METHODS AND APPLICATIONS OF ANALYSIS.
Vol. 12, No. 4, pp. 441–450, December 2005

NONEXISTENCE OF POSITIVE SOLUTIONS FOR SOME FULLY NONLINEAR ELLIPTIC EQUATIONS*

QINIAN JIN , YANYAN LI , AND HAOYUAN XU†

Dedicated to Joel Smoller on his 70th birthday

Key words. Fully nonlinear elliptic equations, nonexistence, comparison principle

AMS subject classifications. 35J60

It is well known that

$$\Delta u \geq u^p \quad \text{in } \mathbb{R}^n \tag{1}$$

has no positive solution if $p > 1$. For a proof, see for example Osserman [9], Loewner and Nirenberg [7] and Brezis [2]. We extend this result to some fully nonlinear elliptic equations. Some related problems will also be studied.

Let us fix some notations. For each $1 \leq k \leq n$ let

$$\sigma_k(\lambda) = \sum_{1\leq i_1<\cdots<i_k\leq n} \lambda_{i_1}\cdots\lambda_{i_k}, \qquad \lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{R}^n,$$

denote the kth elementary symmetric function, and let Γ_k denote the connected component of $\{\lambda \in \mathbb{R}^n : \sigma_k(\lambda) > 0\}$ containing the positive cone $\{\lambda \in \mathbb{R}^n : \lambda_1 > 0, \cdots, \lambda_n > 0\}$. It is well known that $\Gamma_k = \{\lambda \in \mathbb{R}^n : \sigma_l(\lambda) > 0, 1 \leq l \leq k\}$. Let $S^{n\times n}$ denote the set of $n \times n$ real symmetric matrices. For any $A \in S^{n\times n}$ we denote by $\lambda(A)$ the eigenvalues of A.

Throughout this note we will assume that $\Gamma \subset \mathbb{R}^n$ is an open convex symmetric cone with vertex at the origin satisfying $\Gamma_n \subset \Gamma \subset \Gamma_1$. Moreover, we also assume that f is a continuous function defined on $\overline{\Gamma}$ verifying the following properties:

$$f \text{ is homogeneous of degree one on } \Gamma, \tag{2}$$

$$f \text{ is symmetric in } \lambda = (\lambda_1, \cdots, \lambda_n) \in \Gamma, \tag{3}$$

and

$$f \text{ is monotonically increasing in each variable on } \Gamma. \tag{4}$$

Given a smooth positive function u defined in $\mathbb{R}^n$ with $n \geq 3$, we may introduce

$$A^u = -\frac{2}{n-2}u^{-\frac{n+2}{n-2}}D^2u + \frac{2n}{(n-2)^2}u^{-\frac{2n}{n-2}}Du \otimes Du - \frac{2}{(n-2)^2}u^{-\frac{2n}{n-2}}|Du|^2 I, \tag{5}$$

where I is the $n \times n$ identity matrix, and Du and D^2u denote the gradient and the Hessian of u respectively. This operator appears in the recent work on conformally invariant elliptic equations and the σ_k-Yamabe problems in conformal geometry, see for example [4, 11].

*Received October 5, 2005; accepted for publication November 10, 2005.
†Department of Mathematics, Rutgers University, New Brunswick, New Jersey 08903, USA (qjin@math.rutgers.edu; yyli@math.rutgers.edu; hyxu@math.rutgers.edu).

First we have

Theorem 1. *Let $\Gamma \subset \mathbb{R}^n$, $n \geq 3$, be an open convex symmetric cone with vertex at the origin satisfying $\Gamma_n \subset \Gamma \subset \Gamma_1$, and let $f \in C(\overline{\Gamma})$ satisfy (2), (3) and (4). If $\Gamma \supseteq \Gamma_k$ for some $1 \leq k \leq n$, then the problem*

$$f(\lambda(-A^u)) = u^{p-\frac{n+2}{n-2}}, \quad \lambda(-A^u) \in \Gamma \text{ in } \mathbb{R}^n \tag{6}$$

has no positive continuous viscosity subsolution if $p > 1 + \max\left\{0, \frac{2(2k-n)}{(n-2)k}\right\}$.

