 + 2 - Nq_1)/(p_1 - 1))$ *or* $v_0 \in I_b$ $(b < (\sigma_2 + 2 - Np_2)/(q_2 - 1))$, *then no global solution exist.*
(iii) *For any* $\nu > 0$, *there exists large* $C > 0$ *such that no global solutions with* $u_0(x) \geq C \exp(-\nu|x|^2)$ *exist.*

Remark 2.4. Each part (i) in Theorems 2.1-2.3 is so-called the Fujita-type nonexistence result. Each (ii) and (iii) are for the initial data with bad decay and for large initial data, respectively.

We can also rewrite the theorems into the way in Escobedo-Levine [3].

COROLLARY 2.5. *Assume that*

(2.3) $$\frac{p_1 + q_1 - 1}{\sigma_1 + 2} \leq \frac{p_2 + q_2 - 1}{\sigma_2 + 2},$$

and let $p_1 < 1$, $q_2 \neq 1$.
(i) *If* $\max(\alpha, \beta) \geq N/2$, *then no nontrivial global solutions exist.*
(ii) *If* $0 < \max(\alpha, \beta) < N/2$, *then no global solutions exist for large data.*

COROLLARY 2.6. *Assume* (2.3), *and let* $p_1 > 1$, $q_2 \neq 1$.
(i) *If* $p_1 + q_1 \leq 1 + (2 + \sigma_1)/N$, *then no nontrivial global solutions exist.*
(ii) *If* $p_1 + q_1 > 1 + (2 + \sigma_1)/N$, *then no global solutions exist for large data.*

3. Key estimates. In this section, we prepare several estimates for the solutions. To show them, we introduce the system of integral equations associated to (1.1) and (1.2):

$$(3.1) \qquad u(t) = S(t)u_0 + \int_0^t S(t-s)| \cdot |^{\sigma_1} u(s)^{p_1} v(s)^{q_1} ds,$$

$$(3.2) \qquad v(t) = S(t)v_0 + \int_0^t S(t-s)| \cdot |^{\sigma_2} u(s)^{p_2} v(s)^{q_2} ds,$$

where

$$S(t)f(x) = (4\pi t)^{-\frac{N}{2}} \int_{\boldsymbol{R}^N} \exp\left(-\frac{|x-y|^2}{4t}\right) f(y) dy.$$

The following lemma is a well-known estimate for the heat equations.

LEMMA 3.1. *Let* u *and* v *be solutions of the system* (1.1) *and* (1.2). *There exists* $C > 0$ *such that*

$$u(x,t) \geq C(1+t)^{-\frac{N}{2}} \exp\left(-\frac{|x|^2}{2t}\right), \qquad\qquad (t > 0),$$

$$v(x,t) \geq C(1+t)^{-\frac{N}{2}} \exp\left(-\frac{|x|^2}{2t}\right), \qquad\qquad (t > 0).$$

Moreover, we can add logarithmic growth to the bounds in the critical case.

LEMMA 3.2. *(*[3]*) Let* u *and* v *be solutions of the system* (1.1) *and* (1.2). *Assume that*

$$u(x,t) \geq C_1(1+t)^{-\frac{N}{2}} \exp\left(-\frac{|x|^2}{t}\right), \qquad\qquad (t > 0),$$

$$v(x,t) \geq C_2(1+t)^{m} \exp\left(-\frac{C_3|x|^2}{t}\right), \qquad\qquad (t > t_0),$$

where C_1, C_2, $C_3 > 0$, $t_0 \geq 0$ *and* $m \in \boldsymbol{R}$. *If* m *and* σ_1 *satisfy*

$$-\frac{Np_1}{2} + mq_1 + \frac{\sigma_1 + 2}{2} = -\frac{N}{2}, \quad \sigma_1 > \max(-2, -N),$$

then there exist constants C_4, $C_5 > 0$ *and* $t_1 > t_0$ *such that*

$$u(x,t) \geq C_4(1+t)^{-\frac{N}{2}} \log(1+t) \exp\left(-\frac{C_5|x|^2}{t}\right), \qquad\qquad (t > t_1).$$

Proof. See Proposition 1 in [3]. ∎

The following two lemmas are for the sublinear case.

