METHODS AND APPLICATIONS OF ANALYSIS.
Vol. 10, No. 3, pp. 387–412, September 2003
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AN INTRODUCTION TO THE H_q-SEMICLASSICAL ORTHOGONAL POLYNOMIALS *

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Abstract. Orthogonal polynomials associated with H_q- semiclassical linear form will be studied as a generalization of the H_q-classical linear forms. The concept of class and a criterion for determining it will be given. The q-difference equation that the corresponding formal Stieltjes series satisfies is obtained. Also, the structure relation as well as the second order linear q-difference equation are obtained. Some examples of H_q-semiclassical of class 1 were highlighted.

Introduction. The aim of this paper is to present the analysis and characterization of the q-analogues of D-semiclassical orthogonal polynomials. D-semiclassical orthogonal polynomials were introduced in a seminal paper by J. A. Shohat [22] and extensively studied by P. Maroni and coworkers in the last decade[13-19]. Furthermore, the present contribution is a natural continuation of a previous work [9] by me and P. Maroni on q-classical orthogonal polynomials.

In the literature, the extension of classical orthogonal polynomials (Hermite, Laguerre, Jacobi, and Bessel) can be done in the q-case from three basic approaches (see [2] for a comparative analysis).

The first one is related with the so called Askey Tableau, where all the classical families appear in a limiting process from the top of Askey-Wilson polynomials (see[10]).

The second one concerns the hypergeometric character of classical orthogonal polynomials , i.e. as solutions of a second order linear differential equation with polynomial coefficients, the so called Nikiforov-Uvarov approach (see[21]).

The third one is based in the Pearson equation which satisfies the symmetric factor for the above differential equation. This idea appears in several papers but the basic theory was developed by P. Maroni.

The structure of this paper is as follows: The first section contains material of preliminary and introductory character. Instead of the derivative operator, we use the q-operator H_q introduced by Hahn [7]. In particular, we define a H_q-semiclassical linear form u from a functional equation which is the $q-$difference distributional Pearson one. The second section deals with so-called class of H_q-semiclassical linear forms. A criterion for determining it is given. In the third section, we establish the different characterizations of H_q-semiclassical linear forms. We can characterize a H_q-semiclassical linear form through the fact that its Stieltjes function satisfies a first order linear $q-$difference equation with polynomial coefficients. A second characterization is the so-called structure relation that the polynomials $\{P_n\}_{n\geq 0}$orthogonal with respect to u satisfy. It is deduced from theory of finite-type relations between polynomial sequences [19]. A third characterization is the second order linear $q-$difference equation satisfied by P_{n+1}, $n \geq 0$. Lastly, in section 4 we construct some examples of H_q-semiclassical linear forms of class 1 by taking into account a method studied by P. Maroni in [15] for the D-case (see paragraph 5.1 below) and by using some symmetric H_q-classical linear forms in [9].

*Received January 24, 2003; accepted for publication September 24, 2003.
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1. Preliminaries and notations. Let $\mathcal{P}$ be the vector space of polynomials with coefficients in $\mathbb{C}$ and let $\mathcal{P}'$ be its topological dual.We denote by $\langle u, f\rangle$ the effect of $u \in \mathcal{P}'$ on $f \in \mathcal{P}$. In particular, we denote by $(u)_n := \langle u, x^n\rangle$, $n \geq 0$ the moments of u. For any linear form u, any polynomial g , let gu , be the linear form defined by duality

$$\langle gu, f\rangle := \langle u, gf\rangle \quad , \quad f, g \in \mathcal{P}.$$

For $f \in \mathcal{P}$ and $u \in \mathcal{P}'$, the product uf is the polynomial

$$(uf)(x) := \left\langle u, \frac{xf(x) - \zeta f(\zeta)}{x - \zeta}\right\rangle = \sum_{k=0}^{n} \left(\sum_{\nu=k}^{n} f_\nu (u)_{\nu-k}\right) x^k ,$$

where $f(x) = \sum\limits_{k=0}^{n} f_k x^k$. The Stieltjes function of $u \in \mathcal{P}'$ is defined by

$$S(u)(z) := -\sum_{n\geq 0} \frac{(u)_n}{z^{n+1}} .$$

Denoting by Δ the linear space generated by $\left\{\delta^{(n)}\right\}_{n\geq 0}$, where $\delta^{(n)}$ means the n *th* derivative of the Dirac delta in the origin, i.e.,

$$\left\langle \delta^{(n)}, f\right\rangle = (-1)^n f^{(n)}(0) = (-1)^n \frac{d^n}{dx^n} f(0) \ , f \in \mathcal{P} ,$$

and by $\mathcal{F}$ the isomorphism : $\Delta \longrightarrow \mathcal{P}$ defined as follows [14] :
for $u = \sum\limits_{k=0}^{n} (u)_k \dfrac{(-1)^n}{n!} \delta^{(n)}$, $\mathcal{F}(u) = \sum\limits_{k=0}^{n} (u)_k\ z^k$.
Let $\{P_n\}_{n\geq 0}$ be a sequence of monic polynomials with $\deg P_n = n$, $n \geq 0$ (polynomial sequence : PS) and let $\{u_n\}_{n\geq 0}$ be its dual sequence, $u_n \in \mathcal{P}'$ defined by $\langle u_n, P_m\rangle := \delta_{n,m}$, $n, m \geq 0$. Let us recall some results [17] .

Lemma 1.1. *For any $u \in \mathcal{P}'$ and any integer $m \geq 1$, the following statements are equivalent*
i) $\langle u, P_{m-1}\rangle \neq 0$, $\langle u, P_n\rangle = 0$, $n \geq m$,
ii) $\exists \lambda_v \in \mathbb{C}$, $0 \leq \nu \leq m-1$, $\lambda_{m-1} \neq 0$ *such that*

$$u = \sum_{\nu=0}^{m-1} \lambda_\nu u_\nu.$$

Similarly, with the definitions
$\langle h_a u, f\rangle := \langle u, h_a f\rangle = \langle u, f(ax)\rangle$, $u \in \mathcal{P}'$, $f \in \mathcal{P}$, $a \in \mathbb{C} - \{0\}$.
The linear form u is called *regular* if we can associate with it a sequence of polynomials $\{P_n\}_{n\geq 0}$ such that

$$\langle u, P_m P_n\rangle = r_n \delta_{n,m} \quad , \quad n,\ \ m \geq 0 \ ; \ \ r_n \neq 0 \ , \ n \geq 0.$$

The sequence $\{P_n\}_{n\geq 0}$ is then said orthogonal with respect to u. Necessarily, $u = \lambda u_0$, $\lambda \neq 0$ and $\{P_n\}_{n\geq 0}$ is an (OPS) such that any polynomial can be supposed monic (MOPS). In this case, we have $u_n = r_n^{-1} P_n u_0$, $n \geq 0$ and conversely. Also, the (MOPS)$\{P_n\}_{n\geq 0}$ fulfils the recurrence relation

$$\begin{cases} P_0(x) = 1 \ , \ P_1(x) = x - \beta_0 \ , \\ \\ P_{n+2}(x) = (x - \beta_{n+1}) P_{n+1}(x) - \gamma_{n+1} P_n(x) \ , \ \gamma_{n+1} \neq 0 \ , \ n \geq 0. \end{cases}$$

When u is regular , let ϕ be a polynomial such that $\phi u = 0$.Then $\phi = 0$. Indeed, we have $0 = \langle \phi u, P_m \rangle = c \left\langle u, P_m^2 \right\rangle$ if $\phi = cx^m + \dots$.

Lastly, from the linear application $p \mapsto (\theta_c p)(x) = \dfrac{p(x) - p(c)}{x - c}$, $p \in \mathcal{P}$, $c \in \mathbb{C}$, we define $(x-c)^{-1} u$ by $\left\langle (x-c)^{-1} u, p \right\rangle := \langle u, \theta_c p \rangle$.

The Hahn's operator H_q is defined in the linear space $\mathcal{P}$ in the following way [7, 9, 12]

$$(H_q f)(x) = \frac{f(qx) - f(x)}{(q-1)x}, \ f \in \mathcal{P} \ , \ q \in \widetilde{\mathbb{C}} \ ,$$

where $\widetilde{\mathbb{C}} := \mathbb{C} - \left(\{0\} \bigcup \left(\bigcup_{n\geq 0} \{z \in \mathbb{C}, \ z^n = 1\} \right) \right)$. By duality, the image of a linear form using this operator H_q is a linear form such that [9]

$$\langle H_q u, f \rangle = - \langle u, H_q f \rangle \ , \ \forall \ f \in \mathcal{P}. \tag{1.1}$$

In particular, this yields

$$(H_q u)_n = - [n]_q (u)_{n-1} \ , \ n \geq 0 \ , \tag{1.1$'$}$$

where $(u)_{-1} = 0$ and $[n]_q := \dfrac{q^n - 1}{q - 1}$, $n \geq 0$ [9].

As a consequence of lemma 1.1, the dual sequence $\left\{ u_n^{[1]}(q) \right\}_{n\geq 0}$ of $\left\{ P_n^{[1]}(.;q) := \dfrac{H_q P_{n+1}}{[n+1]_q} \right\}_{n\geq 0}$ is given by [9]

$$H_q \left(u_n^{[1]}(q) \right) = - [n+1]_q \, u_{n+1} \ , \ n \geq 0. \tag{1.2}$$

Remark. When $q \to 1$, we meet again the derivative D. The following well known results (see [9, 14, 19])will be useful for our work. We summarize them in

Lemma 1.2. *Let $\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$ be sequences of monic polynomials with $\{u_n\}_{n\geq 0}$ and $\{v_n\}_{n\geq 0}$ their respective dual sequences. Let Φ be a monic polynomial with $\deg \Phi = t \geq 0$ and $\Phi u_n \neq 0$, $n \geq 0$. The following properties are equivalent*

i) There is an integer $s \geq 0$ such that

$$\Phi(x) Q_n(x) = \sum_{\nu=n-s}^{n+t} \lambda_{n,\nu} P_\nu(x) \ , \ n \geq s \ , \tag{1.3}$$

$$\exists r \geq s : \quad \lambda_{r,r-s} \neq 0. \tag{1.4}$$

ii) There are an integer $s \geq 0$ and an application from $\mathbb{N}$ into $\mathbb{N}$: $m \longmapsto \mu_m$ satisfying

$$\max(0, m-t) \leq \mu_m \leq m+s\ ,\quad m \geq 0\ , \tag{1.5}$$

$$\exists m_0 \geq 0 : \quad \mu_{m_0} = m_0 + s\ , \tag{1.6}$$

and such that

$$\Phi u_m = \sum_{\nu=m-t}^{\mu_m} \lambda_{\nu,m} v_\nu\ ,\quad m \geq t\ , \tag{1.7}$$

$$\lambda_{\mu_m,m} \neq 0\ ,\quad m \geq 0. \tag{1.8}$$

LEMMA 1.3. *For $f, g \in \mathcal{P}$, $u \in \mathcal{P}'$ and $c \in \mathbb{C}$, we have*

$$(x-c)\left((x-c)^{-1} u\right) = u\ , \tag{1.9}$$

$$(x-c)^{-1}\left((x-c)\, u\right) = u - (u)_0\, \delta_c\ , \tag{1.10}$$

$$S(fu)(z) = f(z)\, S(u)(z) + (u\theta_0 f)(z)\ , \tag{1.11}$$

$$H_q(fg)(x) = (h_q f)(x)\,(H_q g)(x) + g(x)\,(H_q f)(x)\ , \tag{1.12}$$

$$h_a(gu) = (h_{a^{-1}} g)(h_a u)\ , \tag{1.13}$$

$$h_{q^{-1}} \circ H_q = H_{q^{-1}}\ ,\ H_q \circ h_{q^{-1}} = q^{-1} H_{q^{-1}}\ in\ \mathcal{P}\ , \tag{1.14}$$

$$H_q \circ H_{q^{-1}} = q^{-1} H_{q^{-1}} \circ H_q\ in\ \mathcal{P}\ , \tag{1.14$'$}$$

$$h_{q^{-1}} \circ H_q = q^{-1} H_{q^{-1}}\ ,\ H_q \circ h_{q^{-1}} = H_{q^{-1}}\ in\ \mathcal{P}'\ , \tag{1.15}$$

$$\left(H_q\left(h_{q^{-1}} f\right) g\right)(x) = f(x)\,(H_q g)(x) + q^{-1} g(x)\left(H_{q^{-1}} f\right)(x)\ , \tag{1.16}$$

$$H_q(gu) = \left(h_{q^{-1}} g\right) H_q u + q^{-1}\left(H_{q^{-1}} g\right) u. \tag{1.17}$$