The definition of viscosity subsolutions appeared in Theorem 1 will be given below. In [4, 5] Li and Li established some Liouville type theorems for the fully nonlinear elliptic equation

$$f(\lambda(A^u)) = u^{p-\frac{n+2}{n-2}}, \quad \lambda(A^u) \in \Gamma \text{ and } u > 0 \text{ in } \mathbb{R}^n. \tag{7}$$

They showed that for $-\infty < p < \frac{n+2}{n-2}$ problem (7) has no solution $u \in C^2(\mathbb{R}^n)$, while for $p = \frac{n+2}{n-2}$ any solution $u \in C^2(\mathbb{R}^n)$ of (7) must be of the form

$$u(x) = \left(\frac{a}{1 + b^2|x - \bar{x}|^2}\right)^{\frac{n-2}{2}}, \quad \forall x \in \mathbb{R}^n$$

for some $\bar{x} \in \mathbb{R}^n$ and some positive constants a and b satisfying some suitable conditions. See also [4, 5] for earlier works on the subject. Theorem 1 indicates the sharp contrast between (6) and (7).

The proof of Theorem 1, in the spirit of [2], is based on a comparison principle. Let us work on slightly more general framework. Suppose Ω is an open set in $\mathbb{R}^n$. Then for any mapping $B(\cdot,\cdot,\cdot) : \overline{\Omega} \times \mathbb{R}_+ \times \mathbb{R}^n \to S^{n\times n}$ and any positive function $h(x,t)$ defined on $\Omega \times \mathbb{R}_+$, we may consider the problem

$$f(\lambda(D^2u + B(x,u,Du))) = h(x,u), \ \lambda(D^2u + B(x,u,Du)) \in \Gamma \text{ in } \Omega. \tag{8}$$

A positive function $u \in C^2(\Omega)$ is said to be a classical subsolution of (8) if $\lambda(D^2u + B(x,u,Du)) \in \Gamma$ and

$$f(\lambda(D^2u + B(x,u,Du))) \geq h(x,u) \quad \text{in } \Omega.$$

Similarly we can define the classical supersolutions and classical solutions for (8).

In the following we will recall the well-known definition of viscosity solutions for (8).

Definition 1. *We say a positive function $u \in C(\Omega)$ is a viscosity subsolution of (8) if for each $\bar{x} \in \Omega$ there exists an $\varepsilon > 0$ such that for any $\psi \in C^2(B_\varepsilon(\bar{x}))$ with the properties $\psi(\bar{x}) = u(\bar{x})$ and*

$$\psi > 0, \ \psi \geq u \text{ and } \lambda(D^2\psi + B(x,\psi,D\psi)) \in \Gamma \text{ in } B_\varepsilon(\bar{x}),$$

there holds

$$f(\lambda(D^2\psi(\bar{x}) + B(\bar{x},\psi(\bar{x}),D\psi(\bar{x})))) \geq h(\bar{x},\psi(\bar{x})).$$

Similarly one can define *viscosity supersolution* of (8). A positive function $u \in C(\Omega)$ is called a *viscosity solution* of (8) if u is both a viscosity subsolution and a viscosity supersolution of (8).

It is straightforward to show that if $u \in C^2(\Omega)$ is a positive function satisfying $\lambda(D^2u + B(x,u,Du)) \in \Gamma$ in Ω, then u is a viscosity subsolution of (8) if and only if u is a classical subsolution of (8).

We have the following simple comparison principle.

