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BERGMAN APPROXIMATIONS OF HARMONIC MAPS INTO THE SPACE OF KÄHLER METRICS ON TORIC VARIETIES

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We generalize the results of Song–Zelditch on geodesics in spaces of Kähler metrics on toric varieties to harmonic maps of any compact Riemannian manifold with boundary into the space of Kähler metrics on a toric variety. We show that the harmonic map equation can always be solved and that such maps may be approximated in the C^2 topology by harmonic maps into the spaces of Bergman metrics. In particular, Wess–Zumino–Witten (WZW) maps, or equivalently solutions of a homogeneous Monge–Ampère equation on the product of the manifold with a Riemann surface with S^1 boundary admit such approximations. We also show that the Eells–Sampson flow on the space of Kähler potentials is transformed to the usual heat flow on the space of symplectic potentials under the Legendre transform, and hence it exists for all time and converges.

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1. Introduction

Our main purpose in this article is to prove that the Dirichlet problem for a harmonic map $\varphi : N \to \mathcal{H}(T)$ of any compact Riemannian manifold with boundary N into the infinite-dimensional space $\mathcal{H}(T)$ of toric Kähler metrics on a smooth projective toric variety (M, ω) admits a smooth solution that may be approximated in $C^2(N \times M)$ by a special sequence of harmonic maps $\varphi_k : N \to \mathcal{B}_k(T) \subset \mathcal{H}(T)$ into the finite-dimensional subspaces of Bergman (or Fubini–Study) metrics induced from projective embeddings. As a special case, we show that the WZW (Wess–Zumino–Witten) equation, or equivalently the homogeneous complex Monge–Ampère equation (HCMA) on the product of the manifold with a Riemann surface with S^1 boundary, admits such a solution as well as such approximations. This generalizes previous work of Song–Zelditch in the case of geodesics, i.e., where $N = [0, 1]$.

Before stating our results, we briefly recall the background to our problem. Let (M, ω) be a compact closed Kähler manifold of dimension m with integral Kähler form and let $(L, h_0) \to M$ be an ample Hermitian holomorphic line bundle with $\omega_{h_0} = \omega$ satisfying $[\omega] = c_1(L)$, where $\omega_{h_0} = -\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log h_0$ is the curvature $(1,1)$-form of h_0. Any other hermitian metric on L may be expressed as $h_\varphi = \mathrm{e}^{-\varphi} h_0$, with φ a smooth function on M. Following Mabuchi [**17**], Semmes [**23**] and Donaldson [**10**] one may regard the space

$$\mathcal{H}_\omega := \left\{\varphi \in C^\infty(M) : \omega_\varphi \ = \ \omega + \sqrt{-1}\partial\bar{\partial}\varphi > 0 \ \right\} \tag{1.1}$$

of potentials of Kähler metrics in a fixed cohomology class as an infinite-dimensional symmetric space dual to the group of Hamiltonian diffeomorphisms $\mathrm{Ham}(M, \omega)$. We will henceforth usually identify $\mathcal{H}_\omega$ with the space of Hermitian metrics on L of positive curvature

$$\mathcal{H} := \{\, h \,:\, h = h_0 \mathrm{e}^{-\varphi}, \varphi \in \mathcal{H}_\omega\}.$$

The symmetric space Riemannian metric g_{L^2} is defined by

$$g_{L^2}(\zeta, \eta)_\varphi := \frac{1}{V}\int_M \zeta\eta\,\omega_\varphi^m, \qquad \varphi \in \mathcal{H}_\omega, \quad \zeta, \eta \in T_\varphi\mathcal{H}_\omega \cong C^\infty(M). \tag{1.2}$$

A basic idea, considered by Yau, Tian and Donaldson, is to approximate transcendental objects defined on $\mathcal{H}$ by algebraic objects defined on the finite-dimensional symmetric spaces $\mathcal{B}_k$ of Bergman (or Fubini–Study) metrics on L. To define them, we use the following notation: $H^0(M, L^k)$ is the space of holomorphic sections of the kth power $L^k \to M$, $d_k + 1 = \dim H^0(M, L^k)$ and $\mathcal{B}H^0(M, L^k)$ is the manifold of all bases $\underline{s} = \{s_0, \ldots, s_{d_k}\}$ of $H^0(M, L^k)$. A basis $\underline{s}$ determines a Bergman metric

$$h_{\underline{s}} := (\iota_{\underline{s}}^\star h_{\mathrm{FS}})^{1/k} = \frac{h_0}{\left(\sum_{j=0}^{d_k} |s_j(z)|^2_{h_0^k}\right)^{1/k}}, \tag{1.3}$$

as the pullback of the Fubini–Study metric h_{FS} on the hyperplane bundle $\mathcal{O}(1) \to \mathbb{P}^{d_k}$ under the Kodaira embedding

$$\iota_{\underline{s}} : M \to \mathbb{P}^{d_k}, \ \ z \to [s_0(z), \ldots, s_{d_k}(z)]. \tag{1.4}$$