Furthermore,

LEMMA 1.4. *For $f \in \mathcal{P}$ and $u \in \mathcal{P}'$, the following formulas hold*

$$\left(H_q(uf)\right)(x) = \left(\left(H_{q^{-1}} u\right)(h_q f)\right)(x) + q\left(u\left(H_q f\right)\right)(x) + (u\theta_0 f)(x)\,, \tag{1.18}$$

$$S(H_q u)(z) = q^{-1}\left(H_{q^{-1}}(S(u))\right)(z)\,, \tag{1.19}$$

$$\left(h_q(\theta_0 f)\right)(x) = q^{-1}\left(\theta_0(h_q f)\right)(x)\,, \tag{1.20}$$

$$(u\theta_0 H_q f)(x) = q\left(u\left(H_q(\theta_0 f)\right)\right)(x) + \left(u\theta_0^2 f\right)(x)\,, \tag{1.21}$$

$$H_q(u\theta_0 f)(x) = q^{-1}\left(H_{q^{-1}} u\right)(\theta_0 \circ h_q f)(x) + (u\theta_0 H_q f)(x)\,. \tag{1.22}$$

Let Φ monic and Ψ be two polynomials, $\deg \Phi = t$, $\deg \Psi = p \geq 1$. We suppose that the pair (Φ, Ψ) is *admissible*, i.e. when $p = t-1$, writing $\Psi(x) = a_p x^p + \ldots$, then $a_p \neq [n+1]_q$, $n \in \mathbb{N}$.

DEFINITION 1.5. *A linear form u is called H_q-semiclassical when it is regular and satisfies the equation*

$$H_q(\Phi u) + \Psi u = 0 \,, \tag{1.23}$$

where the pair (Φ, Ψ) is admissible. The corresponding orthogonal sequence $\{P_n\}_{n\geq 0}$ is called H_q-semiclassical.

REMARK. We have the following result (see[9]).
Let $\left\{\widehat{P}_n := a^{-n}(h_a P_n)\right\}_{n\geq 0}$, $a \neq 0$; when u_0 satisfies (1.23) , then $\widehat{u}_0 = h_{a^{-1}} u_0$ fulfils the equation

$$H_q\left(\widehat{\Phi}\widehat{u}_0\right) + \widehat{\Psi}\widehat{u}_0 = 0 \,, \tag{1.24}$$

where $\widehat{\Phi}(x) = a^{-t}\Phi(ax)$, $\widehat{\Psi}(x) = a^{1-t}\Psi(ax)$.

2. Class of a H_q-semiclassical linear form. It is obvious that a H_q-*semiclassical* linear form satisfies an infinity number of equations of type (1.23). Indeed, multiplying (1.23) by a polynomial χ we obtain

$$\begin{aligned}
0 &= \chi H_q(\Phi u) + \chi \Psi u = \left(h_{q^{-1}}(h_q \chi)\right) H_q(\Phi u) + \chi \Psi u \\
&= H_q((h_q \chi)\Phi u) - q^{-1}\left(H_{q^{-1}} \circ h_q \chi\right)\Phi u + \chi \Psi u \text{ (by (1.17))} \\
&= H_q((h_q \chi)\Phi u) + \{\chi \Psi - \Phi(H_q \chi)\} u \text{ (by (1.14)).}
\end{aligned}$$

Then, for any pair (Φ, Ψ) satisfying (1.23) we associate the positive integer $\max(\deg \Phi - 2, \deg \Psi - 1)$. Denoting

$$\mathfrak{h}(u) := \{\max(\deg \Phi - 2, \deg \Psi - 1),\ H_q(\Phi u) + \Psi u = 0\},$$

what leads us to the following definition

DEFINITION 2.1. *Giving a H_q-semiclassical linear form u, we define the class of u, the positive integer s, as*

$$s := \min \mathfrak{h}(u).$$

The corresponding orthogonal sequence $\{P_n\}_{n\geq 0}$ will be said to be of class s.

LEMMA 2.2. *Let u be a H_q-semiclassical linear form satisfying*

$$H_q(\Phi_1 u) + \Psi_1 u = 0 \,, \tag{2.1}$$

and

$$H_q(\Phi_2 u) + \Psi_2 u = 0 \,, \tag{2.2}$$

where $\Phi_1, \Psi_1, \Phi_2, \Psi_2$ *are polynomials,* Φ_1, Φ_2 *monic,* $\deg\Psi_1 \geq 1, \deg\Psi_2 \geq 1$. *Denoting* $s_1 = \max(\deg\Phi_1 - 2, \deg\Psi_1 - 1)$, $s_2 = \max(\deg\Phi_2 - 2, \deg\Psi_2 - 1)$. *Let* $\Phi = \gcd(\Phi_1, \Phi_2)$. *Then, there exists a polynomial* $\Psi, \deg\Psi \geq 1$ *such that*

$$H_q(\Phi u) + \Psi u = 0 \,, \tag{2.3}$$

with

$$(2.3)' \quad \max(\deg\Phi - 2, \deg\Psi - 1) \; = s_1 - \deg\Phi_1 + \deg\Phi = s_2 - \deg\Phi_2 + \deg\Phi.$$

Proof. With $\Phi = \gcd(\Phi_1, \Phi_2)$, there exist two coprime polynomials $\overset{\vee}{\Phi}_1$, $\overset{\vee}{\Phi}_2$ such that

$$\Phi_1 = \Phi\overset{\vee}{\Phi}_1,\ \Phi_2 = \Phi\overset{\vee}{\Phi}_2. \tag{2.4}$$

Taking into account (1.17) equations (2.1) – (2.2) become

$$(2.5)_i \qquad \left(h_{q^{-1}}\overset{\vee}{\Phi}_i\right) H_q(\Phi u) + \left\{\Psi_i + q^{-1}\Phi\left(H_{q^{-1}}\overset{\vee}{\Phi}_i\right)\right\} u = 0 \,,\ i \in \{1,2\}\,.$$

The operation $\left(h_{q^{-1}}\overset{\vee}{\Phi}_2\right) \times (2.5)_1 - \left(h_{q^{-1}}\overset{\vee}{\Phi}_1\right) \times (2.5)_2$ gives

$$\begin{aligned}\Bigg\{\left(h_{q^{-1}}\overset{\vee}{\Phi}_2\right)&\left(\Psi_1 + q^{-1}\Phi\left(H_{q^{-1}}\overset{\vee}{\Phi}_1\right)\right) - \\ &- \left(h_{q^{-1}}\overset{\vee}{\Phi}_1\right)\left(\Psi_2 + q^{-1}\Phi\left(H_{q^{-1}}\overset{\vee}{\Phi}_2\right)\right)\Bigg\} u = 0.\end{aligned}$$

From regularity of u we get

$$(2.6) \; \left(h_{q^{-1}}\overset{\vee}{\Phi}_2\right)\left(\Psi_1 + q^{-1}\Phi\left(H_{q^{-1}}\overset{\vee}{\Phi}_1\right)\right) = \left(h_{q^{-1}}\overset{\vee}{\Phi}_1\right)\left(\Psi_2 + q^{-1}\Phi\left(H_{q^{-1}}\overset{\vee}{\Phi}_2\right)\right).$$

Thus, there exists a polynomial Ψ such that

$$(2.6)' \qquad \begin{cases} \Psi_1 + q^{-1}\Phi\left(H_{q^{-1}}\overset{\vee}{\Phi}_1\right) = \Psi\left(h_{q^{-1}}\overset{\vee}{\Phi}_1\right) \,, \\ \\ \Psi_2 + q^{-1}\Phi\left(H_{q^{-1}}\overset{\vee}{\Phi}_2\right) = \Psi\left(h_{q^{-1}}\overset{\vee}{\Phi}_2\right). \end{cases}$$

Then, formulas (2.1) – (2.2) become

$$\left(h_{q^{-1}}\overset{\vee}{\Phi}_i\right)\{H_q(\Phi u) + \Psi u\} = 0 \,,\ i \in \{1,2\}$$

writing $\overset{\vee}{\Phi}_i(x) = \prod_{k=1}^{l_i}(x - c_{i,k})^{\alpha_{i,k}}$, $i \in \{1,2\}$, which yields

$$H_q(\Phi u) + \Psi u = \sum_{k=1}^{l_1}\beta_{1,k}\,\delta_{qc_{1,k}}^{(\alpha_{1,k})} = \sum_{k=1}^{l_2}\beta_{2,k}\,\delta_{qc_{2,k}}^{(\alpha_{2,k})}.$$

But the polynomials $\overset{\vee}{\Phi}_1$ and $\overset{\vee}{\Phi}_2$ have no common zero, which allows (2.3). With (2.4) and $(2.6)'$ it is easy to prove $(2.3)'$. □

Proposition 2.3. *For any H_q-semiclassical linear form u, the pair (Φ, Ψ) which realizes the minimum of $\mathfrak{h}(u)$ is unique.*

Proof. If $s_1 = s_2$ in $(2.1) - (2.2)$ and $s_1 = s_2 = s = \min \mathfrak{h}(u)$, then $\deg \Phi_1 = \deg \Phi = \deg \Phi_2$. Consequently $\Phi_1 = \Phi = \Phi_2$, $\Psi_1 = \Psi = \Psi_2$. □
Then, it's necessary to give a criterion which allows us to simplify the class.

Proposition 2.4. *A regular form u H_q-semiclassical satisfying* (1.23) *is of class s if and only if*

$$\prod_{c \in \mathcal{Z}_\Phi} \left\{ |q(h_q\Psi)(c) + (H_q\Phi)(c)| + |\langle u, q(\theta_{cq}\Psi) + (\theta_{cq} \circ \theta_c\Phi) \rangle| \right\} > 0, \tag{2.7}$$

where $\mathcal{Z}_\Phi$ is the set of zeros of Φ.