LEMMA 1. *Let $\Omega \subset \mathbb{R}^n$ be an open bounded set, and let $t \to t^{-1}h(x,t)$ be strictly increasing on $(0,\infty)$ for each $x \in \Omega$. Suppose that $u \in C(\overline{\Omega})$ is a positive viscosity subsolution of (8) in Ω and that $v \in C^2(\Omega) \cap C(\overline{\Omega})$ is a positive classical supersolution of (8) with $\lambda(D^2v + t^{-1}B(x,tv,tDv)) \in \Gamma$ for each $t \geq 1$. Suppose also that for each $x \in \Omega$ and $\xi, \mathbf{p} \in \mathbb{R}^n$ the function*

$$t \to t^{-1}\langle B(x,t,t\mathbf{p})\xi,\xi\rangle \tag{9}$$

is non-increasing on $(0,\infty)$. If $u \leq v$ on $\partial\Omega$, then $u \leq v$ on $\overline{\Omega}$.

Proof. Suppose the conclusion is not true. Since u is bounded from above and v is positive on $\overline{\Omega}$, there must exist $a > 1$ such that $u \leq av$ on $\overline{\Omega}$ and $u(\bar{x}) = av(\bar{x})$ for some $\bar{x} \in \overline{\Omega}$. Since $u \leq v$ on $\partial\Omega$ and $a > 1$, $\bar{x}$ must be an interior point of Ω. By assumption,

$$\lambda\left(D^2(av) + B(x,av,D(av))\right) = a\lambda\left(D^2v + a^{-1}B(x,av,aDv)\right) \in \Gamma.$$

Since u is a viscosity subsolution of (8), we have by using the degree one homogeneity of f that

$$af\left(\lambda\left(D^2v(\bar{x}) + a^{-1}B(\bar{x},av(\bar{x}),aDv(\bar{x}))\right)\right) \geq h(\bar{x},av(\bar{x})).$$

By using (9) and the monotonicity of f, noting that v is a classical supersolution of (8), we have

$$f\left(\lambda\left(D^2v + a^{-1}B(x,av,aDv)\right)\right) \leq f\left(\lambda\left(D^2v + B(x,v,Dv)\right)\right) \leq h(x,v).$$

Therefore $ah(\bar{x},v(\bar{x})) \geq h(\bar{x},av(\bar{x}))$. This clearly contradicts the condition that the function $t \to t^{-1}h(\bar{x},t)$ is strictly increasing on $(0,\infty)$. ☐

Now we are in a position to indicate the idea of showing nonexistence of positive viscosity subsolutions of (8) when $\Omega = \mathbb{R}^n$. To this end, let us pick a sequence of bounded open sets $\{\Omega_j\}$ such that

$$\Omega_1 \subset \Omega_2 \subset \cdots \subset \Omega_j \subset \cdots \quad \text{and} \quad \bigcup_{j=1}^{\infty} \Omega_j = \mathbb{R}^n.$$

Suppose we can construct a sequence of positive functions $\{U_j\}$ with $U_j \in C^2(\Omega_j)$ such that

$$\lambda\left(D^2U_j + t^{-1}B(x,tU_j,tDU_j)\right) \in \Gamma \quad \text{in } \Omega_j \text{ for each } t \geq 1, \tag{10}$$

$$f(\lambda(D^2U_j + B(x,U_j,DU_j))) \leq h(x,U_j) \quad \text{in } \Omega_j, \tag{11}$$

$$U_j(x) \to +\infty \text{ uniformly as } d(x,\partial\Omega_j) \to 0 \tag{12}$$

and

$$U_j(x) \to 0 \text{ as } j \to \infty \text{ for each fixed } x \in \mathbb{R}^n. \tag{13}$$

If $u \in C(\mathbb{R}^n)$ is a positive viscosity subsolution of (8) with $\Omega = \mathbb{R}^n$, then we can apply Lemma 1 to conclude that

$$u(x) \le U_j(x) \text{ whenever } x \in \Omega_j \text{ for each } j.$$

Taking $j \to \infty$ and using (13) gives $u(x) \equiv 0$ which is a contradiction.