LEMMA 3.3. *Let $0 \leq q_2 < 1$, $\sigma_2 > \max(-2, -N)$ and define*

$$\bar{v}(x,t) = \widetilde{C} t^{\frac{\sigma_2+2}{2(1-q_2)}} \left(S(t)u_0(x)^\varepsilon\right)^{\frac{p_2}{\varepsilon(1-q_2)}}.$$

for $\widetilde{C}, \varepsilon > 0$. If $\widetilde{C}$ and ε are sufficiently small, then $\bar{v}(x,t)$ is a subsolution for the problem:

$$v_t - \Delta v = |x|^{\sigma_2} u^{p_2} v^{q_2}, \qquad\qquad x \in \mathbf{R}^N, \ \ t > 0,$$
$$v(x,0) = v_0(x), \qquad\qquad x \in \mathbf{R}^N.$$

Proof. Let $k > \max\{(\sigma_2 + N)/N, 1\}$ and $0 < \varepsilon < \min(1, p_2/\{(1-q_2)k\})$. It suffices to prove that

$$\bar{v}(x,t) \leq \int_0^t S(t-s)|x|^{\sigma_2} \left(S(s)u_0(x)\right)^{p_2} \bar{v}(x,s)^{q_2} ds.$$

By Jensen's inequality, we have

$$\int_0^t S(t-s)|x|^{\sigma_2} \left(S(s)u_0(x)\right)^{p_2} \bar{v}(x,s)^{q_2} ds$$

(3.3)
$$\geq \widetilde{C}^{q_2} \int_0^t s^{\frac{q_2(\sigma_2+2)}{2(1-q_2)}} S(t-s)|x|^{\sigma_2} \left(S(s)u_0(x)^\varepsilon\right)^{\frac{p_2}{\varepsilon(1-q_2)}} ds.$$

Using the inverse Hölder inequality and Jensen's inequality again, we have for $k > 1$,

$$S(t-s)|x|^{\sigma_2} \left(S(s)u_0(x)^\varepsilon\right)^{\frac{p_2}{\varepsilon(1-q_2)}}$$
$$\geq \{S(t-s)|x|^{\frac{\sigma_2}{1-k}}\}^{1-k}\{S(t-s)\left(S(s)u_0(x)^\varepsilon\right)^{\frac{p_2}{k\varepsilon(1-q_2)}}\}^k$$
$$\geq \{C_1(t-s)^{\frac{\sigma_2}{2(1-k)}}\}^{1-k}\{S(t-s)\left(S(s)u_0(x)^\varepsilon\right)\}^{\frac{p_2}{\varepsilon(1-q_2)}}$$
(3.4)
$$= C_1^{1-k}(t-s)^{\frac{\sigma_2}{2}} \left(S(t)u_0(x)^\varepsilon\right)^{\frac{p_2}{\varepsilon(1-q_2)}}.$$

Substituting (3.4) into (3.3), we obtain

$$\int_0^t S(t-s)|x|^{\sigma_2} \left(S(s)u_0(x)\right)^{p_2} \bar{v}(x,s)^{q_2} ds$$
$$\geq \widetilde{C}^{q_2} C_1^{1-k} \left(S(t)u_0(x)^\varepsilon\right)^{\frac{p_2}{\varepsilon(1-q_2)}} \int_0^t s^{\frac{q_2(\sigma_2+2)}{2(1-q_2)}} (t-s)^{\frac{\sigma_2}{2}} ds$$
$$\geq \widetilde{C}^{q_2} C_1^{1-k} C_2 t^{\frac{\sigma_2+2}{2(1-q_2)}} \left(S(t)u_0(x)^\varepsilon\right)^{\frac{p_2}{\varepsilon(1-q_2)}}$$
$$= \widetilde{C}^{q_2-1} C_1^{1-k} C_2 \bar{v}(x,t)$$
$$\geq \bar{v}(x,t)$$

for sufficiently small $\widetilde{C} > 0$. This completes the proof. $\square$

LEMMA 3.4. *Let $0 \leq q_2 < 1$, and $\sigma_2 > (-2, -N)$ and let u and v be solutions of the system (1.1) and (1.2). Then there exist constants C_1, $C_2 > 0$ such that*

$$v(x,t) \geq C_1 t^{\frac{\sigma_2+2}{2(1-q_2)}} (1+t)^{-\frac{p_2 N}{2(1-q_2)}} \exp\left(-\frac{C_2|x|^2}{t}\right), \qquad\qquad (t > 0).$$

Proof. Fix arbitrary $s > 0$, and apply Lemma 3.3 to $U(t) = u(t + s)$ and $V(t) = v(t + s)$. Then, we have

$$V(x, t) \geq Ct^{\frac{\sigma_2+2}{2(1-q_2)}} (S(t)U(x, 0)^\varepsilon)^{\frac{p_2}{\varepsilon(1-q_2)}}.$$