The space of all Bergman metrics defined by $\mathcal{B}H^0(M,L^k)$ is denoted by

$$\mathcal{B}_k = \{h_{\underline{s}},\ \ \underline{s} \in \mathcal{B}H^0(M,L^k)\}. \tag{1.5}$$

We may identify the space $\mathcal{B}_k$ with the symmetric space $GL(d_k+1,\mathbb{C})/U(d_k+1)$ since $GL(d_k+1,\mathbb{C})$ acts transitively on the set of bases, while $\iota_{\underline{s}}^\star h_{\mathrm{FS}}$ is unchanged if we replace the basis $\underline{s}$ by a unitary change of basis.

A natural question is: to what extent can the geometry of $\mathcal{H}$ be approximated by that of the spaces $\mathcal{B}_k$ of "algebro-geometric" metrics? At the most basic level of individual points, it follows by the Tian Asymptotic Isometry Theorem [**28**] and its subsequent refinements [**7**, **29**] that a metric $h \in \mathcal{H}$ can be approximated in a canonical way by a sequence of Bergman metrics h_k. The geometry of a Riemannian manifold is reflected to a large extent by its geodesics and more generally by the specification of the harmonic maps into it, involving the analysis of certain nonlinear elliptic PDE. In this article we describe how these PDE can be solved (Proposition 3.1), and furthermore how solutions to these PDE on $\mathcal{H}$ can be approximated in an algebro-geometric manner by a sequence of solutions to PDE on $\mathcal{B}_k$, in the setting of a toric variety (Theorem 1.1). The consideration of harmonic maps into $\mathcal{H}$ is a notion going back to Semmes [**22**], and our article is also partly inspired by his work and draws upon several of his ideas. In line with Semmes thought, a further motivation we have in mind is to interpret the second variation for the Legendre transform in convex analysis in the framework of the space of Kähler metrics (Theorem 3.1, Corollary 3.1 and equation (3.16)). Another motivation for our work is the special case of the WZW equation, that in this context is a homogeneous Monge–Ampère equation and goes back to Semmes [**23**] and Donaldson [**10**], and will be described in detail below (Corollary 1.1).

We will need the canonical sequence of harmonic maps mentioned above to state our results, so we recall how it is constructed. First, observe that $\mathcal{B}_k$ is isomorphic to the symmetric space $\mathcal{I}_k$ of Hermitian inner products on $H^0(M,L^k)$, the correspondence being that a basis is identified with an inner product for which the basis is Hermitian orthonormal. Define the maps

$$\mathrm{Hilb}_k : \mathcal{H} \to \mathcal{I}_k \tag{1.6}$$

by the rule that a Hermitian metric $h \in \mathcal{H}$ induces the metrics h^k on L^k and the inner products on $H^0(M,L^k)$,

$$\|s\|^2_{\mathrm{Hilb}_k(h)} := \frac{1}{V}\int_M |s(z)|^2_{h^k}(k\omega_h)^m, \tag{1.7}$$

where $V = \int_M \omega_h^m$. An inner product $I = (\,\cdot\,,\,\cdot\,)$ on $H^0(M,L^k)$ determines an I-orthonormal basis $\underline{s} = \underline{s}_I$ of $H^0(M,L^k)$, an associated Kodaira embedding (1.4), as well as a Bergman metric given by

$$\mathrm{FS}_k(I) := h_{\underline{s}_I}. \tag{1.8}$$

Tian's asymptotic isometry theorem then states that $\mathrm{FS}_k \circ \mathrm{Hilb}_k(h) \to h$ in the $C^\infty(M)$ topology with complete asymptotic expansions [**7**, **28**, **29**].

