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LOCAL TIME DECAY FOR A NONLINEAR BEAM EQUATION *

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Abstract. Using the Morawetz' Radial Identity, we show that the local energy of a solution is integrable in time and the local L^2 norm of the solution approaches zero as time approaches the infinity for a nonlinear beam equation with the spatial dimension > 5.

1. Introduction. Consider a nonlinear beam equation

$$u_{tt} + \Delta^2 u + f(u) = 0 \tag{1}$$

where $u = u(x,t)$, $x = (x_1, x_2, \ldots, x_n) \in R^n$, R^n is the n-dimensional Euclidean space, $n > 5$, $t \geq 0$, $\Delta =$ Laplacian in x, and $f(u)$ satisfies

$$c_1(uf(u) - 2F(u)) + c_0 u^2 \geq F(u) \geq c_0 u^2 \tag{2}$$

for some positive constants c_0 and c_1, where $F'(u) = f(u)$ with $F(0) = 0$. As usual, the subscript in variables denotes the partial derivative, thus, $u_t = \partial u/\partial t$, etc... We also use the notations $\partial_j = \partial/\partial x_j$, and $u_r = (x/r) \cdot \nabla u$, where ∇ is the gradient in x, and $r = |x|$. Moreover, for a function of one variable $g(s), g'(s) = d(g(s))/ds$ denotes the derivative of g in s. Finally, that a function is C^n means that its n^{th} partial derivatives are continuous. In this work, we show that the local energy of a solution is integrable in time and the local L^2 norm of a solution approaches zero as time approaches the infinity. Our method follows [1] in utilizing the Morawetz' Radial Identity [5].

The global scattering problem was considered in [1] along with several inequalities. It was conjectured in [1] that the local energy is integrable in t and tends to zero as t approaches the infinity. This work proves the first part of this conjecture. The well-posedness, low-energy scattering, stability and instability of solitary and standing waves, and the time decay of solutions for the nonlinear beam equation with a slightly different $f(u)$ can be found in [2, 3]. In the one-spatial dimension, a similar equation to (1) with a different nonlinear term has been studied as a model for a suspension bridge [4].

We shall need the following result from [1]:
The energy $E[u] = \int_{R^n} [(1/2)u_t^2 + (1/2)|\Delta u|^2 + F(u)]dx$ is a constant, and for $n > 5$, assuming that u is a solution that is smooth enough and small enough at the spatial infinity, then there is a positive constant c such that

$$\int_0^\infty \int_{R^n} (1/r)[uf(u) - 2F(u)]dxdt \leq cE[u] \tag{3}$$

$$\int_0^\infty \int_{R^n} (1/r^3)|\nabla u|^2 dxdt \leq cE[u] \text{ , and} \tag{4}$$

$$\int_0^\infty \int_{R^n} (1/r^5)|u|^2 dxdt \leq cE[u] \text{ , provided } n \geq 6 \tag{5}$$

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2. The Morawetz' radial identity. Let ζ be a C^4 radially symmetric function of x. Multiplying (1) by $\zeta(u_r+((n-1)/(2r))u)$ we get the following identity, assuming that u is C^2 in t and C^4 in x,

$$\begin{aligned} 0 &= (u_{tt}+\Delta^2 u+f(u))\zeta(u_r+(n-1)u/(2r)) \\ &= X_t+\nabla\cdot Y+Z \end{aligned} \tag{6}$$

where

$X=u_t\zeta(u_r+(n-1)u/(2r))$,

$Y=-((n-1)u^2/4)(\nabla(\Delta(\zeta/r)))+$ a function depending on ζ, ζ', ζ'', ζ''', x, u, $F(u)$, ∇u, Δu, $\nabla(\Delta u)$, u_r, ∇u_r, u_t, $1/r$, $1/r^2$ and $1/r^3$,

and

$Z=(\zeta'/2)(u_t)^2+(\Delta u)^2[3\zeta'/2]+A(u_r)^2+B(|\nabla u|^2-(u_r)^2)+Cu^2+(n-1)\zeta(uf(u)-2F(u))/(2r)-\zeta' F(u)+(\zeta-r\zeta')P$,

where

$A=-7\zeta'''/2-\zeta'(n-1)(n-3)/(2r^2)+\zeta(n-1)(n-3)/(2r^3)$,
$B=-3\zeta'''/2+\zeta''(n-5)/r-\zeta'(n^2+2n-19)/(2r^2)+\zeta(n^2+2n-19)/(2r^3)$,
$C=((n-1)/2)[\zeta''''/(2r)+\zeta'''(n-3)/(r^2)+\zeta''(n-3)(n-7)/(2r^3)-3\zeta'(n-3)(n-5)/(2r^4)+3\zeta(n-3)(n-5)/(2r^5)]$,
$P=(2/r)[\sum_{i,j}(S_{ij}u)^2-\sum_i(\sum_j(x_j/r)S_{ij}u)^2]\geq 0$
with $S_{ij}u=((x_i)/r^3)\sum_k[x_k(x_k\partial_j-x_j\partial_k)u_r]+\partial_j\sum_k[(x_k/r^2)(x_k\partial_i-x_i\partial_k)u]$.

Remark. If $\zeta=1$, this identity is the identity shown in the proof of Theorem 1 of [1].

3. The integrability of the local energy. We now state the main result of this work.

Theorem. Consider the nonlinear beam equation (1) with the condition (2) on $f(u)$. Assume that the spatial dimension n is >5. Assume also that u is C^2 in t and C^4 in x, u_t is bounded, and u and all its partial derivatives in x up to the 4^{th} order approach zero as $|x|$ approaches the infinity. Then the local energy is integrable in t.