Proof. Let c be a zero of $\Phi : \Phi(x) = (x - c)\Phi_c(x)$. The Euclidean algorithm gives

$$\Phi_c(x) + q\Psi(x) = (x - cq) Q_{cq}(x) + r_{cq}.$$

Then (1.23) becomes

$$(x - cq)\{H_q(\Phi_c u) + Q_{cq}u\} + r_{cq}u = 0,$$

on account of $(1.9) - (1.10)$, the last equation is equivalent to

$$H_q(\Phi_c u) + Q_{cq}u = (H_q(\Phi_c u) + Q_{cq}u)_0\, \delta_{cq} - (x - cq)^{-1} r_{cq}u. \tag{2.8}$$

Moreover, it is easy to see that

$$\Phi_c(cq) = (H_q\Phi)(c) \quad , \quad \Phi_c(x) = (\theta_c\Phi)(x).$$

Finally

$$\begin{cases} r_{cq} = (H_q\Phi)(c) + q(h_q\Psi)(c) \ , \\ Q_{cq}(x) = q(\theta_{cq}\Psi)(x) + (\theta_{cq} \circ \theta_c\Phi)(x) \ , \\ (H_q(\Phi_c u) + Q_{cq}u)_0 = \langle u, Q_{cq} \rangle = \langle u, q(\theta_{cq}\Psi) + (\theta_{cq} \circ \theta_c\Phi) \rangle. \end{cases} \tag{2.9}$$

Necessity. Let us suppose that there exists c, $\Phi(c) = 0$, satisfying

$$r_{cq} = 0 \quad , \quad \langle u, Q_{cq} \rangle = 0.$$

Then by (2.8), u verifies

$$H_q(\Phi_c u) + Q_{cq}u = 0 \quad , \tag{2.10}$$

with $s_c = \max(\deg Q_{cq} - 1, \deg \Phi_c - 2) < s$, what contradicts that $s := \min \mathfrak{h}(u)$.
Sufficiency. Let us suppose that the class of u is $\widetilde{s} < s$. There exist two polynomials , $\widetilde{\Phi}$ (monic), $\deg \widetilde{\Phi} = \widetilde{t} \geq 0$, $\widetilde{\Psi}$, $\deg \widetilde{\Psi} = \widetilde{p} \geq 1$ such that

$$H_q\left(\widetilde{\Phi}u\right) + \widetilde{\Psi}u = 0.$$

Consider $\widehat{\Phi} = \gcd\left(\Phi, \widetilde{\Phi}\right)$, $\deg \widehat{\Phi} = \widehat{t}$. On account of lemma 2.2, there exists a polynomial $\widehat{\Psi}$, $\deg \widehat{\Psi} = \widehat{p} \geq 1$, such that $H_q\left(\widehat{\Phi}u\right) + \widehat{\Psi}u = 0$, $\widehat{s} = \max\left(\widehat{p} - 1, \widehat{t} - 2\right) = s - t + \widehat{t} = \widetilde{s} - \widetilde{t} + \widehat{t}$.
Using proposition 2.3, we easily obtain $\widehat{\Phi} = \widetilde{\Phi}$, $\widehat{\Psi} = \widetilde{\Psi}$. Then, there exists a polynomial χ satisfying

$$\Phi = \chi\widetilde{\Phi} \ , \ \ \Psi = \left(h_{q^{-1}}\chi\right)\widetilde{\Psi} - q^{-1}\left(H_{q^{-1}}\chi\right)\widetilde{\Phi}.$$

Since $\widetilde{s} < s$ hence $\deg \chi \geq 1$. Let c be a zero of $\chi : \chi(x) = (x - c)\chi_c(x)$.
Writing $\Phi(x) = (x - c)\Phi_c(x)$, $\left(\Phi_c = \chi_c\widetilde{\Phi}\right)$, which allows

$$\begin{cases} r_{cq} = \left(H_q\Phi\right)(c) + q\left(h_q\Psi\right)(c) = 0 \ , \\ \left\langle u, q\left(\theta_{cq}\Psi\right) + \left(\theta_{cq} \circ \theta_c\Phi\right)\right\rangle = 0 \ \ , \end{cases}$$

what contradicts (2.7) . Consequently, $\widetilde{s} = s, \widetilde{\Phi} = \Phi$ and $\widetilde{\Psi} = \Psi$. □

Remarks. **1.** When $q \to 1$ we recover again the criterion which allows us to simplify a $D-$semiclassical linear form [17] .
2. When $q \in \widetilde{\mathbb{C}}$ and $s = 0$, the linear form u is usually called H_q-*classical* [9].

Definition 2.5 [4]. *A linear form u is called symmetric if* $\langle u, x^{2n+1}\rangle = 0,\ n \geq 0$.

Proposition 2.6. *Let u be a symmetric H_q-semiclassical linear form of class s satisfying* (1.23) . *The following statements hold*
i) When s is odd then the polynomial Φ is odd and Ψ is even .
ii) When s is even then the polynomial Φ is even and Ψ is odd .

Proof. Writing $\Phi(x) = \Phi^e\left(x^2\right) + x\Phi^o\left(x^2\right)$, $\Psi(x) = \Psi^e\left(x^2\right) + x\Psi^o\left(x^2\right)$, then (1.23) becomes

$$\left\{H_q\left(\Phi^e\left(x^2\right)u\right) + x\Psi^o\left(x^2\right)u\right\} + \left\{H_q\left(x\Phi^o\left(x^2\right)u\right) + \Psi^e\left(x^2\right)u\right\} = 0.$$

Denoting $w^e = H_q\left(\Phi^e\left(x^2\right)u\right) + x\Psi^o\left(x^2\right)u$, $w^o = H_q\left(x\Phi^o\left(x^2\right)u\right) + \Psi^e\left(x^2\right)u$. Then

$$w^e + w^o = 0. \tag{2.11}$$

From (2.11) we get

$$\left(w^e\right)_n = -\left(w^o\right)_n \ , \ n \geq 0. \tag{2.12}$$

From definitions we can write for $n \geq 0$

$$(2.13)\quad \begin{cases} (w^e)_{2n} = \left\langle u, x^{2n+1}\Psi^o\left(x^2\right) - [2n]_q\, x^{2n-1}\Phi^e\left(x^2\right)\right\rangle \\ \\ (w^o)_{2n+1} = \left\langle u, x^{2n+1}\Psi^e\left(x^2\right) - [2n+1]_q\, x^{2n+1}\Phi^o\left(x^2\right)\right\rangle . \end{cases}$$

Now, with u symmetric: $(u)_{2k+1} = 0$, $k \geq 0$, (2.13) gives

$$(2.12)' \qquad (w^e)_{2n} = 0 = (w^o)_{2n+1} \ ,\ n \geq 0.$$

On account of (2.12) and $(2.12)'$we deduce $w^e = w^o = 0$. Consequently, u satisfies two functional equations

$$(2.14) \qquad H_q\left(\Phi^e\left(x^2\right)u\right) + x\Psi^o\left(x^2\right)u = 0\ ,$$

and

$$(2.14)' \qquad H_q\left(x\Phi^o\left(x^2\right)u\right) + \Psi^e\left(x^2\right)u = 0\ .$$

i) When $s = 2k+1$, with $s = \max(t-2, p-1)$ we get $t \leq 2k+3$, $p \leq 2k+2$, then $\deg\left(x\Psi^o\left(x^2\right)\right) \leq 2k+1$, $\deg\left(\Phi^e\left(x^2\right)\right) \leq 2k+2$. So, in accordance with (2.14) , we obtain the contradiction $s = 2k+1 \leq 2k$. Necessary $\Phi^e = \Psi^o = 0$.
ii) When $s = 2k$, with $s = \max(t-2, p-1)$ we get $t \leq 2k+2$, $p \leq 2k+1$, then $\deg\left(\Psi^e\left(x^2\right)\right) \leq 2k$, $\deg\left(x\Phi^o\left(x^2\right)\right) \leq 2k+1$. So, in accordance with $(2.14)'$, we obtain the contradiction $s = 2k \leq 2k-1$. Necessary $\Phi^o = \Psi^e = 0$.Hence the desired result. □

Remark. When $q \to 1$ we recover again the same result for the $D-$semiclassical case [1] .

3. Different characterizations of H_q-semiclassical linear forms. One of the most important characterizations of the H_q-semiclassical linear forms is given in terms of a non homogeneous first order linear q-difference equation which its formal Stieltjes series satisfies. See also [6, 14] for the $D-$case and [11] for the $D_\omega-$one.

Proposition 3.1. *The linear form u is H_q-semiclassical of class s, if and only if, it is regular and there exist three coprime polynomials A (monic),C, D such that*

$$(3.1) \qquad A(z)\ H_{q^{-1}}(S(u))(z) = C(z)\,S(u)(z) + D(z)\ ,$$

with

$$(3.2) \qquad s = \max(\deg C - 1, \deg D)\,.$$

Proof. Necessity. From (1.23) , we have $0 = H_q(\Phi u) + \Psi u = \left(h_{q^{-1}}\Phi\right)(H_q u)$ $+ \left\{\Psi + q^{-1}H_{q^{-1}}\Phi\right\}u$ (with (1.17)). The isomorphism $\mathcal{F}$ yields

$$\mathcal{F}\left(\left(h_{q^{-1}}\Phi\right)(H_q u) + \left\{\Psi + q^{-1}H_{q^{-1}}\Phi\right\}u\right)(z) = 0.$$

From definition of $S(u)$, we obtain

$$(3.3)\qquad S\left(\left(h_{q^{-1}}\Phi\right)\left(H_q u\right)\right)(z)+S\left(\Psi u\right)(z)+q^{-1}S\left(\left(H_{q^{-1}}\Phi\right)u\right)(z)=0.$$

On account of (1.11), (3.3) becomes

$$\begin{aligned}&\left(h_{q^{-1}}\Phi\right)(z)\,S\left(H_q u\right)(z)+\left(\left(H_q u\right)\left(\theta_0\circ h_{q^{-1}}\Phi\right)\right)(z)+\Psi(z)\,S(u)(z)+\\ &\qquad+\left(u\theta_0\Psi\right)(z)+q^{-1}\left(H_{q^{-1}}\Phi\right)(z)\,S(u)(z)+q^{-1}\left(u\theta_0 H_{q^{-1}}\Phi\right)(z)=0.\end{aligned}$$

Then ,with (1.19)

$$\begin{aligned}&q^{-1}\left(h_{q^{-1}}\Phi\right)(z)\left(H_{q^{-1}}S(u)\right)(z)=\\ &\qquad-\left\{\Psi(z)+q^{-1}\left(H_{q^{-1}}\Phi\right)(z)\right\}S(u)(z)-\\ &\quad-\left\{\left(\left(H_q u\right)\left(\theta_0\circ h_{q^{-1}}\Phi\right)\right)(z)+\left(u\theta_0\Psi\right)(z)+q^{-1}\left(u\theta_0 H_{q^{-1}}\Phi\right)(z)\right\}.\end{aligned}$$