Putting $s = t$ and using Lemma 3.1, we obtain

$$v(x, 2t) \geq Ct^{\frac{\sigma_2+2}{2(1-q_2)}} (S(t)u(x, t)^\varepsilon)^{\frac{p_2}{\varepsilon(1-q_2)}}$$

$$\geq Ct^{\frac{\sigma_2+2}{2(1-q_2)}}(1 + t)^{-\frac{p_2 N}{2(1-q_2)}} \left\{ (4\pi t)^{-\frac{N}{2}} \int \exp\left(-\frac{|x - y|^2}{4t} - \frac{\varepsilon|y|^2}{2t}\right) dy \right\}^{\frac{p_2}{\varepsilon(1-q_2)}}$$

$$\geq Ct^{\frac{\sigma_2+2}{2(1-q_2)}}(1 + t)^{-\frac{p_2 N}{2(1-q_2)}} \exp\left(-\frac{C|x|^2}{t}\right).$$

This completes the proof. $\square$

4. Proof of Theorem 2.1. NECESSARY CONDITION FOR THE GLOBAL EXISTENCE   Assume that (u, v) are global solutions for (1.1) and (1.2). Since $p_1 < 1$, $q_2 < 1$ and $p_2 q_1 - (1 - p_1)(1 - q_2) > 0$, we can take a positive constant $k > 0$ such that $(1 - q_2)/p_2 < k < q_1/(1 - p_1)$. For this k, fix positive constants $r_1, r_2 > 0$ satisfying

$$r_2 = kr_1,$$
$$r_1 < \min\{1 - p_1, \ p_2\},$$
$$r_2 < \min\{1 - q_2, \ q_1\},$$
$$r_1 \sigma_1 < \frac{N(q_1 - k(1 - p_1))}{k},$$
$$r_2 \sigma_2 < \frac{N(kp_2 - (1 - q_2))}{k}.$$

For $\varepsilon > 0$, define the cut off function

$$\rho_\varepsilon(x) = \begin{cases} \varepsilon^{\frac{N}{2}} \exp\left(-\dfrac{1}{1 - \varepsilon|x|^2}\right) & (|x| < \varepsilon^{-\frac{1}{2}}) \\ 0 & (|x| \geq \varepsilon^{-\frac{1}{2}}), \end{cases}$$

and set

$$(4.1) \qquad\qquad F_\varepsilon(t) = \int_{\mathbf{R}^N} u(x, t)^{r_1} \rho_\varepsilon(x) dx,$$

$$(4.2) \qquad\qquad G_\varepsilon(t) = \int_{\mathbf{R}^N} v(x, t)^{r_2} \rho_\varepsilon(x) dx.$$

Then the following inequalities hold.

LEMMA 4.1. *Let $p_1 < 1$, $q_2 < 1$ and $\sigma_j > -N$ $(j = 1, 2)$. Then there exist constants $C_1, C_2, C_3, C_4 > 0$ such that*

$$(4.3) \qquad F'_\varepsilon(t) \geq -C_1 \varepsilon F_\varepsilon(t) + C_2 \varepsilon^{-\frac{\sigma_1}{2}} F_\varepsilon(t)^{-\frac{(1-p_1)-r_1}{r_1}} G_\varepsilon(t)^{\frac{q_1}{r_2}},$$

$$(4.4) \qquad G'_\varepsilon(t) \geq -C_3 \varepsilon G_\varepsilon(t) + C_4 \varepsilon^{-\frac{\sigma_2}{2}} F_\varepsilon(t)^{\frac{p_2}{r_1}} G_\varepsilon(t)^{-\frac{(1-q_2)-r_2}{r_2}}.$$

Proof. Multiplying (1.1) by $u^{r_1-1}\rho_\varepsilon$, and integrating over $\mathbf{R}^N$ with respect to x, we obtain the desired inequality (4.3). Indeed, integration by parts implies that

$$\int_{\mathbf{R}^N} \rho_\varepsilon u^{r_1-1} u_t dx = \frac{1}{r_1}\frac{d}{dt}F_\varepsilon(t),$$

$$\int_{\mathbf{R}^N} \rho_\varepsilon u^{r_1-1}\Delta u dx \geq -\int_{\mathbf{R}^N} \nabla\rho_\varepsilon \cdot u^{r_1-1}\nabla u dx$$

$$= -\frac{1}{r_1}\int_{\mathbf{R}^N} \nabla\rho_\varepsilon \cdot \nabla(u^{r_1}) dx$$

$$= \frac{1}{r_1}\int_{\mathbf{R}^N} u^{r_1}\Delta\rho_\varepsilon dx$$

$$\geq -\frac{C\varepsilon}{r_1}F_\varepsilon(t).$$

Here, we have used the property of ρ_ε that there exists a constant $C > 0$ depending only on N such that $\Delta\rho_\varepsilon \geq -C\varepsilon\rho_\varepsilon$. The normal and inverse Hölder inequalities also imply that

$$\int_{\mathbf{R}^N} \rho_\varepsilon |x|^{\sigma_1} u^{r_1-(1-p_1)} v^{q_1} dx$$

