Methods and Applications of Analysis.
Vol. 11, No. 1, pp. 069–076, March 2004

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DARBOUX EQUATIONS IN EXTERIOR DOMAINS *

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Abstract. We give sufficient conditions ensuring existence and regularity of a radial solution to the following equation

$$\det(\phi_{ij}) = F(|x|, \phi, |\nabla\phi|), \text{ in } \Omega$$
$$\phi_{|\partial\Omega} = c$$

when Ω is an exterior domain.

1. Introduction. In this work, we consider the Dirichlet problem for real Monge-Ampère equations in exterior domains. More precisely, let $B \subset \mathbb{R}^n$ be an open ball, centered at the origin, that can be supposed, without loss of generality, to be the unit ball . Our purpose is to establish the existence of radial, convex solution $u \in C^2(\mathbb{R}^n \setminus B)$ of radially symmetric Monge-Ampère equation

$$\begin{cases} \det(\phi_{ij}) = F(|x|, \phi, |\nabla\phi|), & \text{in } \mathbb{R}^n \setminus \overline{B} \\ \phi_{|\partial B} = c \end{cases} \tag{1}$$

where F is a nonnegative continuous function. As usual, $|x|$ denotes the Euclidean length of $x = (x_1, ..., x_n)$ and n is (all over this paper) the dimension of our Euclidean space. Additional hypothesis on F are described in §2.

When Ω is a strictly convex domain, this problem has received considerable study. Not many results are known about the solutions in unbounded domains. In the case when $F > 0$, F.Finster and O.C. Schnürer [2] proved the existence of smooth, strictly convex solution to (1) under some restrictions on F. We can also cite the work of T. Kusano and Ch.A. Swanson [3] related to radially symmetric two-dimensional elliptic Monge-Ampère equations.

Our attention will be directed toward the construction of radial solutions $u(x) = u(t)$ of (1), $t = |x|$. Direct computation (see [1]), shows that solving the equation (1) in C^2 is equivalent to solving the ordinary differential equation

$$\begin{cases} \left[(y')^n\right]' = nt^{n-1}F(t, y, y'), & \text{if } t > 1 \\ y(1) = c \end{cases} \tag{2}$$

Without loss of generality, we can take $c = 0$.

If we take as initial condition $y'(1) = 0$, we can easily transform (2) into the following integro-differential equation

$$y(r) = \int_1^r \left[\int_1^\rho nt^{n-1}F(t, y(t), y'(t))\, dt\right]^{\frac{1}{n}} d\rho \tag{3}$$

*Received April 22, 2003; accepted for publication May 18, 2004.
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Example. Let $F(r) = (r-1)^{n-1-\varepsilon} r^{1-n}$, with $\varepsilon > 0$ small enough. Then, $u'(r) = \left[\frac{n}{n-\varepsilon}\right]^{\frac{1}{n}} (r-1)^{1-\frac{\varepsilon}{n}}$. In this case $F \in C^0$, but $u \notin C^2$.

This example shows that even when F depends only on r, it may not yield a C^2 solution, if F is allowed to vanish in the domain. This implies that we should place some restrictions on F.

Throughout this work, F satisfies some hypothesis be selected from the following list:

(H_1):

i) $F(t,y,z)$ is a nonincreasing function with respect to both y and z for each fixed (t,z) and (t,y), respectively.

ii) $\int_1^{+\infty} t^{n-1} F(t,0,0)\,dt < +\infty$

(H_2): $F(t,y,z) \le C_0 t^{-n-\alpha} |y|^{\beta} |z|^{\theta}$, with $\beta \ge 0$, $\alpha > \beta$, $\theta \ge 0$ and $C_0 \le \frac{\alpha-\beta}{n}$ if $\beta + \theta = n$.