By using (1.22) the last equation becomes

$$\begin{aligned}(3.4)\quad&\left(h_{q^{-1}}\Phi\right)(z)\left(H_{q^{-1}}S(u)\right)(z)=\\ &\qquad-\left\{q\Psi(z)+\left(H_{q^{-1}}\Phi\right)(z)\right\}S(u)(z)-\\ &\qquad\qquad-\left\{H_{q^{-1}}\left(u\theta_0\Phi\right)(z)+q\left(u\theta_0\Psi\right)(z)\right\}.\end{aligned}$$

From (3.4) denoting

$$(3.5)\quad\begin{cases}A(z)=q^{\deg\Phi}\left(h_{q^{-1}}\Phi\right)(z)\ ,\\[2mm] C(z)=-q^{\deg\Phi}\left(q\Psi(z)+\left(H_{q^{-1}}\Phi\right)(z)\right)\ ,\\[2mm] D(z)=-q^{\deg\Phi}\left(H_{q^{-1}}\left(u\theta_0\Phi\right)(z)+q\left(u\theta_0\Psi\right)(z)\right).\end{cases}$$

Let c be a zero of Φ. From the first relation in (3.5), we remark that cq is a zero of A. As u is of class s, in accordance with (2.7) we get

$$q\left(h_q\Psi\right)(c)+\left(H_q\Phi\right)(c)\neq 0\ \text{ or }\ \left\langle u,q\left(\theta_{cq}\Psi\right)+\left(\theta_{cq}\circ\theta_c\Phi\right)\right\rangle\neq 0.$$

But with definitions of H_q , θ_ζ, uf and formula (1.14), it is easy to see that

$$(3.6)\qquad\begin{cases}C(cq)=-q^{\deg\Phi}\left(h_{q^{-1}}\left(q\left(h_q\Psi\right)+H_q\Phi\right)\right)(cq)\ ,\\[2mm] D(cq)=-q^{\deg\Phi}\left\langle u,q\left(\theta_{cq}\Psi\right)+\left(\theta_{cq}\circ\theta_c\Phi\right)\right\rangle.\end{cases}$$

Consequently, A, C and D have no common zero. Then A, C, and D are co-prime.
Sufficiency. Let $u\in\mathcal{P}'$ regular with its formal Stieltjes series $S(u)$ satisfying (3.1). From (1.11) and (1.19) formula (3.1) becomes

$$(3.1)'\quad S\left(A\left(H_q u\right)-q^{-1}Cu\right)(z)=\left(H_q u\theta_0 A\right)(z)-q^{-1}\left(u\theta_0 C\right)(z)+q^{-1}D(z).$$

But

$$S\left(A\left(H_q u\right)-q^{-1}Cu\right)(z)=S\left(\left(h_{q^{-1}}\left(h_q A\right)\right)\left(H_q u\right)-q^{-1}Cu\right)z$$

$$= S\left(H_q\left(\left(h_qA\right)u\right) - q^{-1}\left(H_{q^{-1}}\circ h_qA\right)u - q^{-1}Cu\right)(z) \quad \text{(by (1.17))}$$
$$= S\Big(H_q\left(\left(h_qA\right)u\right) - \left\{\left(H_qA\right)+q^{-1}C\right\}u\Big)(z) \quad \text{(with (1.14)).}$$

Then, $(3.1)'$ could be written as

$$S(H_q\left(\left(h_qA\right)u\right) - \left\{\left(H_qA\right)+q^{-1}C\right\}u)(z) = \\ \left(H_qu\theta_0A\right)(z) - q^{-1}\left(u\theta_0C\right)(z) + q^{-1}D(z)\,,$$

which implies

$$\begin{cases} H_q\left(\left(h_qA\right)u\right) - \left\{\left(H_qA\right)+q^{-1}C\right\}u = 0\,, \\ D(z) = \left(u\theta_0C\right)(z) - q\left(H_qu\theta_0A\right)(z)\,. \end{cases}$$

Denoting

$$\begin{cases} \Phi(x) = q^{-\deg A}\left(h_qA\right)(x)\;, \\ \Psi(x) = -q^{-\deg A}\left\{\left(H_qA\right)(x) + q^{-1}C(x)\right\}. \end{cases} \tag{3.7}$$

Now, it is easy to see that

$$H_q(\Phi u) + \Psi u = 0 \quad \text{with } \; s = \max\left(\deg\Phi - 2, \deg\Psi - 1\right). \ \square$$

Two structure relations for the H_q-semiclassical polynomials can be deduced from theory of finite-type relations between polynomial sequences [19].

Proposition 3.2. *For any monic polynomial* Φ *and any orthogonal sequence* $\{P_n\}_{n\geq 0}$, *the following statements are equivalent*
a) There exists an integer $s \geq 0$ *such that*

$$\Phi(x)\,P_n^{[1]}(x;q) = \sum_{\nu=n-s}^{n+t} \lambda_{n,\nu}P_\nu(x)\;,\; n\geq s\;,\; t=\deg\Phi\;, \tag{3.8}$$

$$\lambda_{n,n-s} \neq 0\;,\; n \geq s+1. \tag{3.9}$$

b) There exists a polynomial Ψ, $\deg\Psi = p \geq 1$ *such that*

$$H_q\left(\Phi u_0\right) + \Psi u_0 = 0\;, \tag{3.10}$$

where the pair (Φ,Ψ) *is admissible.*
c) There exist an integer $s\geq 0$ *and a polynomial* Ψ, $\deg\Psi = p \geq 1$ *such that*

$$\Phi(x)\left(H_q\circ h_{q^{-1}}P_m\right)(x) - \Psi(x)\left(h_{q^{-1}}P_m\right)(x) = \sum_{\nu=m-t}^{m+s_m}\widetilde{\lambda}_{m,\nu}P_{\nu+1}(x)\;, \\ m\geq t\;, \tag{3.11}$$

$$\widetilde{\lambda}_{m,m-t} \neq 0\;,\; m\geq t\;, \tag{3.12}$$

where $s = \max(p-1, t-2)$, *the pair* (Φ, Ψ) *being admissible and*

$$s_m = \begin{cases} p-1\ , & m = 0\ , \\ s\ \ , & m \geq 1. \end{cases}$$

We may write

$$\widetilde{\lambda}_{m,\nu} = -\left[\nu+1\right]_q \frac{\left\langle u_0, P_m^2 \right\rangle}{\left\langle u_0, P_{\nu+1}^2 \right\rangle} \lambda_{\nu,m}\ ,\ 0 \leq \nu \leq m+s. \tag{3.13}$$

Proof. *a)⇒b),c).* Supposing *a)*, then Lemma 1.2 *i)* is fulfilled with $Q_n = P_n^{[1]}(.;q)$. But (3.9) implies $\mu_m = m+s$, $m \geq 1$, and (1.7) becomes $\Phi u_m = \sum_{\nu=0}^{\mu_m} \lambda_{\nu,m} u_\nu^{[1]}(q)\ ,\ m \geq 0$.
By virtue of (1.2), we have

$$H_q\left(\Phi u_m\right) = -\sum_{\nu=0}^{\mu_m} \lambda_{\nu,m} \left[\nu+1\right]_q u_{\nu+1}\ ,\ m \geq 0.$$

In accordance with the orthogonality of $\{P_n\}_{n\geq 0}$, we get

$$H_q\left(P_m \Phi u_0\right) = -\Psi_{\mu_m+1} u_0\ ,\ m \geq 0\ , \tag{3.14}$$

with

$$\Psi_{\mu_m+1}(x) = \sum_{\nu=0}^{\mu_m} \left[\nu+1\right]_q \frac{\left\langle u_0, P_m^2 \right\rangle}{\left\langle u_0, P_{\nu+1}^2 \right\rangle} \lambda_{\nu,m} P_{\nu+1}(x)\ ,\ m \geq 0. \tag{3.15}$$

Further, with (1.17), we obtain for (3.14)

$$\left(h_{q^{-1}} P_m\right) H_q\left(\Phi u_0\right) + q^{-1}\left(H_{q^{-1}} P_m\right) \Phi u_0 = -\Psi_{\mu_m+1} u_0\ ,\ m \geq 0. \tag{3.16}$$

Taking $m = 0$ into (3.16), we have

$$H_q\left(\Phi u_0\right) + \Psi_{\mu_0+1} u_0 = 0. \tag{3.17}$$

Inserting (3.17) into (3.16), with (1.14) and according to the regularity of u_0, we get

$$\Phi\left(H_q \circ h_{q^{-1}} P_m\right) - \Psi_{\mu_0+1}\left(h_{q^{-1}} P_m\right) = -\Psi_{\mu_m+1}\ ,\ m \geq 0.$$

The consideration of the degrees of both sides leads to : when $t-1 > \mu_0 + 1$ (which implies $t \geq 3$), then $t = s+2$, $\mu_0 < s$ and when $t-1 \leq \mu_0+1$, then $\mu_0 = s$, $t \leq s+2$.Obviously, the pair (Φ, Ψ_{μ_0+1}) is admissible and putting $p = \mu_0+1$, we have $s = \max(p-1, t-2)$. So (3.11) and (3.12) are valid on account of (3.13). Thus, we have proved that *a)⇒b)* and *a)⇒c).*
b)⇒c). Consider for $m \geq 0$

$$q^{-1}\Phi(x)\left(H_{q^{-1}} P_m\right)(x) - \Psi(x)\left(h_{q^{-1}} P_m\right)(x) = \sum_{\nu=0}^{m+s_m+1} \lambda'_{m,\nu} P_\nu(x)\ .$$

We successively derive from this

$$\left\langle u_0, \left(q^{-1}\Phi\left(H_{q^{-1}}P_m\right) - \Psi\left(h_{q^{-1}}P_m\right)\right)P_\mu\right\rangle = \lambda'_{m,\mu}\ \left\langle u_0, P_\mu^2\right\rangle\ ,\ 0 \le \mu \le m+s+1.$$

But

$$\begin{aligned}&\left\langle u_0, \left(q^{-1}\Phi\left(H_{q^{-1}}P_m\right) - \Psi\left(h_{q^{-1}}P_m\right)\right)P_\mu\right\rangle\\ &\qquad = \left\langle \Phi u_0, q^{-1}\left(H_{q^{-1}}P_m\right)P_\mu\right\rangle + \left\langle -\Psi u_0, \left(h_{q^{-1}}P_m\right)P_\mu\right\rangle\\ &\qquad = \left\langle \Phi u_0, q^{-1}\left(H_{q^{-1}}P_m\right)P_\mu\right\rangle + \left\langle H_q\left(\Phi u_0\right), \left(h_{q^{-1}}P_m\right)P_\mu\right\rangle \text{ (by (3.10))}\\ &\qquad = \left\langle \Phi u_0, q^{-1}\left(H_{q^{-1}}P_m\right)P_\mu - H_q\left(\left(h_{q^{-1}}P_m\right)P_\mu\right)\right\rangle \text{ (by (1.1))}\\ &\qquad = -\left\langle \left(H_qP_\mu\right)\Phi u_0, P_m\right\rangle \text{ (by (1.16))}.\end{aligned}$$