$$\geq \left(\int_{|x|<\varepsilon^{-\frac{1}{2}}} \rho_\varepsilon v^{r_2} dx\right)^{\frac{q_1}{r_2}} \left(\int_{|x|<\varepsilon^{-\frac{1}{2}}} \rho_\varepsilon |x|^{\frac{r_2\sigma_1}{r_2-q_1}} u^{\frac{r_2(r_1-(1-p_1))}{r_2-q_1}} dx\right)^{\frac{r_2-q_1}{r_2}}$$

$$\geq G_\varepsilon^{\frac{q_1}{r_2}} \left(\int_{|x|<\varepsilon^{-\frac{1}{2}}} \rho_\varepsilon u^{r_1} dx\right)^{\frac{r_1-(1-p_1)}{r_1}} \left(\int_{|x|<\varepsilon^{-\frac{1}{2}}} \rho_\varepsilon |x|^{-\frac{r_1r_2\sigma_1}{r_1q_1-r_2(1-p_1)}} dx\right)^{-\frac{r_1q_1-r_2(1-p_1)}{r_1r_2}}$$

$$= C\varepsilon^{-\frac{\sigma_1}{2}} F_\varepsilon(t)^{-\frac{(1-p_1)-r_1}{r_1}} G_\varepsilon(t)^{\frac{q_1}{r_2}}.$$

Multiplying (1.2) by $v^{r_2-1}\rho_\varepsilon$, and integrating over $\mathbf{R}^N$ with respect to x, we can also get (4.4). ☐

Setting

$$\widetilde{F_\varepsilon}(t) = F_\varepsilon^{\frac{1-p_1}{r_1}}(t),$$

$$\widetilde{G_\varepsilon}(t) = G_\varepsilon^{\frac{1-q_2}{r_2}}(t),$$

we simplify the inequalities (4.3) and (4.4).

LEMMA 4.2. *Let* $p_1 < 1$, $q_2 < 1$ *and* $\sigma_j > -N$ $(j = 1, 2)$. *Then there exist constants* C_5, C_6, C_7, $C_8 > 0$ *such that*

$$\widetilde{F_\varepsilon}'(t) \geq -C_5\varepsilon\widetilde{F_\varepsilon}(t) + C_6\varepsilon^{-\frac{\sigma_1}{2}}\widetilde{G_\varepsilon}(t)^{\frac{q_1}{1-q_2}},$$

$$\widetilde{G_\varepsilon}'(t) \geq -C_7\varepsilon\widetilde{G_\varepsilon}(t) + C_8\varepsilon^{-\frac{\sigma_2}{2}}\widetilde{F_\varepsilon}(t)^{\frac{p_2}{1-p_1}}.$$

From the phase field argument in [4], we get upper bounds of $F_\varepsilon(t)$ and $G_\varepsilon(t)$ as follows:

PROPOSITION 4.3. *Let* $p_1 < 1$, $q_2 < 1$ *and* $\sigma_j > -N$ $(j = 1, 2)$.
(i) *There exist constants* $A > 0$ *and* $B > 0$ *such that*

(4.5) $$\widetilde{F_\varepsilon}(t) \leq A\varepsilon^{\alpha(1-p_1)},$$

(4.6) $$\widetilde{G_\varepsilon}(t) \leq B\varepsilon^{\beta(1-q_2)},$$

for all $t > 0$ and $\varepsilon > 0$, where α and β are defined in (2.1).

(ii) *(upperbounds) There exist constants $A > 0$ and $B > 0$ such that*

$$(4.7) \qquad\qquad F_\varepsilon(t) \leq A\varepsilon^{\alpha r_1},$$

$$(4.8) \qquad\qquad G_\varepsilon(t) \leq B\varepsilon^{\beta r_2},$$

for all $t > 0$ and $\varepsilon > 0$.

Proof of Theorem 2.1 (i). We consider the case $\alpha \geq N/2$. Lemmas 3.1, 3.2, 3.4, and the definition of F_ε in (4.1) give lower bounds of $F_\varepsilon(\varepsilon^{-1})$:

$$(4.9) \qquad F_\varepsilon(\varepsilon^{-1}) \geq \begin{cases} C_5 \varepsilon^{\frac{Nr}{2}}, & (\alpha > \frac{N}{2}), \\ C_6 \varepsilon^{\frac{Nr}{2}} \log(1 + \varepsilon^{-1}), & (\alpha = \frac{N}{2}). \end{cases}$$