By using the degree one homogeneity of f, one can see that (6) can be written in the form of (8) with

$$h(x,t) = \frac{n-2}{2} t^p \quad \text{and} \quad B(x,t,\mathbf{p}) = -\frac{n}{n-2} t^{-1} \mathbf{p} \otimes \mathbf{p} + \frac{1}{n-2} t^{-1} |\mathbf{p}|^2 I$$

for $(x,t,\mathbf{p}) \in \Omega \times \mathbb{R}_+ \times \mathbb{R}^n$. Therefore Lemma 1 applies to (6). Now we are ready to give the proof of Theorem 1.

Proof of Theorem 1. Let B_j denote the ball of radius j with center at the origin. It suffices to show the existence of a sequence of positive functions $U_j \in C^2(B_j)$ satisfying (10), (11), (12) and (13) with $\Omega_j := B_j$.

Step 1. Let $\alpha = \frac{2}{p-1}$ and consider the function

$$U(x) = \left(1 - |x|^2\right)^{-\alpha} \quad \text{in } B_1. \tag{14}$$

We will show that $\lambda(-A^U)(x) \in \Gamma_k \subset \Gamma$ for all $x \in B_1$. Let $r = |x|$, then $U(x) = \varphi(r)$ with $\varphi(r) = (1 - r^2)^{-\alpha}$. We need only to verify the claim at $x = (r, 0, \cdots, 0)$ for $0 \le r < 1$. Let us perform the computation as in [6]. By straightforward calculation one has

$$DU(x) = (\varphi'(r), 0, \cdots, 0) \quad \text{and} \quad D^2U(x) = \operatorname{diag}\left[\varphi''(r), \frac{\varphi'(r)}{r}, \cdots, \frac{\varphi'(r)}{r}\right].$$

Therefore it follows from the definition of A^U that

$$A^U(x) = \operatorname{diag}\left[\lambda_1^U(r), \lambda_2^U(r), \cdots, \lambda_n^U(r)\right],$$

where

$$\begin{cases} \lambda_1^U(r) = -\frac{2}{n-2} \varphi^{-\frac{n+2}{n-2}} \varphi''(r) + \frac{2(n-1)}{(n-2)^2} \varphi^{-\frac{2n}{n-2}} \left[\varphi'(r)\right]^2, \\ \lambda_2^U(r) = \cdots = \lambda_n^U(r) = -\frac{2}{n-2} \varphi^{-\frac{n+2}{n-2}} \frac{\varphi'(r)}{r} - \frac{2}{(n-2)^2} \varphi^{-\frac{2n}{n-2}} \left[\varphi'(r)\right]^2. \end{cases}$$

But for the function φ it is easy to see that

$$\varphi'(r) = 2\alpha r \left[\varphi(r)\right]^{\frac{\alpha+1}{\alpha}} \quad \text{and} \quad \varphi''(r) = 2\alpha \varphi^{\frac{\alpha+2}{\alpha}} \left[1 + (2\alpha+1) r^2\right].$$

Thus

$$\begin{cases} \lambda_1^U(r) = \frac{4\alpha}{n-2} \varphi^{\frac{\alpha+2}{\alpha} - \frac{n+2}{n-2}} \left[-1 + \frac{2\alpha - (n-2)}{n-2} r^2\right], \\ \lambda_l^U(r) = \frac{4\alpha}{n-2} \varphi^{\frac{\alpha+2}{\alpha} - \frac{n+2}{n-2}} \left[-1 - \frac{2\alpha - (n-2)}{n-2} r^2\right], \quad l = 2, \cdots, n. \end{cases}$$

Consequently

$$\lambda(-A^U) = \frac{4\alpha}{n-2} \varphi^{\frac{\alpha+2}{\alpha} - \frac{n+2}{n-2}} \lambda(r), \tag{15}$$

where

$$\lambda(r)=\left(1-\frac{2\alpha-(n-2)}{n-2}r^2,1+\frac{2\alpha-(n-2)}{n-2}r^2,\cdots,1+\frac{2\alpha-(n-2)}{n-2}r^2\right). \quad (16)$$