Then

$$-\left\langle \left(H_qP_\mu\right)\Phi u_0, P_m\right\rangle = \lambda'_{m,\mu}\ \left\langle u_0, P_\mu^2\right\rangle\ .$$

Consequently, $\lambda'_{m,\mu} = 0\ ,\ 0 \le \mu \le m-t\ ,\ \lambda'_{m,0} = 0\ ,\ m \ge 0$. Moreover, for $\mu = m-t+1,\ m \ge t$
$-\left\langle u_0, \left(H_qP_{m-t+1}\right)\Phi P_m\right\rangle = -\left[m-t+1\right]_q\left\langle u_0, P_m^2\right\rangle = \lambda'_{m,m-t+1}\ \left\langle u_0, P_{m-t+1}^2\right\rangle.$
Therefore, for $m \ge t$,

$$\Phi(x)\left(H_q \circ h_{q^{-1}}P_m\right)(x) - \Psi(x)\left(h_{q^{-1}}P_m\right)(x) = \sum_{\nu=m-t}^{m+s_m}\lambda'_{m,\nu+1}P_{\nu+1}(x)\ ,\lambda'_{m,m-t+1} \ne 0.$$

c)⇒a). From (3.11), we have

$$\left\langle u_n, \Phi\left(H_q \circ h_{q^{-1}}P_m\right) - \Psi\left(h_{q^{-1}}P_m\right)\right\rangle = \sum_{\nu=0}^{m+s_m}\widetilde{\lambda}_{m,\nu}\delta_{n,\nu+1}\ ,$$

$$\left\langle q^{-1}H_{q^{-1}}\left(\Phi u_n\right) + h_{q^{-1}}\left(\Psi u_n\right), P_m\right\rangle = -\sum_{\nu=0}^{m+s_m}\widetilde{\lambda}_{m,\nu}\delta_{n,\nu+1}\ ,\ m,n \ge 0.$$

For $n=0$, $\left\langle q^{-1}H_{q^{-1}}\left(\Phi u_n\right) + h_{q^{-1}}\left(\Psi u_n\right), P_m\right\rangle = 0\ ,\ m \ge 0$, therefore

$$q^{-1}H_{q^{-1}}\left(\Phi u_0\right) + h_{q^{-1}}\left(\Psi u_0\right) = 0. \tag{3.18}$$

Further, making $n \to n+1$, we obtain

$$\begin{cases}\left\langle q^{-1}H_{q^{-1}}\left(\Phi u_{n+1}\right) + h_{q^{-1}}\left(\Psi u_{n+1}\right), P_m\right\rangle = 0\ ,\ m \ge n+1+t\ ,\ n \ge 0\ ,\\[2ex] \left\langle q^{-1}H_{q^{-1}}\left(\Phi u_{n+1}\right) + h_{q^{-1}}\left(\Psi u_{n+1}\right), P_{n+t}\right\rangle = -\widetilde{\lambda}_{n+t,n} \ne 0\ ,\ n \ge 0.\end{cases}$$

According to Lemma 1.1,

$$q^{-1}H_{q^{-1}}\left(\Phi u_{n+1}\right) + h_{q^{-1}}\left(\Psi u_{n+1}\right) = -\sum_{\nu=n-s}^{n+t}\widetilde{\lambda}_{\nu,n}u_\nu\ ,\ n \ge s.$$

The orthogonality of $\{P_n\}_{n\ge 0}$ leads to

$$q^{-1}H_{q^{-1}}\left(\Phi P_{n+1}u_0\right) + h_{q^{-1}}\left(\Psi P_{n+1}u_0\right) = -\sum_{\nu=n-s}^{n+t}\widetilde{\lambda}_{\nu,n}\frac{\left\langle u_0, P_{n+1}^2\right\rangle}{\left\langle u_0, P_\nu^2\right\rangle}P_\nu u_0\ ,\ n \ge 0.$$

By virtue of (3.18) and on account of regularity of u_0, we finally obtain (3.8) – (3.9) in accordance with (3.13). □

Likewise the $D-$semiclassical case, see [16], we can easily establish a writing more simplified of (3.8) on account of the three-term recurrence relation. We get

$$\Phi(x)(H_qP_{n+1})(x) = \tfrac{1}{2}(C_{n+1}(x) - C_0(x))P_{n+1}(x) - \gamma_{n+1}D_{n+1}(x)P_n(x), \quad n \geq 0, \tag{3.19}$$

where

$$C_{n+1}(x) = -C_n(x) + 2(x-\beta_n)D_n(x) + 2x(q-1)\Sigma_n(x), \quad n \geq 0, \tag{3.20}$$

(3.21)

$$\gamma_{n+1}D_{n+1}(x) = -\Phi(x) + \gamma_n D_{n-1}(x) + (x-\beta_n)^2 D_n(x) - \left(\tfrac{q+1}{2}x - \beta_n\right)C_n(x) + \\ + x(q-1)\left\{\tfrac{1}{2}C_0(x) + (x-\beta_n)\Sigma_n(x)\right\}, \quad n \geq 0,$$

with

$$C_0(x) = q^{-\deg\Phi}C(x), \; D_0(x) = q^{-\deg\Phi}D(x) \quad (\text{see } (3.5)), \; D_{-1}(x) := 0, \tag{3.22}$$

and

$$\Sigma_n(x) := \sum_{k=0}^{n} D_k(x), \quad n \geq 0. \tag{3.22$'$}$$

It is easy to see that $\deg C_n \leq s+1$ and $\deg D_n \leq s, \; n \geq 0$.

On the other hand, from (3.20) – (3.21) ,by elimination of the terms $(x-\beta_n)D_n(x)$, $(x-\beta_n)^2 D_n(x)$ and after some calculations we get the important formula

$$\tfrac{1}{4}\left(C_{n+1}^2(x) - C_0^2(x)\right) - \gamma_{n+1}D_n(x)D_{n+1}(x) - \\ -\tfrac{1}{2}(C_{n+1}(x) - C_0(x))x(q-1)\Sigma_n(x) = \Phi(x)\Sigma_n(x), \quad n \geq 0. \tag{3.23}$$

REMARKS.**1.** When $q \to 1$ in (3.19) – (3.23) we recover again the $D-$case [5, 16].
2. The sequence $\{D_{n+1}\}_{n\geq 0}$ gives us some informations about zeros of polynomials P_{n+1}. In fact, when $P_{n+1}(c) = 0$, $n \geq 1$ and $\left(H_q^\nu P_{n+1}\right)(c) = 0$, $1 \leq \nu \leq \mu$ with $\mu \geq 2$ then $\mu \leq s+1$ and $D_{n+1}(c) = 0$, $\left(H_q^\nu D_{n+1}\right)(c) = 0$, $1 \leq \nu \leq \mu - 1$.
3. When $s = 0$, writing $\Phi(x) = \frac{1}{2}\Phi''(0)x^2 + \Phi'(0)x + \Phi(0), \Psi(x) = \Psi'(0)x + \Psi(0)$, we can easily determine the coefficients of the structure relation (3.19) (see also [20])

$$\left\{\begin{array}{l} \frac{1}{2}(C_{n+1}(x) - C_0(x)) = \frac{1}{2}\Phi''(0)([n+1]_q x - q^{-n}S_n) + \\ \qquad +q^{-n}(\Psi'(0) - \frac{1+q^n}{2}\Phi''(0)[n+1]_q)\beta_{n+1} + \\ \qquad\quad +q^{-n}(\Psi(0) - \Phi'(0)[n+1]_q) - q^{-n}(q-1)\Psi'(0)S_n, \; n \geq 0, \\ D_{n+1}(x) = q^{-n}(\frac{1}{2}\Phi''(0)[2n+1]_q - \Psi'(0)), \; n \geq 0, \end{array}\right. \tag{3.19'}$$

with $\quad S_n := \sum_{k=0}^{n} \beta_k, \; n \geq 0.$

Regarding the relation (3.11), we are going to give the characterization of a H_q-semiclassical linear form in term of a second order linear $q-$difference equation, satisfied by the corresponding (MOPS), which is the extension of the Bochner one[3]. This result is the $q-$analog of the Hahn characterization [8] for the $D-$semiclassical case, see also $[5-6]$ for the $D-$case and [11] for the $D_\omega-$one.

PROPOSITION 3.3. *Let* $\{P_n\}_{n\geq 0}$ *be a* (MOPS) *with respect to the linear form* u. *If The linear form* u *is* H_q*-semiclassical of class* s, *then there exist polynomials* $J_q(.,n), K_q(.,n), L_q(.,n)$, *with coefficients depending on* n *and degree at most* $2s+2, 2s+1, 2s$, *respectively, for which*

(3.24)

$$J_q(x,n)\left(H_q \circ H_{q^{-1}} P_{n+1}\right)(x) + K_q(x,n)\left(H_{q^{-1}} P_{n+1}\right)(x) + L_q(x,n) P_{n+1}(x) = 0,$$
$$n \geq 0.$$

Proof. Let write (3.19) in the following way

$$\Phi(x)(H_q P_{n+1})(x) = A(x,n) P_{n+1}(x) + B(x,n) P_n(x),\ n \geq 0, \tag{3.25}$$

$$\Phi(x)(H_q P_{n+2})(x) = A_1(x,n) P_{n+1}(x) + B_1(x,n) P_n(x), n \geq 0, \tag{3.25$'$}$$

so that

$$\begin{cases} A(x,n) = \frac{1}{2}\left(C_{n+1}(x) - C_0(x)\right)\ ,\ B(x,n) = -\gamma_{n+1} D_{n+1}(x)\ , \\ A_1(x,n) = \frac{1}{2}\left(C_{n+2}(x) - C_0(x)\right)(x - \beta_{n+1}) - \gamma_{n+2} D_{n+2}(x)\ , \\ B_1(x,n) = -\frac{1}{2}\left(C_{n+2}(x) - C_0(x)\right)\gamma_{n+1} \quad , \quad n \geq 0. \end{cases} \tag{3.26}$$

If we multiply in (3.25) by $B_1(x,n)$, in equation $(3.25)'$ by $B(x,n)$ and subtract the resulting expressions we have for $n \geq 0$

$$B_1(x,n)\Phi(x)(H_q P_{n+1})(x) - B(x,n)\Phi(x)(H_q P_{n+2})(x) = \Delta_n(x) P_{n+1}(x), \tag{3.27}$$

with

$$\Delta_n(x) = B_1(x,n) A(x,n) - B(x,n) A_1(x,n)\ ,\ n \geq 0. \tag{3.28}$$

From the three-term recurrence relation and by virtue of (1.12), the relation (3.27) becomes

$$\begin{aligned} &\Big(B_1(x,n) - (qx - \beta_{n+1}) B(x,n)\Big)\Phi(x)(H_q P_{n+1})(x) = \\ &\qquad \Big(\Delta_n(x) + \Phi(x) B(x,n)\Big) P_{n+1}(x) - \\ &\qquad\qquad -\gamma_{n+1} B(x,n)\Phi(x)(H_q P_n)(x)\ ,\ n \geq 0. \end{aligned} \tag{3.27$'$}$$

Applying the operator H_q to (3.25), taking into account (1.12) and multiplying the result by $(-\gamma_{n+1} B(x,n)\Phi(x))$ we get