Indeed, in the critical case $\alpha = N/2$, we have

$$u(x,t) \leq C(1+t)^{-\frac{N}{2}} \exp\left(-\frac{|x|^2}{t}\right), \qquad\qquad (t > 0),$$

$$v(x,t) \leq C(1+t)^{\frac{\sigma_2+2-p_2 N}{2(1-q_2)}} \exp\left(-\frac{C|x|^2}{t}\right), \qquad (t > 1)$$

from Lemmas 3.1 and 3.4. Applying Lemma 3.2, we have

$$(4.10) \qquad u(x,t) \leq C(1+t)^{-\frac{N}{2}} \log(1+t) \exp\left(-\frac{|x|^2}{t}\right), \qquad (t > t_0)$$

for some $t_0 > 1$. Substituting (4.10) into (4.1), we obtain (4.9). This contradicts (4.7) for small $\varepsilon > 0$. This completes the proof. $\square$

Proof of Theorem 2.1 (ii). We consider the case $\liminf_{|x|\to\infty} |x|^a u_0(x) > 0$ ($a < 2\alpha$). Then we have

$$\begin{aligned} F_\varepsilon(0) &= \int_{\mathbf{R}^N} u_0(x)^{r_1} \rho_\varepsilon(x)\,dx \\ &= \int_{|x|<1} u_0(\varepsilon^{-\frac{1}{2}}x)^{r_1} \exp\left(-\frac{1}{1-|x|^2}\right) dx. \end{aligned}$$

Hence, for the constant A in Proposition 4.3, we can choose sufficiently small $\varepsilon > 0$ such that

$$\begin{aligned} \varepsilon^{-\alpha r_1} & F_\varepsilon(0) \\ &\geq \varepsilon^{-\alpha r_1} \int_{1/2<|x|<1} u_0(\varepsilon^{-\frac{1}{2}}x)^{r_1} \exp\left(-\frac{1}{1-|x|^2}\right) dx \\ &\geq C\varepsilon^{(-\alpha+\frac{a}{2})r_1} \int_{1/2<|x|<1} |x|^{-ar_1} \exp\left(-\frac{1}{1-|x|^2}\right) dx \\ &> A. \end{aligned}$$

Therefore, we get

$$F_\varepsilon(0) > A\varepsilon^{\alpha r_1},$$

which contradicts (4.7). This completes the proof. $\square$

Proof of Theorem 2.1 (iii). We assume that $u_0(x) \geq \widetilde{C} \exp(-\nu|x|^2)$ for sufficiently large $\widetilde{C} > 0$. Letting $\varepsilon = 1$ and $t = 0$, we get for the constant A in Proposition 4.3,

$$F_1(0) = \widetilde{C}^{r_1} \int_{|x|<1} \exp\left(-\nu r_1 |x|^2\right) \exp\left(-\frac{1}{1-|x|^2}\right) dx$$

$$> A,$$

which contradicts (4.7). This completes the proof. $\square$

5. Proofs of Theorems 2.2 and 2.3. In this section we prove Theorems 2.2 and 2.3. In order to prove the theorems, it suffices to show the following propositions.

PROPOSITION 5.1. *Let $p_1 > 1$, $q_2 < 1$. (i) If $\alpha \geq N/2$, then no nontrivial global solutions exist.*
(ii) If $u_0 \in I_a$ $(a < -\{q_1(\sigma_2 + 2) + (1 - q_2)(\sigma_1 + 2) - p_2 q_1 N\}/\{(1 - p_1)(1 - q_2)\})$, then no global solutions exist.
(iii) For any $\nu > 0$, there exists large $C > 0$ such that no global solutions with $u_0(x) \geq C \exp(-\nu|x|^2)$ exist.

PROPOSITION 5.2. *Let $p_1 > 1$. (i) If $p_1 + q_1 \leq 1 + (2 + \sigma_1)/N$, then no nontrivial global solutions exist.*
(ii) If $u_0 \in I_a$ $(a < (\sigma_1 + 2 - Nq_1)/(p_1 - 1))$, then no global solutions exist.
(iii) For any $\nu > 0$, there exists large $C > 0$ such that no global solutions with $u_0(x) \geq C \exp(-\nu|x|^2)$ exist.

NECESSARY CONDITION FOR THE GLOBAL EXISTENCE Assume that (u, v) are global solutions for (1.1) and (1.2). For $\varepsilon > 0$, define

(5.1) $$F_\varepsilon(t) = \int_{\mathbf{R}^N} u(x,t)^r \rho_\varepsilon(x) dx,$$

where $r > 0$ satisfying $r\sigma_1 < \frac{N}{p_1 - 1}$.