Therefore we need only to show that $\lambda(r)\in\Gamma_k$ for all $0\le r<1$. It is obvious that $\lambda(0)=(1,1,\cdots,1)\in\Gamma_k$. So by the convexity of Γ_k it suffices to show that $\lambda(1)\in\Gamma_k$. Note that $\lambda(1)$ is a positive multiple of the vector $\beta=\left(\frac{n-2}{\alpha}-1,1,\cdots,1\right)$, it suffices to show $\beta\in\Gamma_k$.

It is well known that

$$\det\left(tI+\mathrm{diag}\left[\frac{n-2}{\alpha}-1,1,\cdots,1\right]\right)=\sum_{l=0}^{n}\sigma_{n-l}(\beta)t^l.$$

Therefore, by letting $g(t)$ denote the function on the left hand side in the above equation, we have

$$\sigma_l(\beta)=\frac{1}{(n-l)!}\frac{d^{n-l}g}{dt^{n-l}}(0).$$

Noting that

$$g(t)=\left(t+\frac{n-2}{\alpha}-1\right)(t+1)^{n-1}=(t+1)^n+\left(\frac{n-2}{\alpha}-2\right)(t+1)^{n-1}.$$

We thus obtain

$$\sigma_l(\beta)=\frac{(n-1)\cdots(l+1)}{(n-l)!}\left[n+\left(\frac{n-2}{\alpha}-2\right)l\right]>0$$

for all $1\le l\le k$ since $p>1+\max\left\{0,\frac{2(2k-n)}{(n-2)k}\right\}$. The claim therefore follows.

Step 2. For each j consider the function

$$U_j(x)=Cj^{\alpha}\left(j^2-|x|^2\right)^{-\alpha}\quad\text{in } B_j, \quad (17)$$

where C is a positive constant. Such functions have been used in [9, 7, 2] to deal with problems similar to (1). We claim that one can choose a suitably large C independent of j such that for all j there hold

$$\lambda(-A^{U_j})\in\Gamma_k\subset\Gamma \quad (18)$$

and

$$f\left(\lambda(-A^{U_j})\right)\le U_j^{p-\frac{n+2}{n-2}}\quad\text{in } B_j. \quad (19)$$

In fact, by writing $U_j(x)=Cj^{-\alpha}U\left(\frac{x}{j}\right)$ we can see that

$$A^{U_j}(x)=C^{-\frac{4}{n-2}}j^{\frac{4\alpha}{n-2}-2}A^U\left(\frac{x}{j}\right),\quad x\in B_j. \quad (20)$$

This gives (18) by the corresponding property for A^U. In order to show (19), by using the degree one homogeneity of f it follows from (15) and (20) that

$$f\left(\lambda(-A^{U_j}(x))\right)=\frac{4\alpha}{n-2}C^{-\frac{4}{n-2}}j^{\frac{4\alpha}{n-2}-2}\left[U\left(\frac{x}{j}\right)\right]^{\frac{\alpha+2}{\alpha}-\frac{n+2}{n-2}}f\left(\lambda\left(\frac{|x|}{j}\right)\right),$$

where $\lambda(r)$ is defined by (16). Since $\{\lambda(r) : 0 \le r \le 1\}$ is a compact subset of Γ_k, one can find a positive constant C_0 such that $f(\lambda(r)) \le C_0$ for $0 \le r \le 1$. Moreover, since $\alpha = \frac{2}{p-1}$, we have

$$\frac{\alpha+2}{\alpha} = p \quad \text{and} \quad \frac{4\alpha}{n-2} - 2 = -\left(p - \frac{n+2}{n-2}\right)\alpha.$$