(3.29)

$$\begin{aligned}
&-\gamma_{n+1}B(x,n)\,\Phi(x)\,(h_q\Phi)(x)\left(H_q^2P_{n+1}\right)(x)\text{-}\\
&\qquad-\gamma_{n+1}B(x,n)\,\Phi(x)\Big((H_q\Phi)(x)\text{-}(h_qA)(x,n)\Big)(H_qP_{n+1})(x)+\\
&\qquad\qquad+\gamma_{n+1}B(x,n)\,\Phi(x)\,(H_qA)(x,n)\,P_{n+1}(x)=\\
&-\gamma_{n+1}B(x,n)\,\Phi(x)\,(h_qB)(x,n)\,(H_qP_n)(x)\text{-}\\
&\qquad\qquad\text{-}\gamma_{n+1}B(x,n)\,\Phi(x)\,(H_qB)(x,n)\,P_n(x)\,,\ n\geq 0.
\end{aligned}$$

Using the expressions for P_n , H_qP_n from (3.25) and $(3.27)'$, we obtain

$(3.29)'$

$$\begin{aligned}
&-B(x,n)\,\Phi(x)\,(h_q\Phi)(x)\left(H_q^2P_{n+1}\right)(x)-\\
&\quad-\Phi(x)\Big\{B(x,n)\Big((H_q\Phi)(x)-(h_qA)(x,n)\Big)+\\
&\qquad+\tfrac{1}{\gamma_{n+1}}(h_qB)(x,n)\Big(B_1(x,n)-(qx-\beta_{n+1})B(x,n)\Big)-\\
&\quad-\Phi(x)\,(H_qB)(x,n)\Big\}(H_qP_{n+1})(x)+\\
&\quad+\Big\{\Phi(x)\Big(B(x,n)\,(H_qA)(x,n)-A(x,n)\,(H_qB)(x,n)\Big)+\\
&\quad+\tfrac{1}{\gamma_{n+1}}(h_qB)(x,n)\Big(\Delta_n(x)+\Phi(x)\,B(x,n)\Big)\Big\}P_{n+1}(x)=0\,,\ n\geq 0.
\end{aligned}$$

But

$$\begin{aligned}
\frac{\Delta_n(x)}{\gamma_{n+1}}&=\frac{1}{2}\left(C_{n+2}(x)-C_0(x)\right)\Big(-\tfrac12\left(C_{n+1}(x)-C_0(x)\right)+D_{n+1}(x)\,(x-\beta_{n+1})\Big)-\\
&\qquad-\gamma_{n+2}D_{n+1}(x)\,D_{n+2}(x)\ \text{(from (3.26))}\\
&=\tfrac12\left(C_{n+2}(x)-C_0(x)\right)\left(\tfrac12\left(C_{n+2}(x)+C_0(x)\right)-x\,(q-1)\,\Sigma_{n+1}(x)\right)-\\
&\qquad-\gamma_{n+2}D_{n+1}(x)\,D_{n+2}(x)\ \text{(from (3.20))}\\
&=\tfrac14\left(C_{n+2}^2(x)-C_0^2(x)\right)-\tfrac12x\,(q-1)\left(C_{n+2}(x)-C_0(x)\right)\Sigma_{n+1}(x)-\\
&\qquad-\gamma_{n+2}D_{n+1}(x)\,D_{n+2}(x)\\
&=\Phi(x)\ \Sigma_{n+1}(x)\quad\text{(from (3.23))}\,,\ n\geq 0.
\end{aligned}$$

Applying the operator $h_{q^{-1}}$ to $(3.29)'$, taking into account (1.14), (3.25) ,definitions of h_q and H_q and after some calculations we obtain (3.24) with (compare with [6])

$$(3.30)\quad\left\{\begin{aligned}
&J_q(x,n)=q\Phi(x)\,D_{n+1}(x)\ ,\\
&K_q(x,n)=D_{n+1}\left(q^{-1}x\right)\left(H_{q^{-1}}\Phi\right)(x)-\left(H_{q^{-1}}D_{n+1}\right)(x)\,\Phi\left(q^{-1}x\right)+\\
&\qquad\qquad\qquad\qquad+C_0\left(q^{-1}x\right)D_{n+1}(x)\ ,\\
&L_q(x,n)=\tfrac12\left(C_{n+1}\left(q^{-1}x\right)-C_0\left(q^{-1}x\right)\right)\left(H_{q^{-1}}D_{n+1}\right)(x)-\\
&-\tfrac12\left(H_{q^{-1}}\left(C_{n+1}-C_0\right)\right)(x)\,D_{n+1}\left(q^{-1}x\right)-D_{n+1}(x)\,\Sigma_n\left(q^{-1}x\right)\ ,\ n\geq 0.
\end{aligned}\right.$$

From $\deg C_n\leq s+1$, $\deg D_n\leq s$, $n\geq 0$, $\deg\Phi\leq s+2$ and (3.30), it is easy to see that $\deg J_q\leq 2s+2$, $\deg K_q\leq 2s+1$ and $\deg L_q\leq 2s$. □

Remark. The converse is not proved.

4. Examples. 4.1. Let v be a regular linear form. Denoting by $\{P_n\}_{n\geq 0}$ its (MOPS) sequence

$$(4.1)\quad \begin{cases} P_0(x)=1 \;,\; P_1(x)=x-\beta_0 \;, \\ P_{n+2}(x)=(x-\beta_{n+1})\,P_{n+1}(x)-\gamma_{n+1}P_n(x) \;,\; n\geq 0. \end{cases}$$

Let $u\in\mathcal{P}'$ satisfying

$$(4.2)\quad xu=\lambda v \;,\; \lambda\in\mathbb{C}.$$

Equation (4.2) is equivalent to

$$(4.3)\quad u=\delta+\lambda x^{-1}v.$$

Suppose u regular and let $\left\{\widetilde{P}_n\right\}_{n\geq 0}$ its (MOPS) sequence

$$(4.4)\quad \begin{cases} \widetilde{P}_0(x)=1 \;,\; \widetilde{P}_1(x)=x-\widetilde{\beta}_0 \;, \\ \widetilde{P}_{n+2}(x)=\left(x-\widetilde{\beta}_{n+1}\right)\widetilde{P}_{n+1}(x)-\widetilde{\gamma}_{n+1}\widetilde{P}_n(x) \;,\; n\geq 0. \end{cases}$$

From (4.2) and by virtue of Lemma 1.2 we have

$$(4.5)\quad \widetilde{P}_0(x)=1 \;,\; \widetilde{P}_{n+1}(x)=P_{n+1}(x)+a_nP_n(x) \;,\; n\geq 0 \;,$$

with $a_n\neq 0 \;,\; n\geq 0.$

Let us recall the fundamental result [15, **Théorème 1.2**].

Proposition 4.1. *Let v be a regular linear form. The following statements are equivalent*
i) The linear form $u=\delta+\lambda x^{-1}v$ is regular for any $\lambda\neq 0$.
ii) v is symmetric.

We may write

$$(4.5)'\quad \tfrac{\gamma_{n+1}}{a_n}+a_{n+1}=0 \;,\; n\geq 0 \;,$$

$$(4.6)\quad \begin{cases} a_{2n}=-\lambda\prod_{\nu=0}^{n}\tfrac{\gamma_{2\nu}}{\gamma_{2\nu-1}} \;,\; n\geq 0 \;,\; \gamma_{-1}=1 \;, \\ a_{2n+1}=\tfrac{1}{\lambda}\prod_{\nu=0}^{n}\tfrac{\gamma_{2\nu+1}}{\gamma_{2\nu}} \;,\; n\geq 0 \;, \end{cases}$$

$$(4.7)\quad \widetilde{\beta}_0=-a_0=\lambda \;,\; \widetilde{\beta}_{n+1}=a_n-a_{n+1} \;,\; \widetilde{\gamma}_{n+1}=-a_n^2 \;,\; n\geq 0 \;,$$

$$(4.8)\quad \begin{cases} xP_n(x)=\widetilde{P}_{n+1}(x)-a_n\widetilde{P}_n(x) \;,\; n\geq 0 \;, \\ xP_{n+1}(x)=(x-a_n)\,\widetilde{P}_{n+1}(x)+a_n^2\widetilde{P}_n(x) \;,\; n\geq 0. \end{cases}$$

4.2. Suppose v be a symmetric H_q-classical linear form satisfying (1.23)

$$H_q(\Phi v)+\Psi v=0 \;,\; \deg\Phi\leq 2 \;,\; \deg\Psi=1.$$

Multiplying the last equation by λ and on account of (4.2) we get

$$H_q\left(\widetilde{\Phi}u\right)+\widetilde{\Psi}u=0 \tag{4.9}$$

with

$$\widetilde{\Phi}(x)=x\Phi(x)\ ,\ \widetilde{\Psi}(x)=x\Psi(x)\,. \tag{4.10}$$

In accordance with Proposition 2.4, the linear form u is H_q-semiclassical of class 1.
Now, we are going to give the structure relation of $\left\{\widetilde{P}_n\right\}_{n\geq 0}$.
From $(3.19)'$ with $\beta_n=0,\ n\geq 0$ the structure relation of $\{P_n\}_{n\geq 0}$ is

$$\Phi(x)\left(H_qP_{n+1}\right)(x)=\tfrac{1}{2}\left(C_{n+1}(x)-C_0(x)\right)P_{n+1}(x)-\gamma_{n+1}D_{n+1}(x)\,P_n(x)\ ,\quad n\geq 0, \tag{4.11}$$

where

$$\begin{cases}\tfrac{1}{2}\left(C_{n+1}(x)-C_0(x)\right)=q^{-n}\left\{\tfrac{1}{2}\Phi''(0)\,q^n[n+1]_qx+\Psi(0)-\Phi'(0)\,[n+1]_q\right\}\ ,\\[2ex] D_{n+1}(x)=q^{-n}\left(\tfrac{1}{2}\Phi''(0)\,[2n+1]_q-\Psi'(0)\right),\ n\geq 0.\end{cases} \tag{4.12}$$

From $(4.5)\,,(4.11)$ and (5.1) we have

$$\Phi(x)\left(H_q\widetilde{P}_{n+1}\right)(x)=u_n(x)\,P_{n+1}(x)+v_n(x)\,P_n(x)\ ,\ n\geq 0, \tag{4.13}$$

with for $n\geq 0$

$$\begin{cases}u_n(x)=\tfrac{1}{2}\left(C_{n+1}(x)-C_0(x)\right)+a_nD_n(x)\quad ,\\[2ex] v_n(x)=\left\{-\tfrac{1}{2}\left(C_{n+1}(x)-C_0(x)\right)-C_0(x)\,+\,x\,(q-1)\,\Sigma_n(x)\right\}a_n-\gamma_{n+1}D_{n+1}(x)\,.\end{cases} \tag{4.14}$$