(H_3):

i) $F(t,y,z)$ is a nondecreasing function with respect to both y and z for each fixed (t,z) and (t,y), respectively.

ii) There exists a constant $a > 0$ such that

$$\int_1^{+\infty} n t^{n-1} F(t,(t-1)a,a)\,dt \le a^n$$

(H_4): $F(t,y,z) = (t-1)^l \widetilde{F}(t,y,z)$, with $\widetilde{F}(1,y,0) \ne 0$, for $y \ge 0$, $l \ge n-1$, $\widetilde{F} \in C^0$.

An example of a Monge-Ampère equation satisfying (H_3) is the Gauss curvature equation

$$\begin{cases} \det(u_{ij}) = p(|x|)\, u^{\gamma} \left(1 + |\nabla u|^2\right)^{\delta}, & x \in \mathbb{R}^n \setminus B \\ u_{|\partial B} = 1 \end{cases}$$

with $\gamma, \delta \ge 0$, $2\delta + \gamma < n$ and p is a non-négative function satisfying:

$$\int_1^{+\infty} t^{n+\gamma-1} p(t)\,dt < +\infty$$

In the following, $\widetilde{F}$ is used as introduced in (H_4). We shall prove

Theorem A. If (H_4) and either (H_1), (H_2) or (H_3) holds, equation (1) has an infinitude of radial convex solutions $u \in C^2$ such that $\frac{u(x)}{|x|}$ has a positive finite limit at ∞.

Theorem B. If we suppose, in addition to the hypothesis of Theorem A, that

$$\widetilde{F} \in C^k\left((\mathbb{R}^n \setminus B) \times \mathbb{R}^2\right) \tag{4}$$

and

$$\text{either } \frac{l+1}{n} \in \mathbb{N} \text{ or } \frac{l+1}{n} \geq k+1 \tag{5}$$

then the solutions given by theorem A are in C^{k+2}

2. Proof of theorem A. To prove the existence of a radially symmetric convex solution to the problem (1), we need to introduce the Frechet space C^1 of all continuously differentiable functions in $[1,+\infty[$, with the topology of uniform convergence of functions and their first derivatives on compact intervals. Consider now the closed convex subset $\mathcal{K}_R$ of C^1

$$\mathcal{K}_R = \left\{ y \in C^1 \mid y(1) = 0, 0 \leq y'(t) \leq R \right\} \tag{6}$$

and the operator $T : \mathcal{K}_\mathcal{R} \rightarrow C^1$ defined by

$$T(y)(r) = \int_1^r \left[\int_1^\rho n t^{n-1} F(t, y(t), y'(t))\, dt \right]^{\frac{1}{n}} d\rho,\ r \geq 1 \tag{7}$$

In order to prove that T has a fixed point $y \in \mathcal{K}_R$, we need to verify that T maps $\mathcal{K}_R$ continuously into a relatively compact subset of $\mathcal{K}_R$.

If $y \in \mathcal{K}_R$, (7) implies that $T(y)(1) = 0$ and

$$0 \leq (Ty)'(r) = \left[\int_1^r n t^{n-1} F(t, y(t), y'(t))\, dt \right]^{\frac{1}{n}}$$

We shall need to verify that we can find a constant $R > 0$ such that

$$\left[\int_1^{+\infty} n s^{n-1} F(s, y(s), y'(s))\, ds \right]^{\frac{1}{n}} \leq R,\ \forall y \in \mathcal{K}_R \tag{8}$$

* If F satisfies (H_1), we can write using $(H_1)(i)$,

$$(Ty)'(r) \leq \left[\int_1^r n s^{n-1} F(s, 0, 0)\, ds \right]^{\frac{1}{n}}$$

by $(H_1)(ii)$, it suffices then to take

$$R = \left[n \int_1^{+\infty} s^{n-1} F(s, 0, 0)\, ds \right]^{\frac{1}{n}}$$

and we get

$$(Ty)'(r) \leq R$$

* When F satisfies (H_2), then, since

$$y(r) = \int_1^r y'(t)\, dt,$$

we get by (6),

$$|y(r)| \leq (r-1)\, R,$$

so,

$$(Ty)'(r) \leq \left[\int_1^r nC_0 s^{-\alpha-1}(s-1)^{\beta} R^{\beta+\theta} ds\right]^{\frac{1}{n}} \leq \left(\frac{n}{\alpha-\beta}C_0\right)^{\frac{1}{n}} R^{\frac{\beta+\theta}{n}}$$