Therefore

$$\begin{aligned}
&[U_j(x)]^{p-\frac{n+2}{n-2}} - f\left(\lambda(-A^{U_j}(x))\right) \\
&\quad \ge C^{p-\frac{n+2}{n-2}} j^{-(p-\frac{n+2}{n-2})\alpha} \left[U\left(\frac{x}{j}\right)\right]^{p-\frac{n+2}{n-2}} - \frac{4\alpha}{n-2} C_0 C^{-\frac{4}{n-2}} j^{\frac{4\alpha}{n-2}-2} \left[U\left(\frac{x}{j}\right)\right]^{p-\frac{n+2}{n-2}} \\
&\quad = C^{-\frac{4}{n-2}} \left[U\left(\frac{x}{j}\right)\right]^{p-\frac{n+2}{n-2}} j^{\frac{4\alpha}{n-2}-2} \left[C^{p-1} - \frac{4\alpha C_0}{n-2}\right].
\end{aligned}$$

This gives (19) if we choose C such that $C^{p-1} > \frac{4\alpha C_0}{n-2}$ which is always possible since $p > 1$. The proof is complete. ▯

Next we will apply the argument in the proof of Theorem 1 to show a nonexistence result of positive solutions for some Hessian equations in $\mathbb{R}^n$. There has been much work on Hessian equations, see e.g. [3, 10] and the references therein.

THEOREM 2. *Let $\Gamma \subset \mathbb{R}^n$ be an open convex symmetric cone with vertex at the origin satisfying $\Gamma_n \subset \Gamma \subset \Gamma_1$, let $f \in C(\overline{\Gamma})$ satisfy (2), (3) and (4). Then the equation*

$$f(\lambda(D^2u)) = u^p, \quad \lambda(D^2u) \in \Gamma \text{ in } \mathbb{R}^n \text{ with } n \ge 1 \tag{21}$$

has no positive continuous viscosity subsolution for any $p > 1$.

Proof. Consider the function U in B_1 defined by (14). Then the computation in the proof of Theorem 1 indicates that

$$\lambda(D^2U(x)) = 2\alpha[U(x)]^{\frac{\alpha+2}{\alpha}} \widetilde{\lambda}(r), \qquad x \in B_1,$$

where $r = |x|$ and

$$\widetilde{\lambda}(r) = \left(1 + (2\alpha+1)r^2, 1 - r^2, \cdots, 1 - r^2\right).$$

Therefore $\lambda(D^2U(x)) \in \Gamma_n \subset \Gamma$ for $x \in B_1$.

Next consider the function U_j on B_j defined by (17). Recall that $U_j(x) = Cj^{-\alpha}U\left(\frac{x}{j}\right)$, we have for $x \in B_j$ that $\lambda(D^2U_j(x)) \in \Gamma$ and

$$\lambda\left(D^2U_j(x)\right) = 2C\alpha j^{-\alpha-2}\left[U\left(\frac{x}{j}\right)\right]^{\frac{\alpha+2}{\alpha}} \widetilde{\lambda}\left(\frac{|x|}{j}\right).$$

Therefore by the degree one homogeneity of f it follows that

$$f\left(\lambda\left(D^2U_j(x)\right)\right) = 2C\alpha j^{-\alpha-2}\left[U\left(\frac{x}{j}\right)\right]^{\frac{\alpha+2}{\alpha}} f\left(\widetilde{\lambda}\left(\frac{|x|}{j}\right)\right).$$

Since $\{\widetilde{\lambda}(r) : 0 \le r \le 1\}$ is a compact subset of $\overline{\Gamma}$, we can choose a constant C_0 such that $f(\widetilde{\lambda}(r)) \le C_0$ for $0 \le r \le 1$. Therefore

$$\begin{aligned} -f\left(\lambda\left(D^2U_j(x)\right)\right) + [U_j(x)]^p & \\ \ge & -2\alpha C_0 C j^{-\alpha-2}\left[U\left(\frac{x}{j}\right)\right]^{\frac{\alpha+2}{\alpha}} + C^p j^{-p\alpha}\left[U\left(\frac{x}{j}\right)\right]^p \\ = & C j^{-p\alpha}\left[U\left(\frac{x}{j}\right)\right]^p\left\{C^{p-1} - 2C_0\alpha\right\} \\ \ge & 0 \end{aligned}$$

if we choose C so large that $C^{p-1} \ge 2\alpha C_0$.