Multiplying (1.1) by $\rho_\varepsilon(x) u^{r-1}$ and integrating by parts, we have

$$F_\varepsilon(t)' \geq -C_1 \varepsilon F_\varepsilon(t) + C_2 \varepsilon^{-\frac{\sigma_1}{2}} t^{\frac{q_1(\sigma_2+2)-p_2 q_1 N}{2(1-q_2)}} F_\varepsilon(t)^{\frac{r+p_1-1}{r}} \qquad (t \geq \varepsilon^{-1}),$$

where C_1 and $C_2 > 0$. Indeed, from the inverse Hölder inequality and Lemma 3.4,

$$\int_{\mathbf{R}^N} \rho_\varepsilon |x|^{\sigma_1} u^{r+p_1-1} v^{q_1} dx$$

$$\geq \left(\int_{\mathbf{R}^N} \rho_\varepsilon u^r dx\right)^{\frac{r+p_1-1}{r}} \left(\int_{\mathbf{R}^N} \rho_\varepsilon |x|^{\frac{r\sigma_1}{1-p_1}} v^{\frac{rq_1}{1-p_1}} dx\right)^{\frac{1-p_1}{r}}$$

$$\geq F_\varepsilon(t)^{\frac{r+p_1-1}{r}} \cdot C\varepsilon^{-\frac{\sigma_1}{2}} t^{\frac{q_1(\sigma_2+2)-p_2 q_1 N}{2(1-q_2)}}.$$

Putting

$$\begin{cases} \widetilde{F_\varepsilon}(s) = \varepsilon^{\frac{q_1(\sigma_2+2)+(1-q_2)(\sigma_1+2)-p_2 q_1 N}{2(1-p_1)(1-q_2)} - r} F_\varepsilon(t), \\ s = \varepsilon t, \end{cases}$$

yields the following inequality:

$$\widetilde{F_\varepsilon}(s)' \geq -C_1\widetilde{F_\varepsilon}(s) + C_2 s^{\frac{q_1(\sigma_2+2)-p_2q_1N}{2(1-q_2)}}\widetilde{F_\varepsilon}(s)^{\frac{r+p_1-1}{r}} \qquad (s \geq 1).$$

A comparison argument and the global existence of $\widetilde{F_\varepsilon}(s)$ imply that

$$\widetilde{F_\varepsilon}(1) \leq K,$$

where $K > 0$ is independent of $0 < \varepsilon \leq 1$. Hence,

$$(5.2) \qquad\qquad F_\varepsilon(\varepsilon^{-1}) \leq K\varepsilon^{-\frac{q_1(\sigma_2+2)+(1-q_2)(\sigma_1+2)-p_2q_1N}{2(1-p_1)(1-q_2)}r},$$

for $0 < \varepsilon \leq 1$.

Proof of Proposition 5.1 (i). Lemmas 3.1 and 3.2, and the definition of F_ε in (5.1) give lower bounds of $F_\varepsilon(\varepsilon^{-1})$:

$$F_\varepsilon(\varepsilon^{-1}) \geq \begin{cases} C_3\varepsilon^{\frac{Nr}{2}}, & (\alpha > \frac{N}{2}), \\ C_4\varepsilon^{\frac{Nr}{2}}\log(1+\varepsilon^{-1}), & (\alpha = \frac{N}{2}), \end{cases}$$

which contradicts (5.2) for small $\varepsilon > 0$. Indeed, one can see that $\alpha \geq \frac{N}{2}$ is equivalent to

$$-\frac{q_1(\sigma_2+2)+(1-q_2)(\sigma_1+2)-p_2q_1N}{2(1-p_1)(1-q_2)} \geq \frac{N}{2}.$$

This completes the proof. □

Proof of Proposition 5.1 (ii). From the definition of F_ε in (5.1), we obtain

$$F_\varepsilon(\varepsilon^{-1})$$
$$\geq \int_{|x|<\varepsilon^{-\frac{1}{2}}} \left(S(\varepsilon^{-1})u_0(x)\right)^r \rho_\varepsilon(x)dx$$
$$\geq \int_{|x|<\varepsilon^{-\frac{1}{2}}} \left((4\pi\varepsilon^{-1})^{-\frac{N}{2}} \int_{\mathbf{R}^N} \exp\left(-\frac{\varepsilon|x|^2}{2}\right)\exp\left(-\frac{\varepsilon|y|^2}{2}\right)u_0(y)dy\right)^r \rho_\varepsilon(x)dx$$
$$\geq \int_{|x|<1} \exp\left(-\frac{r|x|^2}{2}-\frac{1}{1-|x|^2}\right)dx \int_{\mathbf{R}^N} \exp\left(-\frac{r|y|^2}{2}\right)u_0(\varepsilon^{-\frac{1}{2}}y)^r dy$$
$$= C\int_{\mathbf{R}^N} \exp\left(-\frac{r|y|^2}{2}\right)u_0(\varepsilon^{-\frac{1}{2}}y)^r dy.$$