In order to get (8), it suffices to take R small enough when $(\beta+\theta) > n$, big enough when $(\beta+\theta) < n$. In the case when $\beta+\theta = n$ and $C_0 \leq \frac{\alpha-\beta}{n}$, any positive constant R lead to

$$(Ty)'(r) \leq R$$

* Finally, if F satisfies (H_3), then, assumption $(H_3)\,(i)$ shows that

$$(Ty)'(r) \leq \left[\int_1^r ns^{n-1} F(s,(s-1)\,R,R)\, ds\right]^{\frac{1}{n}}$$

it suffices then to take $R = a$ to ensure by $(H_3)\,(ii)$ the inequality (8).

To establish the continuity of T, let (y_k) be a sequence in $\mathcal{K}_R$ with $\lim\limits_{k\to+\infty} y_k = y \in C^1$ in the C^1-topology. By the dominated convergence theorem, we have then

$$\lim_{k\to+\infty}\int_1^r ns^{n-1}F(s,y_k(s),y_k'(s))\,ds = \int_1^r ns^{n-1}F(s,y(s),y'(s))\,ds$$

uniformly on $[1,+\infty[$, from which Ty_k and $(Ty_k)'$ converge uniformly to Ty and $(Ty)'$, respectively, on compact intervals in $[1,+\infty[$. this means that Ty_k converges to Ty in the C^1-topology.

The relative compactness of $T(\mathcal{K}_R)$ is a consequence of Ascoli's Theorem; we need only verify the local uniform boundedness and local equicontinuity of the sets $T(\mathcal{K}_R)$ and $T(\mathcal{K}_R)' = \left\{(Ty)',\, y \in \mathcal{K}_R\right\}$.

Let us denote $G(t) = nF(t,u(t),u'(t))$ and $\widetilde{G}(t) = n\widetilde{F}(t,u(t),u'(t))$.

For every $y \in \mathcal{K}_R$, $1 \leq t_1 \leq t_2$, the inequality $a^{\frac{1}{n}} - b^{\frac{1}{n}} \leq (a-b)^{\frac{1}{n}}$, true for $a \geq b \geq 0$, implies

$$\begin{aligned}(Ty)'(t_2) - (Ty)'(t_1) &= \left(\int_1^{t_2} t^{n-1}G(t)\,dt\right)^{\frac{1}{n}} - \left(\int_1^{t_1} t^{n-1}G(t)\,dt\right)^{\frac{1}{n}} \\ &\leq \left(\int_{t_1}^{t_2} t^{n-1}G(t)\,dt\right)^{\frac{1}{n}}\end{aligned}$$

* If F satisfies (H_1), then

$$G(t) \leq nF(t,0,0)$$

and

$$(Ty)'(t_2) - (Ty)'(t_1) \leq \left(\int_{t_1}^{t_2} nt^{n-1}F(t,0,0)\,dt\right)^{\frac{1}{n}} \to 0, \text{ as } t_1,t_2 \to \infty$$

* If F satisfies (H_2), then,

$$\begin{aligned}(Ty)'(t_2) - (Ty)'(t_1) &\leq \left(\int_{t_1}^{t_2} nC_0 t^{n-1} t^{-n-\alpha}(t-1)^{\beta} R^{\beta+\theta} dt\right)^{\frac{1}{n}} \\ &\leq C_1\left(\int_{t_1}^{t_2} t^{\beta-\alpha-1} dt\right)^{\frac{1}{n}} \to 0, \text{ as } t_1,t_2 \to \infty\end{aligned}$$