We have therefore constructed a sequence of positive functions $U_j \in C^2(B_j)$ satisfying (10), (11), (12) and (13). The proof is thus complete. □

We remark that for the equation (21) with $p = 0$ some Bernstein type theorems have been established for some specific function f in the literature. The well-known theorem of Jörgen, Calabi and Pogorelov says that any convex solution of $\det(D^2u) = 1$ in $\mathbb{R}^n$ must be a quadratic polynomial. In [1] it is shown that any convex solution of $\sigma_k(\lambda(D^2u)) = 1$ in $\mathbb{R}^n$ satisfying a quadratic growth condition is a quadratic polynomial; similar result is established for the Hessian quotient equation $\frac{\sigma_n}{\sigma_k}(\lambda(D^2u)) = 1$ in $\mathbb{R}^n$ for some $1 \le k \le n-1$. Combining these facts with Theorem 2 it seems interesting to study the existence of positive solutions of (21) for $0 < p \le 1$.

We now consider some analogous problems in half Euclidean space $\mathbb{R}^n_+$. In [8] Lou and Zhu considered the problem

$$\begin{cases} \Delta u = u^p \quad \text{in } \mathbb{R}^n_+, \\ u > 0 \quad \text{in } \mathbb{R}^n_+, \\ \frac{\partial u}{\partial x_n} = u^q \quad \text{on } \partial\mathbb{R}^n_+, \end{cases} \tag{22}$$

where $\mathbb{R}^n_+ := \{x = (x', x_n) \in \mathbb{R}^n : x_n > 0\}$, and showed that (22) has no solution if $p > 1$ and $q > 1$. We extend below this result to some fully nonlinear elliptic equations.

To set up our framework, let $\Omega \subset \mathbb{R}^n$ be an open set with smooth boundary $\partial\Omega \ne \emptyset$ and let ν be the unit inner normal to $\partial\Omega$. Let $\Sigma \subset \partial\Omega$ be an open subset of $\partial\Omega$. Then for any functions $h(x,t)$ and $g(x,t)$ defined on $\Omega \times (0,\infty)$ and $\Sigma \times [0,\infty)$ respectively, we may consider the problem

$$\begin{cases} f\left(\lambda\left(D^2u + B(x,u,Du)\right)\right) = h(x,u) \quad \text{in } \Omega, \\ \lambda\left(D^2u + B(x,u,Du)\right) \in \Gamma \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = g(x,u) \quad \text{on } \Sigma. \end{cases} \tag{23}$$

We say a function $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ is a classical subsolution of (23) if $u > 0$ in Ω, u is a classical subsolution of (8) in Ω and $\frac{\partial u}{\partial \nu} \ge g(x,u)$ on Σ.

We also introduce the concept of viscosity subsolution for (23).

Definition 2. *Let $u \in C(\Omega \cup \Sigma)$ be such that $u > 0$ in Ω. We say u is a viscosity subsolution of (23) if u is a viscosity subsolution of (8) in Ω, and for each $\bar{x} \in \Sigma$ there is a neighborhood O of $\bar{x}$ such that for any $\psi \in C^1(O \cap \overline{\Omega})$ with the properties*

$$\psi(\bar{x}) = u(\bar{x}) \quad \textit{and} \quad \psi \ge u \textit{ in } O \cap \overline{\Omega}$$