On account of (4.8), we have for (4.13)

$$\widetilde{\Phi}(x)\left(H_q\widetilde{P}_{n+1}\right)(x)=\tfrac{1}{2}\left(\widetilde{C}_{n+1}(x)-\widetilde{C}_0(x)\right)\widetilde{P}_{n+1}(x)-\widetilde{\gamma}_{n+1}\widetilde{D}_{n+1}(x)\,\widetilde{P}_n(x)\ ,\quad n\geq 0, \tag{4.15}$$

where

$$\begin{cases}\tfrac{1}{2}\left(\widetilde{C}_{n+1}(x)-\widetilde{C}_0(x)\right)=(x-a_n)\,u_n(x)+v_n(x)\\[2ex] \widetilde{\gamma}_{n+1}\widetilde{D}_{n+1}(x)=\left(v_n(x)-a_nu_n(x)\right)a_n\end{cases}\ ,\ n\geq 0. \tag{4.16}$$

From (3.22) and (3.5) we have

$$\begin{cases} \widetilde{C}_0(x) = -\left(q\widetilde{\Psi}(x) + \left(H_{q^{-1}}\widetilde{\Phi}\right)(x)\right), \\ \widetilde{D}_0(x) = -\left(H_{q^{-1}}\left(u\theta_0\widetilde{\Phi}\right)(x) + q\left(u\theta_0\widetilde{\Psi}\right)(x)\right). \end{cases}$$

By virtue of (4.10) and (1.12) we get

$$\begin{cases} \widetilde{C}_0(x) = q^{-1}xC_0(x) - ((q-1)x\Psi(x) + \Phi(x)), \\ \widetilde{D}_0(x) = C_0(x) + \lambda D_0(x), \end{cases} \tag{4.17}$$

because

$$\left(u\theta_0\widetilde{\Psi}\right)(x) = \left\langle u, \frac{\widetilde{\Psi}(x) - \widetilde{\Psi}(\zeta)}{x-\zeta} \right\rangle = \Psi(x) + \left\langle \lambda\zeta^{-1}v, \frac{\widetilde{\Psi}(x) - \widetilde{\Psi}(\zeta)}{x-\zeta} \right\rangle$$

$$= \Psi(x) + \lambda\left\langle v, \left\{\frac{\widetilde{\Psi}(x) - \widetilde{\Psi}(\zeta)}{x-\zeta} - \Psi(x)\right\}\frac{1}{\zeta}\right\rangle = \Psi(x) + \lambda(v\theta_0\Psi)(x).$$

In addition, From (4.14) – (4.17) and by taking into account $(4.5)'$ and (3.20) we get for $n \geq 0$

$$\widetilde{\Sigma}_n(x) := \sum_{\nu=0}^{n}\widetilde{D}_\nu(x) = -\tfrac{1}{2}(C_{n+1}(x) - C_0(x)) - a_nD_n(x) + qx\Sigma_n(x). \tag{4.18}$$

Now we are able to give the coefficients of the second order linear $q-$difference equation satisfied by $\widetilde{P}_{n+1},\ n \geq 0$

(4.19)

$$\begin{cases} \widetilde{J}_q(x,n) = qx\Phi(x)\left(v_n\left(q^{-1}x\right) - a_nu_n\left(q^{-1}x\right)\right), \\ \widetilde{K}_q(x,n) = -q^{-1}x\Phi\left(q^{-1}x\right)\left(\left(H_{q^{-1}}v_n\right)(x) - a_n\left(H_{q^{-1}}u_n\right)(x)\right) - \\ -\left(v_n(x) - a_nu_n(x)\right)\left(x\Psi\left(q^{-1}x\right) + \Phi\left(q^{-1}x\right) + q^{-2}x\left(H_{q^{-1}}\Phi\right)\left(q^{-1}x\right)\right) + \\ \quad + \left(v_n\left(q^{-1}x\right) - a_nu_n\left(q^{-1}x\right)\right)\left(\Phi(x) + q^{-1}x\left(H_{q^{-1}}\Phi\right)(x)\right), \\ \widetilde{L}_q(x,n) = q^{-1}xu_n\left(q^{-1}x\right)\left(\left(H_{q^{-1}}v_n\right)(x) - a_n\left(H_{q^{-1}}u_n\right)(x)\right) - \\ -\left(v_n\left(q^{-1}x\right) - a_nu_n\left(q^{-1}x\right)\right)\left(u_n\left(q^{-1}x\right) + x\left(H_{q^{-1}}u_n\right)(x)\right) - \\ \qquad - \left(v_n(x) - a_nu_n(x)\right)\widetilde{\Sigma}_n\left(q^{-1}x\right). \end{cases}$$

Finally, suppose that the function V represents the regular linear form v

$$\langle v, f\rangle = \int_{-\infty}^{+\infty} V(x)f(x)\,dx\ ,\ f \in \mathcal{P}\ ,\ \text{with}\ \int_{-\infty}^{+\infty} V(x)\,dx = 1.$$

In view of (4.3), we may write

$$(4.20)\quad \langle u,f\rangle=\left\{1-\lambda P\int_{-\infty}^{+\infty}\frac{V(x)}{x}dx\right\}f(0)+\lambda P\int_{-\infty}^{+\infty}\frac{V(x)}{x}f(x)\,dx\ ,\ f\in\mathcal{P}\ ,$$

where

$$(4.21)\quad P\int_{-\infty}^{+\infty}\frac{V(x)}{x}dx:=\lim_{\varepsilon\rightsquigarrow 0^{+}}\left(\int_{-\infty}^{-\varepsilon}\frac{V(x)}{x}dx+\int_{+\varepsilon}^{+\infty}\frac{V(x)}{x}dx\right).$$

4.3. Before giving examples of H_q-semiclassical linear form of class 1, let us recall the following standard material [4, 9, 10]

$$(4.22)\quad (a;q)_n:=\begin{cases}1\ ,\ n=0\\ \prod_{\nu=1}^{n}\left(1-aq^{\nu-1}\right)\ ,\ n\geq 1.\end{cases}$$

$$(4.22)'\quad \left[\begin{smallmatrix}n\\k\end{smallmatrix}\right]_q:=\frac{[n]_q!}{[k]_q![n-k]_q!}=\frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}\ ,\ 0\leq k\leq n.$$

$$(4.23)\quad (a;q)_\infty=\prod_{\nu=0}^{+\infty}(1-aq^{\nu})\ ,\ |q|<1.$$

$$(4.24)\quad \sum_{k=0}^{+\infty}\frac{q^{\frac{1}{2}k(k+1)}}{(q;q)_k}(-1)^k z^k=(qz;q)_\infty\ ,\ |q|<1.$$

4.3.1. Consider the symmetric H_q-classical linear form v which is the $q-$analog of Hermite. We have [9]

$$(4.25)\quad \begin{cases}\beta_n=0\ ,\ \gamma_{n+1}=\frac{1}{2}q^n[n+1]_q\ ,\ n\geq 0,\\ \Phi(x)=1\ ,\ \Psi(x)=2x,\end{cases}$$

$$(4.26)\quad (v)_{2n}=\frac{[2n]_q![2n+2]_q}{2^n\prod_{\nu=0}^{n}[2\nu+2]_q}\ ,\ (v)_{2n+1}=0\ ,\ n\geq 0,$$

(4.27)

$$\langle v,f\rangle=\begin{cases}\frac{\sqrt{2}}{\pi}(q-1)^{\frac{1}{2}}\frac{\left(q^{-2};q^{-2}\right)_\infty}{\left(q^{-1};q^{-2}\right)_\infty}\int_{-\infty}^{+\infty}\frac{f(x)}{\left(-2(q-1)x^2;q^{-2}\right)_\infty}dx\ ,\\ \qquad f\in\mathcal{P}\ ,\ q>1,\\ K_1\int_{-\frac{1}{q\sqrt{2(1-q)}}}^{+\frac{1}{q\sqrt{2(1-q)}}}\left(2q^2(1-q)x^2;q^2\right)_\infty f(x)\,dx\ ,\\ \qquad f\in\mathcal{P}\ ,\ 0<q<1,\end{cases}$$

with $\quad K_1=\frac{1}{2}\left(\int_{0}^{+\frac{1}{q\sqrt{2(1-q)}}}\left(2q^2(1-q)t^2;q^2\right)_\infty dt\right)^{-1}.$

REMARK. Taking into account (4.24), we may write

$$K_1^{-1} = \frac{1}{q}\sqrt{\frac{2}{1-q}}\sum_{k=0}^{+\infty}\frac{(-1)^k}{(2k+1)}\frac{q^{k(k+1)}}{(q^2;q^2)_k}.$$

From (3.22) – (3.22)$'$ and (4.12) we get

$$(4.28)\qquad \begin{cases} C_0(x) = -2qx\ ,\ D_0(x) = -2q\ , \\ \frac{1}{2}\left(C_{n+1}(x) - C_0(x)\right) = 0\ ,\ D_{n+1}(x) = -2q^{-n}\ , \\ \Sigma_n(x) = -2q^{1-n}\,[n+1]_q\ ,\ n \geq 0. \end{cases}$$

Consequently, for any $\lambda \neq 0$, the linear form u defined by (4.2) is H_q-semiclassical of class 1 and from (4.10) we get

$$(4.29)\qquad \widetilde{\Phi}(x) = x\ ,\ \widetilde{\Psi}(x) = 2x^2.$$

In accordance of (4.2) and (4.26), the moments of u are

$$(4.30)\qquad \begin{cases} (u)_0 = 1\ ,\ (u)_{2n} = 0\ ,\ n \geq 1, \\ (u)_{2n+1} = \lambda\dfrac{[2n]_q!\,[2n+2]_q}{2^n\prod_{\nu=0}^{n}[2\nu+2]_q}\ ,\ n \geq 0. \end{cases}$$

By virtue of (4.6) and (4.25) we obtain

$$(4.31)\qquad \begin{cases} a_{2n} = -\lambda q^n\dfrac{(q^2;q^2)_n}{(q;q^2)_n}\ ,\ n \geq 0, \\ a_{2n+1} = \frac{1}{2\lambda}q^n\dfrac{(q^3;q^2)_n}{(q^2;q^2)_n}\ ,\ n \geq 0. \end{cases}$$

Then with (4.7) we get

$$(4.32)\qquad \begin{cases} \widetilde{\beta}_0 = \lambda\ ,\ \widetilde{\beta}_{2n+1} = -q^n\left\{\lambda\dfrac{(q^2;q^2)_n}{(q;q^2)_n} + \dfrac{1}{2\lambda}\dfrac{(q^3;q^2)_n}{(q^2;q^2)_n}\right\}\ ,\ n \geq 0, \\ \widetilde{\beta}_{2n+2} = q^n\left\{\lambda q\dfrac{(q^2;q^2)_{n+1}}{(q;q^2)_{n+1}} + \dfrac{1}{2\lambda}\dfrac{(q^3;q^2)_n}{(q^2;q^2)_n}\right\}\ ,\ n \geq 0, \\ \widetilde{\gamma}_{2n+1} = -\lambda^2 q^{2n}\dfrac{(q^2;q^2)_n^2}{(q;q^2)_n^2}\ ,\ \widetilde{\gamma}_{2n+2} = -\dfrac{1}{4\lambda^2}q^{2n}\dfrac{(q^3;q^2)_n^2}{(q^2;q^2)_n^2}\ , n \geq 0. \end{cases}$$