Proof. Assume that ζ and ζ' are non-negative functions and ζ, ζ', ζ'', and $\zeta'''$are bounded functions. We integrate both sides of (6) with respect to x in R^n and t from 0 to T. With the assumption on the smoothness and smallness of u at the spatial infinity, we have $\int_{R^n}\nabla\cdot Y dx=0$ for n >5.

Thus

$$0\geq\int_{R^n}X(x,T)dx-\int_{R^n}X(x,0)dx+\int_0^T\int_{R^n}Zdxdt.$$

Now we can rewrite X into two ways:

$$\begin{aligned} X =& -(\zeta/2)(u_t^2+|\nabla u|^2)+(\zeta/2)[u_t^2+|W|^2+2((x/r)\cdot W)u_t] \\ & -\nabla\cdot[(n-1)\zeta u^2x/(4r^2)]+(n-1)(n-3)\zeta u^2/(8r^2)+(n-1)\zeta' u^2/(4r), \end{aligned} \tag{7}$$

and

$$\begin{aligned} X =&(\zeta/2)(u_t^2+|\nabla u|^2)-(\zeta/2)[u_t^2+|W|^2-2((x/r)\cdot W)u_t] \\ &+\nabla\cdot[(n-1)\zeta u^2 x/(4r^2)]-(n-1)(n-3)\zeta u^2/(8r^2)-(n-1)\zeta' u^2/(4r), \end{aligned} \tag{8}$$

where W $=\nabla u+(n-1)ux/(2r^2)$.

Using the first way (7) for $X(x,T)$ and the second way (8) for $X(x,0)$, we get

$$\int_{R^n}(1/2)\zeta(u_t^2+|\nabla u|^2)(x,0)dx+\int_{R^n}(1/2)\zeta(u_t^2+|\nabla u|^2)(x,T)dx \geq \int_0^T\int_{R^n} Zdxdt.$$

Let T approach the infinity, we get

$$\int_0^\infty\int_{R^n} Zdxdt \leq c_2E[u]\text{ , for some positive constant } c_2, \tag{9}$$

where c_2 depends on c_0, c_1 and the bound for ζ.

Now, let $\zeta(x)=\zeta(r)=1-1/(r^2+4)^2$, where $r=|x|$. Since ζ and ζ' are non-negative and ζ,ζ',ζ'', and ζ''' are bounded functions, the inequality (9) holds. Substituting ζ into (9), we get

$$\begin{aligned} &\int_0^\infty\int_{R^n}[(u_t^2+(\Delta u)^2)r/(r^2+4)^3+|\nabla u|^2/(r^3(r^2+4)^5)+P/(r^2+4)^3 \\ &\qquad\qquad +u^2/(r^5(r^2+4)^6)+(uf(u)-2F(u))/(r(r^2+4)^2)]dxdt \\ &\leq \int_0^\infty\int_{R^n} 4F(u)r/(r^2+4)^3dxdt+c_2E[u] \\ &\leq \int_0^\infty\int_{R^n}[4c_1(uf(u)-2F(u))+4c_0u^2]r/(r^2+4)^3dxdt+c_2E[u] \\ &\leq \int_0^\infty\int_{R^n}[4c_1(uf(u)-2F(u))/r+4c_0u^2/r^5]dxdt+c_2E[u] \\ &\leq c_3E[u] \end{aligned}$$

for some positive constant c_3. Note that we have used (3) and (5) in the above inequality.

Let $h>b>0$, we get

$$\int_0^\infty\int_{b\leq|x|\leq h}[u_t^2+(\Delta u)^2+|\nabla u|^2+u^2+(uf(u)-2F(u))]dxdt \leq c_4E[u]$$

for some positive contant c_4 depending on b and h.

Therefore, $\int_0^\infty\int_{b\leq|x|\leq h}[u_t^2+(\Delta u)^2+F(u)]dxdt \leq c_5E[u]$, for some positive constant c_5 depending on c_0, c_1, b and h. Since the equation (1) is invariant under spatial translation, we get

$$\int_0^\infty\int_{|x|\leq h}[u_t^2+(\Delta u)^2+F(u)]dxdt \leq c_6E[u]\text{ , for some positive constant } c_6.$$

Therefore the local energy is integrable in time.

4. Time decay of the local L^2 norm. Now, we are going to show that the local L^2 norm of u approaches 0 as t approaches the infinity. The idea of the proof is from [5]. Let $h > 0$ and $t > t_1 > 0$, then

$$\begin{aligned}(t-t_1)\int_{|x|\le h} u^2(x,t)dx &= \int_{t_1}^{t} \partial[(\tau-t_1)\int_{|x|\le h} u^2(x,\tau)dx]/\partial\tau d\tau \\ &= \int_{t_1}^{t}\int_{|x|\le h} u^2(x,\tau)dxd\tau + \int_{t_1}^{t}(\tau-t_1)\int_{|x|\le h} 2uu_t(x,\tau)dxd\tau.\end{aligned}$$

Let $t_1 = t-1$, we get

$$\int_{|x|\le h} u^2(x,t)dx \le \int_{t-1}^{t}\int_{|x|\le h} u^2(x,\tau)dxd\tau + \int_{t-1}^{t}\int_{|x|\le h}(u^2+u_t^2)(x,\tau)dxd\tau.$$

Hence $\int_{|x|\le h} u^2(x,t)dx$ approaches 0 as t approaches the infinity since the local energy is integrable in time.

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