there holds

$$\frac{\partial \psi}{\partial \nu}(\bar{x}) \ge g(\bar{x}, \psi(\bar{x})).$$

Theorem 3. *Let $\Gamma \subset \mathbb{R}^n$ be an open convex symmetric cone with vertex at the origin satisfying $\Gamma_n \subset \Gamma \subset \Gamma_1$, let $f \in C(\overline{\Gamma})$ satisfy (2), (3) and (4), and let $g(x,t)$ be a function defined on $\partial \mathbb{R}^n_+ \times [0, \infty)$ such that $g(x,t) > 0$ for all $x \in \partial \mathbb{R}^n_+$ if $t > 0$. If $n \ge 3$ and $\Gamma \subset \Gamma_k$ for some $1 \le k \le n$, then the problem*

$$\begin{cases} f(\lambda(-A^u)) = u^{p-\frac{n+2}{n-2}} & \textit{in } \mathbb{R}^n_+, \\ \lambda(-A^u) \in \Gamma & \textit{in } \mathbb{R}^n, \\ \frac{\partial u}{\partial x_n} = g(x,u) & \textit{on } \partial \mathbb{R}^n_+ \end{cases} \tag{24}$$

has no positive continuous viscosity subsolution for $p > 1 + \max\left\{0, \frac{2(2k-n)}{(n-2)k}\right\}$.

Proof. For each j consider the function U_j on B_j defined by (17). We have shown that U_j satisfies (18) and (19). Note that

$$\frac{\partial U_j}{\partial x_n} = 0 \quad \text{on } B_j \cap \partial \mathbb{R}^n_+.$$

Suppose (24) has a positive continuous viscosity subsolution u. We will derive a contradiction by showing that for each j

$$u(x) \le U_j(x) \quad \text{whenever } x \in B_j^+, \tag{25}$$

where $B_j^+ := B_j \cap \mathbb{R}^n_+$. Suppose (25) is not true, then one can find a number $a > 1$ such that $u \le aU_j$ on B_j^+ and $u(\bar{x}) = aU_j(\bar{x})$ for some $\bar{x} \in \overline{B_j^+}$. Since $U_j(x) \to +\infty$ as $d(x, \partial B_j) \to 0$, we must have $\bar{x} \in B_j \cap \overline{\mathbb{R}^n_+}$. If $\bar{x} \in \partial \mathbb{R}^n_+$, then from Definition 2 we have

$$0 = \frac{\partial (aU_j)}{\partial x_n}(\bar{x}) \ge g(\bar{x}, aU_j(\bar{x})) > 0$$

which is absurd. Therefore $\bar{x}$ must be in the interior of B_j^+. But this can be excluded again by imitating the proof of Lemma 1. □

By the same argument we can also establish the following nonexistence result.

Theorem 4. *Let $\Gamma \subset \mathbb{R}^n$ be an open convex symmetric cone with vertex at the origin satisfying $\Gamma_n \subset \Gamma \subset \Gamma_1$, let $f \in C(\overline{\Gamma})$ satisfy (2), (3) and (4), and let $g(x,t)$*

be a function defined on $\partial\mathbb{R}^n_+ \times [0,\infty)$ *such that* $g(x,t) > 0$ *for all* $x \in \partial\mathbb{R}^n_+$ *if* $t > 0$. *Then the problem*

$$\begin{cases} f(\lambda(D^2u)) = u^p \quad \text{in } \mathbb{R}^n_+ \\ \lambda(D^2u) \in \Gamma \quad \text{in } \mathbb{R}^n_+, \\ \frac{\partial u}{\partial x_n} = g(x,u) \quad \text{on } \partial\mathbb{R}^n_+ \end{cases}$$

has no positive continuous viscosity subsolution if $p > 1$.

REMARK 1. *If we take* $f(\lambda) = \sum_{i=1}^n \lambda_i$, *then it follows from Theorem 3 (or Theorem 4) that problem (22) has no solution if* $p > 1$, *without any condition on* q. *This improves the above mentioned result of Lou and Zhu.*

Acknowledgement. Part of the work of the second author is supported by NSF-DMS-0401118.

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