On the other hand, from (4.14) and (4.16) – (4.18) we have

$$
(4.33)\begin{cases}
u_n(x) = -2q^{1-n}a_n \ , \ v_n(x) = 2q\left(1-q+q^{-n}\right)a_n x + [n+1]_q \ , \ n \geq 0, \\
\frac{1}{2}\left(\widetilde{C}_{n+1}(x) - \widetilde{C}_0(x)\right) = 2q(1-q)a_n x + 2q^{1-n}a_n^2 + [n+1]_q \ , \ n \geq 0, \\
\widetilde{D}_{n+1}(x) = -\left\{2q\left(1-q+q^{-n}\right)a_n x + 2q^{1-n}a_n^2 + [n+1]_q\right\}a_n^{-1} \ , \ n \geq 0, \\
\widetilde{C}_0(x) = -2qx^2 - 1 \quad , \quad \widetilde{D}_0(x) = -2q(x+\lambda) \ , \\
\widetilde{\Sigma}_n(x) = 2q^{1-n}\left(a_n - q[n+1]_q x\right) \ , \ n \geq 0.
\end{cases}
$$

Then, with (4.19), the second order linear $q-$difference equation satisfied by $\widetilde{P}_{n+1}$, $n \geq 0$ is

(4.34)

$$
\begin{aligned}
&\left\{\left(1-q+q^{-n}\right)x + q^{-n}\left(qa_n - a_{n+1}\right)\right\}\left(H_q \circ H_{q^{-1}}\widetilde{P}_{n+1}\right)(x) - \\
&-\left\{\left(1-q+q^{-n}\right)\left(2q^{-1}x^2+1\right) + 2q^{-2-n}\left(qa_n - a_{n+1}\right)x\right\}\left(H_{q^{-1}}\widetilde{P}_{n+1}\right)(x) + \\
&+2q^{1-n}\left\{\left(1-q+q^{-n}\right)\left(q[n+1]_q x - a_n\right) + \right. \\
&\qquad\qquad \left. +q^{-n}[n+1]_q\left(qa_n - a_{n+1}\right)\right\}\widetilde{P}_{n+1}(x) = 0.
\end{aligned}
$$

From the definition (4.21), and (4.27), it easy to see that

$$
P\int_{-\infty}^{+\infty} \frac{dx}{x\left(-2(q-1)x^2; q^{-2}\right)_\infty} = 0 \ , \ q > 1, \text{ and}
$$

$$
P\int_{-\frac{1}{q\sqrt{2(1-q)}}}^{+\frac{1}{q\sqrt{2(1-q)}}} \frac{\left(2q^2(1-q)x^2; q^2\right)_\infty}{x}\,dx = 0 \ , \ 0 < q < 1.
$$

Therefore, with (4.20), and choosing

$$
\lambda^{-1} = \begin{cases}
\dfrac{\sqrt{2}}{\pi}(q-1)^{\frac{1}{2}}\dfrac{\left(q^{-2}; q^{-2}\right)_\infty}{\left(q^{-1}; q^{-2}\right)_\infty} \ , \ q > 1 \\
K_1 \ , \quad 0 < q < 1
\end{cases}
$$

we obtain the integral representation of u

(4.35)

$$
\langle u, f\rangle = \begin{cases}
f(0) + P\displaystyle\int_{-\infty}^{+\infty} \frac{f(x)}{x\left(-2(q-1)x^2; q^{-2}\right)_\infty}dx \ , \\
\qquad f \in \mathcal{P} \ , \ q > 1, \\
f(0) + P\displaystyle\int_{-\frac{1}{q\sqrt{2(1-q)}}}^{+\frac{1}{q\sqrt{2(1-q)}}} \frac{\left(2q^2(1-q)x^2; q^2\right)_\infty}{x}f(x)\,dx \ , \\
\qquad f \in \mathcal{P} \ , \ 0 < q < 1.
\end{cases}
$$

4.3.2. Consider the symmetric H_q-classical linear form v which is in the family of $q-$Jacobi, we have [9]

$$(4.36)\quad \begin{cases} \beta_n = 0\ ,\ \gamma_{n+1} = \left(1-q^{n+1}\right) q^{-(2n+1)}\ ,\ n\geq 0, \\ H_q\left(\left(x^2+1\right)v\right) - (q-1)^{-1}xv = 0, \end{cases}$$

$$(4.37)\quad (v)_{2n} = q^{-n^2}\left(q;q^2\right)_n\ ,\ (v)_{2n+1} = 0\ ,\ n\geq 0,$$

$$(4.38)\quad \langle v,f\rangle = \frac{\left(q^2;q^2\right)_\infty}{\pi\left(q;q^2\right)_\infty}\int_{-\infty}^{+\infty}\frac{1}{\left(-x^2;q^2\right)_\infty} f(x)\,dx,\ f\in\mathcal{P}, 0<q<1.$$

Taking into account $(3.22)-(3.22)'$ and (4.12) we get

$$(4.39)\quad \begin{cases} C_0(x) = (q(q-1))^{-1}x\ ,\ D_0(x) = (q-1)^{-1}\ , \\ \frac{1}{2}\left(C_{n+1}(x) - C_0(x)\right) = [n+1]_q x\ ,\ D_{n+1}(x) = q^{n+1}(q-1)^{-1}\ , \\ \Sigma_n(x) = (q-1)^{-1}[n+1]_q\ ,\ n\geq 0. \end{cases}$$

The linear form u defined by (4.2) is H_q-semiclassical of class 1 for any $\lambda\neq 0$ and fulfils

$$(4.40)\qquad H_q\left(x\left(x^2+1\right)u\right) - (q-1)^{-1}x^2u = 0.$$

From (4.2) and (4.37), the moments of u are

$$(4.41)\qquad \begin{cases} (u)_0 = 1\ ,\ (u)_{2n} = 0\ ,\ n\geq 1, \\ (u)_{2n+1} = \lambda q^{-n^2}\left(q;q^2\right)_n\ ,\ n\geq 0. \end{cases}$$

By virtue of (4.6) and (4.36) we obtain

$$(4.42)\qquad \begin{cases} a_{2n} = -\lambda q^{-n(n+1)}\dfrac{\left(q^2;q^2\right)_n}{\left(q;q^2\right)_n}\ ,\ n\geq 0, \\ a_{2n+1} = \frac{1}{\lambda}(1-q)\,q^{-n(n+1)-1}\dfrac{\left(q^3;q^2\right)_n}{\left(q^2;q^2\right)_n}\ ,\ n\geq 0. \end{cases}$$

Then with (4.7) we get for $n\geq 0$

(4.43)

$$\begin{cases} \widetilde{\beta}_0 = \lambda\ ,\ \widetilde{\beta}_{2n+1} = -q^{-n(n+1)-1}\left\{\lambda q\dfrac{\left(q^2;q^2\right)_n}{\left(q;q^2\right)_n} + \dfrac{1}{\lambda}(1-q)\dfrac{\left(q^3;q^2\right)_n}{\left(q^2;q^2\right)_n}\right\}\ , \\ \widetilde{\beta}_{2n+2} = q^{-n(n+1)-1}\left\{\lambda q^{-2n-1}\dfrac{\left(q^2;q^2\right)_{n+1}}{\left(q;q^2\right)_{n+1}} + \dfrac{1}{\lambda}(1-q)\dfrac{\left(q^3;q^2\right)_n}{\left(q^2;q^2\right)_n}\right\}\ , \\ \widetilde{\gamma}_{2n+1} = -\lambda^2 q^{-2n(n+1)}\dfrac{\left(q^2;q^2\right)_n^2}{\left(q;q^2\right)_n^2}\ , \\ \widetilde{\gamma}_{2n+2} = -\dfrac{1}{\lambda^2}q^{-2\left(n^2+n+1\right)}(1-q)^2\dfrac{\left(q^3;q^2\right)_n^2}{\left(q^2;q^2\right)_n^2}\ . \end{cases}$$

In accordance of (4.14) and $(4.16)-(4.18)$ we obtain for $n\geq 0$

(4.44)

$$\begin{cases} u_n(x) = (q-1)^{-1}\{(q^{n+1}-1)x + q^n a_n\} \ , \\ v_n(x) = -(q-1)^{-1}\{q^{-1}a_n x + q^{-n}(1-q^{n+1})\} \ , \\ \frac{1}{2}\left(\widetilde{C}_{n+1}(x) - \widetilde{C}_0(x)\right) = (q-1)^{-1}\Big\{(q^{n+1}-1)x^2 + \\ +(1-q^{-1}+q^n-q^{n+1})a_n x - q^n a_n^2 - q^{-n}(1-q^{n+1})\Big\} \ , \\ \widetilde{D}_{n+1}(x) = (q-1)^{-1}\Big\{(q^{-1}+q^{n+1}-1)a_n x + q^n a_n^2 + q^{-n}(1-q^{n+1})\Big\}a_n^{-1} \ , \\ \widetilde{C}_0(x) = q^{-2}(q-1)^{-1}x^2 - 1 \ , \ \widetilde{D}_0(x) = (q-1)^{-1}\left(q^{-1}x + \lambda\right) \ , \\ \widetilde{\Sigma}_n(x) = -(q-1)^{-1}\Big\{(2q-1)[n+1]_q x + q^n a_n\Big\} \ . \end{cases}$$

Therefore, with (4.19), the second order linear $q-$difference equation satisfied by $\widetilde{P}_{n+1},\ n \geq 0$ is

(4.45)

$$\begin{aligned} &\left(x^2+1\right)\Big\{(1-q^{-1}-q^{n+1})x + q^{n+1}(qa_{n+1}-a_n)\Big\}\left(H_q \circ H_{q^{-1}}\widetilde{P}_{n+1}\right)(x) + \\ &+\Big\{q^{-2}(q-1)^{-1}(q^{-1}+q^{n+1}-1)(1-q^{-2}-q^2)x^2 + q^{n-2}(a_n - qa_{n+1})(1+q+ \\ &+q^2-q^{-2}(q-1)^{-1})x + q^{-1}+q^{n+1}-1\Big\}\left(H_{q^{-1}}\widetilde{P}_{n+1}\right)(x) + (q-1)^{-2}\Big\{(1-q^{-1}- \\ &-q^{n+1})x + q^{n-1}(1-q^{n+2})a_n + q^{n+1}(q-2)(1-q^{n+1})a_{n+1}\Big\}\widetilde{P}_{n+1}(x) = 0. \end{aligned}$$

Lastly, from the definition (4.21), and (4.38), we have

$$P\int_{-\infty}^{+\infty} \frac{1}{x\left(-x^2;q^2\right)_\infty}dx = 0 \ , \ 0 < q < 1.$$

Therefore, with (4.20),and choosing $\lambda^{-1} = \dfrac{\left(q^2;q^2\right)_\infty}{\pi\left(q;q^2\right)_\infty}$, for $f \in \mathcal{P}$, $0 < q < 1$ we obtain the integral representation of u

$$\langle u, f\rangle = f(0) + P\int_{-\infty}^{+\infty} \frac{1}{x\left(-x^2;q^2\right)_\infty} f(x)\,dx. \tag{4.46}$$

Acknowledgment. I would like to thank the referees for their valuable reviews and for bringing certain references to my attention.

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