* Finally, when F satisfies (H_3) , then by (i), since $R = a$,

$$G(t) \leq nF(t,(t-1)a,a)$$

and

$$(Ty)'(t_2) - (Ty)'(t_1) \leq \left(\int_{t_1}^{t_2} nt^{n-1}F(t,(t-1)a,a)\,dt\right)^{\frac{1}{n}} \to 0, \text{ as } t_1,t_2 \to \infty$$

Then, in all these cases, for any compact interval I in $[1,+\infty[$ and arbitrary $\varepsilon > 0$, there is a corresponding $\delta > 0$, independent of t_1, t_2 and $y \in \mathcal{K}_R$, such that

$$\left|(Ty)'(t_2) - (Ty)'(t_1)\right| \leq \varepsilon$$

for all $t_1,t_2 \in I$ with $|t_1 - t_2| < \delta$.

The local equicontinuity of $T(\mathcal{K}_R)$ can be verified in the same way, and the local uniform boundedness is obvious.

Therefore the Schauder-Tychonoff fixed point theorem ([5]; lemma 1 and [6]; Theorem 4.5.1.) implies that T has a fixed point $u \in \mathcal{K}_R$, satisfying the integro-differential equation (3) for any R such that (8) holds. It remains to prove that $u' \in C^1$.

For $t > 1$, we have

$$u'(t) = \left[\int_1^t s^{n-1}(s-1)^l \widetilde{G}(s)\,ds\right]^{\frac{1}{n}}$$

Since $\widetilde{G}(1) \neq 0$, then $u' \in C^1\,]1,+\infty[$ and

$$u''(t) = t^{n-1}(t-1)^l \widetilde{G}(t)\left[\int_1^t s^{n-1}(s-1)^l \widetilde{G}(s)\,ds\right]^{\frac{1}{n}-1}$$

$$= t^{n-1}(t-1)^{\frac{l+1}{n}-1}\widetilde{G}(t)\left[\int_0^1 [(t-1)s+1]^{n-1} s^l \widetilde{G}((t-1)s+1)\,ds\right]^{\frac{1}{n}-1}$$

which gives

$$\lim_{t\to 1^+} u''(t) = \begin{cases} 0, & \text{if } l > n-1 \\ \left[\frac{1}{l+1}\right]^{\frac{1}{n}-1}\widetilde{G}(1)^{\frac{1}{n}} & \text{if } l = n-1 \end{cases}$$

Hence, $u \in C^2\,[1,+\infty[$. It is not to be noted that u is a solution of (1) satisfying $u(1) = 0$ and $u'(1) = 0$.

Furthermore, the relation (3) and the inequality (8) imply that the limit

$$\lim_{t\to+\infty}\frac{u(t)}{t} = \lim_{t\to+\infty} u'(t) = \left[\int_1^{+\infty} ns^{n-1}F(s,u(s),u'(s))\,ds\right]^{\frac{1}{n}}$$

is positive and finite, proving the asymptotic property in theorem A.

Since any non-negative constant b will serve as initial value $y'(1) = b$, there exists an infinitude of radial convex solutions to our problem.

3. Proof of theorem B. In this section, we study the regularity of the solution u given by theorem A. To prove the C^{k+2} regularity of u, let us proceed by induction on $k \in \mathbb{N}$. For $k = 0$, we have established in section 2, that $u \in C^2$. Suppose that

$$\widetilde{F} \in C^{k-1} \Rightarrow u \in C^{k+1}$$

for some fixed $k \geq 1$. Assume now that $\widetilde{F} \in C^k$. It follows in particular that $u \in C^{k+1}$. Hence, from the integral formula (7) and the hypothesis (H_4), we get $u \in C^{k+2}(]1,+\infty[)$. It remains to check the regularity of u at the boundary $t = 1$.