Hence, for the constant K in (5.2), we can choose sufficiently small $\varepsilon > 0$ such that

$$\varepsilon^{\frac{q_1(\sigma_2+2)+(1-q_2)(\sigma_1+2)-p_2q_1N}{2(1-p_1)(1-q_2)}r}F_\varepsilon(\varepsilon^{-1})$$
$$\geq C\varepsilon^{\frac{q_1(\sigma_2+2)+(1-q_2)(\sigma_1+2)-p_2q_1N}{2(1-p_1)(1-q_2)}r}\int_{\mathbf{R}^N}\exp\left(-\frac{r|y|^2}{2}\right)u_0(\varepsilon^{-\frac{1}{2}}y)^r dy$$
$$\geq C\varepsilon^{\left\{\frac{q_1(\sigma_2+2)+(1-q_2)(\sigma_1+2)-p_2q_1N}{2(1-p_1)(1-q_2)}+\frac{a}{2}\right\}r}$$
$$> K.$$

Therefore, we get

$$F_\varepsilon(\varepsilon^{-1}) > K\varepsilon^{-\frac{q_1(\sigma_2+2)+(1-q_2)(\sigma_1+2)-p_2q_1N}{2(1-p_1)(1-q_2)}r},$$

which contradicts (5.2). This completes the proof. $\square$

Proof of Theorem 5.1 (iii). We assume that $u_0(x) \geq \widetilde{C}\exp(-\nu|x|^2)$ for sufficiently large $\widetilde{C} > 0$. In the same way as the previous proof, we obtain

$$F_\varepsilon(\varepsilon^{-1}) \geq C\int_{\mathbf{R}^N} \exp\left(-\frac{r|y|^2}{2}\right) u_0(\varepsilon^{-\frac{1}{2}}y)^r dy.$$

Letting $\varepsilon = 1$, we get for the constant K in (5.2)

$$\begin{aligned}
F_1(1) &\geq C\int_{\mathbf{R}^N} \exp\left(-\frac{r|y|^2}{2}\right) u_0(y)^r dy \\
&\geq C\widetilde{C}^r \int_{\mathbf{R}^N} \exp\left(-\frac{r|y|^2}{2}\right) \exp\left(-\nu r|y|^2\right) dy \\
&> K,
\end{aligned}$$

which contradicts (5.2). This completes the proof. $\square$

Proof of Proposition 5.2 (i), (ii) *and* (iii). Using Lemma 3.1 instead of Lemma 3.4 for the estimate of $v(x,t)$, we can prove Proposition 5.2(i), (ii) and (iii) in the same way as the proof of Proposition 5.1(i), (ii) and (iii), respectively. $\square$

6. Appendix. In this section, we give a comparison theorem and a local existence theorem.

COMPARISON PRINCIPLE

LEMMA 6.1. *Let $f(u,v)$ and $g(u,v)$ be strictly monotone increasing in u and v for u, $v \geq 0$. Assume that $\bar{u}$, $\bar{v}$, $\underline{u}$, $\underline{v}$ are nonnegative and satisfy on $\mathbf{R}^N \times (0,T)$,*

$$\begin{aligned}
\bar{u}_t - \Delta\bar{u} &\geq |x|^{\sigma_1} f(\bar{u},\bar{v}), \\
\bar{v}_t - \Delta\bar{v} &\geq |x|^{\sigma_2} g(\bar{u},\bar{v}), \\
\underline{u}_t - \Delta\underline{u} &\leq |x|^{\sigma_1} f(\underline{u},\underline{v}), \\
\underline{v}_t - \Delta\underline{v} &\leq |x|^{\sigma_2} g(\underline{u},\underline{v}),
\end{aligned}$$

and that on $\mathbf{R}^N$,

$$\begin{aligned}
\bar{u}(x,0) - \underline{u}(x,0) &\geq 0, \not\equiv 0, \\
\bar{v}(x,0) - \underline{v}(x,0) &\geq 0, \not\equiv 0.
\end{aligned}$$

Then we have $\bar{u}(x,t) \geq \underline{u}(x,t)$ and $\bar{v}(x,t) \geq \underline{v}(x,t)$ on $\mathbf{R}^N \times (0,T)$.