The following preliminary result will be needed

Lemma ([4] corollary 4.2). The kth derivative of $g^{\frac{1}{n}}$, can be written as a sum of terms of the form

$$g^{\frac{1}{n}-\lambda}P_\lambda\left(g', g'', ..., g^{(k+1-\lambda)}\right)$$

where P_λ is a monomial of degree $\lambda \le k$ and of weighted degree k.

Now, using the notation

$$H_y(t) = \int_0^1 [(t-1)s+1]^{n-1} s^l \widetilde{G_y}((t-1)s+1)\,ds, \tag{9}$$

we can write

$$u'(t) = (t-1)^{\frac{l+1}{n}} H_u^{\frac{1}{n}}(t)$$

where, by the induction hypothesis, $H_u \in C^k$. Then,

$$u^{(k+1)}(t) = \sum_{i=0}^{k} \left(\begin{array}{c} k \\ i \end{array}\right) \left[(t-1)^{\frac{1+l}{n}}\right]^{(i)} \left(H_u^{\frac{1}{n}}\right)^{(k-i)}(t)$$

furthermore, applying the above lemma, we get the following

$$\left(H_u^{\frac{1}{n}}\right)^{(l)} = \sum_{i=2}^{l} c_i H_u^{\frac{1}{n}-i} P_i\left(H_u', ..., H_u^{(l+1-i)}\right) + \frac{1}{n} H_u^{\frac{1}{n}-1} H_u^{(l)}, \ \ \forall l \le k$$

Since, $\forall j \le k$,

$$(H_u)^{(j)}(t) = \sum_{i=0}^{j} c_{i,j} \int_0^1 \left([(t-1)s+1]^{n-1}\right)^{(j-i)} s^{l+i} \widetilde{G_u}^{(i)}((t-1)s+1)\,ds$$

it suffices then to prove that

$$f(t) = (t-1)^{\frac{1+l}{n}} h_k \in C^1([1,+\infty[)$$

where $h_k(t) = \int_0^1 [(t-1)s+1]^{n-1} s^{l+k} \widetilde{G_u}^{(k)}((t-1)s+1)\,ds$.
Differentiating f, yields

$$\forall t > 1,\ f'(t) = (t-1)^{\frac{1+l}{n}-1} \left[t^{n-1} \widetilde{G_u}^{(k)}(t) + c h_k(t)\right]$$

which implies

$$\lim_{r \to 1^+} f'(t) = \begin{cases} 0 & \text{if } l > n-1 \\ \widetilde{G_u}^{(k)}(1)\left[1 + c\int_0^1 s^{l+k} ds\right], & \text{if } l = n-1 \end{cases}$$

Consequently, $f \in C^1([1,+\infty[)$. Which completes the proof of theorem B.

Acknowledgement. The author would like to thank the referee for several helpful suggestions.

REFERENCES

[1] Ph. Delanoë, *Radially symmetric boundary value problems for real and complex elliptic Monge-Ampère equations*, Journal of Differential equations, 58 (1985), pp. 318–344.
[2] F. Finster and O.C. Schnürer, *Hypersurfaces of prescribed Gauss curvature in exterior domains*, Calc. Var., 15 (2002), pp. 67–80.
[3] T. Kusano and Ch.A. Swanson, *Existence theorems for elliptic Monge-Ampère equations in the plane*, Differential and Integral Equations, 3:3 (1990), pp. 487–493.
[4] D. Monn, *Regularity of the complex Monge-Ampère equation for radially symmetric functions of the unit ball*, Math. Ann., 275 (1986), pp. 501–511.

[5] A.J.B. Potter, *An elementary version of the Leray-Schauder theorem*, J. Lond. Math. Soc., 2:5 (1972), pp. 414–416.
[6] D.R. Smart, *Fixed point Theorems*, Cambridge Tracts in Mathematics, No. 66, Cambridge Univ. Press, London/New York, 1974.