LOCAL EXISTENCE RESULT

THEOREM 6.2. *Let δ_1 and δ_2 be defined in (2.2). Assume that $(u_0, v_0) \in I^{\delta_1} \times I^{\delta_2}$ and that $0 \leq \delta_1, \delta_2 < N$. Then there exist $(u(t), v(t)) \in P_T = \{(u,v) \in E_T; u \geq 0, v \geq 0\}$ satisfying the integral equations (3.1) and (3.2) for some $T > 0$.*

To prove the theorem, we define $\{u_n(x,t)\}$ and $\{v_n(x,t)\}$ $(n = 1, 2 \cdots)$ inductively by:

$$u_{n+1}(t) = S(t)u_0 + \int_0^t S(t-s)| \cdot |^{\sigma_1} u_n(s)^{p_1} v_n(s)^{q_1} ds,$$

$$v_{n+1}(t) = S(t)v_0 + \int_0^t S(t-s)| \cdot |^{\sigma_2} u_n(s)^{p_2} v_n(s)^{q_2} ds,$$

$$u_1 = S(t)u_0,$$

$$v_1 = S(t)v_0.$$

At first, we introduce two lemmas.

LEMMA 6.3.   ([4]) (i) *Let* $(u_0, v_0) \in I^{\delta_1} \times I^{\delta_2}$ *and* $0 \le \delta_1, \delta_2 < N$. *Then* $(S(\cdot)u_0, S(\cdot)v_0) \in E_T$ *for all* $T > 0$, *and we have*

$$\sup_{s \in [0,T]} \|S(s)u_0\|_{\infty,\delta_1} \le C\|u_0\|_{\infty,\delta_1},$$

$$\sup_{s \in [0,T]} \|S(s)v_0\|_{\infty,\delta_2} \le C\|v_0\|_{\infty,\delta_2}.$$

(ii) *For* $(u, v) \in E_T$, *define* $\Phi_1(u,v)$ *and* $\Phi_2(u,v)$ *by*

$$\Phi_1(u,v) = \int_0^t S(t-s)| \cdot |^{\sigma_1} u(s)^{p_1} v(s)^{q_1} ds,$$

$$\Phi_2(u,v) = \int_0^t S(t-s)| \cdot |^{\sigma_2} u(s)^{p_2} v(s)^{q_2} ds.$$

Then $(\Phi_1(u,v), \Phi_2(u,v)) \in E_T$, *and we have*

$$\sup_{s \in [0,T]} \|\Phi_1(u,v)(s)\|_{\infty,\delta_1} \le CT \left(\sup_{s \in [0,T]} \|u(s)\|_{\infty,\delta_1}^{p_1} \sup_{s \in [0,T]} \|v(s)\|_{\infty,\delta_2}^{q_1} \right),$$

$$\sup_{s \in [0,T]} \|\Phi_2(u,v)(s)\|_{\infty,\delta_2} \le CT \left(\sup_{s \in [0,T]} \|u(s)\|_{\infty,\delta_1}^{p_2} \sup_{s \in [0,T]} \|v(s)\|_{\infty,\delta_2}^{q_2} \right).$$

This lemma leads to uniform estimates for the solutions.

LEMMA 6.4. *Suppose that* $(u_0, v_0) \in I^{\delta_1} \times I^{\delta_2}$. *Then there exist* $K > 0$ *and* $T > 0$ *such that*

$$\sup_{t \in [0,T]} \|u_n(t)\|_{\infty,\delta_1} < K,$$

$$\sup_{t \in [0,T]} \|v_n(t)\|_{\infty,\delta_2} < K,$$

for all n.

Proof.   Let $C > 0$ be as in Lemma 6.3, (i) and (ii).   Put $R = \max(\|u_0\|_{\infty,\delta_1}, \|v_0\|_{\infty,\delta_2})$. Taking $K > 0$ and $T > 0$ such that

$$K > 2CR, \ T < \frac{K - CR}{C(K^{p_1+q_1} + K^{p_2+q_2})},$$

we can get the desired estimates. This completes the proof. □

Now, we can prove Theorem 6.2.

Proof of Theorem 6.2. From Lemma 6.4, one can see that

$$\sup_{t\in[0,T]} \|u_n(t)\|_\infty < K,$$

$$\sup_{t\in[0,T]} \|v_n(t)\|_\infty < K$$

for all n. The monotonicity of the heat kernel gives

$$u_n \le u_{n+1}, v_n \le v_{n+1}$$

for all n. Therefore, there exist $\widetilde{u}(x,t) = \lim_{n\to\infty} u_n(x,t)$, $\widetilde{v}(x,t) = \lim_{n\to\infty} v_n(x,t)$ on $\mathbf{R}^N \times [0,T]$, and we have

$$\sup_{t\in[0,T]} \|\widetilde{u}(t)\|_{\infty,\delta_1} \le K,$$

$$\sup_{t\in[0,T]} \|\widetilde{v}(t)\|_{\infty,\delta_2} \le K.$$

Moreover, from Lebesgue's monotone convergence theorem, we can easily see that $(\widetilde{u}, \widetilde{v})$ are local solutions for (3.1) and (3.2). This completes the proof of Theorem 6.2. □

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