The question was then raised [**2**, **11**, **19**] whether geodesics of $\mathcal{H}$ could be well approximated by the one-parameter subgroup geodesics of $\mathcal{B}_k$. Geodesics of $\mathcal{H}$ are given by solutions of

$$\ddot{\varphi} - \tfrac{1}{2}|\nabla\dot{\varphi}|^2 = 0, \quad \varphi(0,\,\cdot\,) = \varphi_0,\ \varphi(1,\,\cdot\,) = \varphi_1, \quad \varphi_0, \varphi_1 \in \mathcal{H}, \tag{1.9}$$

where φ is considered as a map from $[0,1]$ to $\mathcal{H}$, or equivalently as a function on $[0,1] \times M$. Phong–Sturm studied this problem in depth and proved that a sequence of geodesics in $\mathcal{B}_k$ converges weakly (almost everywhere) to a prescribed geodesic of $\mathcal{H}$ [**19**]. Song–Zelditch proved that the same sequence converges in $C^2([0,1]\times M)$ when the manifold is toric and one restricts to the torus-invariant metrics [**25**, **26**], and Berndtsson used a different argument to prove that geodesics in $\mathcal{H}$ can be C^0-approximated by geodesics in spaces of Bergman metrics induced by embeddings by sections of $L^k \otimes K_M$, where K_M is the canonical bundle of M [**4**]. In addition, Phong–Sturm and Song–Zelditch proved approximation results for geodesic rays constructed from test configurations [**20**, **27**].

A harmonic map between two Riemannian manifolds (N, f) and $(\tilde{N}, \tilde{f})$ is a critical point of the energy functional

$$E(a) = \int_N |da|^2_{f\otimes a^\star \tilde{f}}\, dV_{N,f},$$

on the space of smooth maps a from N to $\tilde{N}$ [**13**]. Note that this notion may also be defined when the target manifold $(\tilde{N}, \tilde{f})$ is an infinite-dimensional weakly Riemannian manifold. When the target is taken to be $(\mathcal{H}, g_{L^2})$, the energy functional takes the form

$$E(\varphi) = \int_N |d\varphi|^2 dV_{N,f} = \int_{N\times M} f^{ab}\frac{\partial\varphi}{\partial y^a}\frac{\partial\varphi}{\partial y^b}\omega^n_\varphi \wedge dV_{N,f}. \tag{1.10}$$

Definition 1.1. By a smooth map φ from N to $\mathcal{H}$ we mean a function $\varphi \in C^\infty(N \times M)$ such that $\varphi(q,\cdot) \in \mathcal{H}$, for each $q \in N$. A harmonic map from a Riemannian manifold (N, f) into $(\mathcal{H}, g_{L^2})$ is a smooth map from N to $\mathcal{H}$ that is a critical point of (1.10).

The problem we study in this article is whether higher dimensional harmonic maps of general compact Riemannian manifolds N with boundary ∂N into $\mathcal{H}$ admit similar kinds of "algebro-geometric" approximations. For maps into the space of toric Kähler metrics on toric manifolds, we obtain an affirmative solution at the same level of precision as in the case of geodesics studied by Song–Zelditch.

To describe our results, let us define our setting more precisely. We recall that a toric variety M of complex dimension m carries the holomorphic action of a complex torus $(\mathbb{C}^\star)^m$ with an open dense orbit. We let $T = (S^1)^m$

be the associated real torus. Objects associated to M are called toric if they are invariant with respect to T. We let ω denote a toric integral Kähler form on M, and let L be an ample line bundle with $[\omega] = c_1(L)$. We then define the space of toric Hermitian metrics on L,

$$\mathcal{H}(T) = \{h \in \mathcal{H} : t^\star h = h, \ \ \forall t \in T\}. \tag{1.11}$$

This is a flat submanifold of $\mathcal{H}$. As before we will frequently identify an element of $\mathcal{H}(T)$ with the corresponding element of $\mathcal{H}_\omega$. Moreover, we will often identify an element with the local Kähler potential defined on the open orbit of the complex torus; see Section 3. We also denote by $\mathcal{B}_k(T) \subset \mathcal{B}_k$ the subspace of Bergman metrics defined by T-invariant inner products, or equivalently by T-equivariant embeddings. Any such embedding is induced by the basis of toric monomials $\{\chi_\alpha\}_{\alpha \in kP \cap \mathbb{Z}^m}$ of $H^0(M, L^k)$, where P denotes the moment polytope associated to the action, and kP denotes its dilation by a factor of k. Finally, let (N, f) be a compact oriented Riemannian manifold with smooth boundary, let $G(y, q)$ denote the positive Dirichlet Green kernel for the Laplacian $\Delta_N := \Delta_{N,f}$ (see Section 3), and let $dV_{\partial N,f}$ denote the induced measure on ∂N from the restriction of the Riemannian volume form $dV_{N,f}$ from N to ∂N. The main result of this article is:

Theorem 1.1. *Let (M, L, ω) be a polarized toric Kähler manifold, and let (N, f) be a compact oriented smooth Riemannian manifold with smooth boundary ∂N. Let $\psi : \partial N \to \mathcal{H}(T)$ denote a fixed smooth map. There exists a harmonic map $\varphi : N \to \mathcal{H}(T)$ with $\varphi|_{\partial N} = \psi$ and harmonic maps $\varphi_k : N \to \mathcal{B}_k(T)$ with $\varphi_k|_{\partial N} = \mathrm{FS}_k \circ \mathrm{Hilb}_k(\psi)$, given on the open orbit by*

$$\begin{aligned}\varphi_k(y, z) = \frac{1}{k} \log \sum_{\alpha \in kP \cap \mathbb{Z}^m} &|\chi_\alpha(z)|^2_{h_0^k} \\ &\times \exp\left(\int_{\partial N} \partial_{\nu_q} G(y, q) \log ||\chi_\alpha||^2_{h^k_{\psi(q)}} dV_{\partial N,f}(q) \right),\end{aligned} \tag{1.12}$$

and one has

$$\lim_{k \to \infty} \varphi_k = \varphi,$$

in the $C^2(N \times M)$ topology. In fact, for each $\epsilon \in (0, 1/3)$ there exists $C = C(\epsilon) > 0$ independent of k such that

$$||\varphi_k - \varphi||_{C^2(N \times M)} \leq C k^{-1/3+\epsilon}.$$

A motivating special case is the unit disc $N = D := \{z \in \mathbb{C} : |z| \leq 1\}$. It has been the subject of intensive studies (e.g., [**8**, **9**, **12**]). Then the map φ corresponds to certain foliations by holomorphic discs arising from a solution of a certain HCMA equation. To describe this case, let $\pi_2 : D \times M \to M$

denote the projection onto the second factor and consider the HCMA equation,

$$(\pi_2^\star\omega + \sqrt{-1}\partial\bar\partial\varphi)^{n+1} = 0 \quad \text{on} \quad D\times M, \tag{1.13}$$

$$(\pi^\star\omega + \sqrt{-1}\partial\bar\partial\varphi)|_{\{t\}\times M} > 0, \quad \forall\, t\in D, \tag{1.14}$$

$$\varphi = \psi \quad \text{on} \quad \partial D\times M. \tag{1.15}$$

One may show that this HCMA is the Euler–Lagrange equation of an infinite-dimensional version of a WZW model, given by the energy functional

$$E_\sigma^{\mathrm{WZW}}(b) = \frac{1}{2}\int_D |\nabla b|^2 + \int_Z \theta,$$

on the space of maps $b\in C^\infty(D, G^{\mathbb{C}}/G)$, where $\sigma : \partial D\to G^{\mathbb{C}}/G$ is a fixed map σ [**10**]. The Lie bracket of G determines a three-form θ and Z is any cochain with boundary $b(D) - b_0(D)$ for some fixed reference map b_0 with the same boundary conditions ψ. The Euler–Lagrange equations for this functional are the WZW equations

$$d^\star d\, b + [b_\star\frac{\partial}{\partial q}, b_\star\frac{\partial}{\partial s}] = 0, \tag{1.16}$$

in Euclidean coordinates $q+\sqrt{-1}s\in D$, and where $d^\star$ maps sections of $T^\star D\otimes b^\star TG^{\mathbb{C}}/G$ to sections of $b^\star TG^{\mathbb{C}}/G$. Finally, when G and $G^{\mathbb{C}}/G$ are replaced by $\mathrm{Ham}(M,\omega)$ and $\mathcal{H}$, the Christoffel symbols are given by $\Gamma(\zeta,\eta)|_\varphi = -\frac{1}{2}g_\varphi(\nabla\zeta,\nabla\eta)$ (see Lemma 3.1), and the WZW equation is

$$\varphi_{qq} + \varphi_{ss} - \frac{1}{2}|\nabla\varphi_q|^2 - \frac{1}{2}|\nabla\varphi_s|^2 + \{\varphi_q,\varphi_s\}_{\omega_\varphi} = 0.$$

It is a perturbation of the usual harmonic map equation by a Poisson bracket term. Coming back to the toric situation and restricting to the space of torus-invariant Kähler potentials $\mathcal{H}(T)\subseteq\mathcal{H}$ the functions φ_q and φ_s are commuting Hamiltonians and hence the WZW equation reduces to the harmonic map equation. The finite-dimensional WZW equation on $GL(d_k+1,\mathbb{C})/U(d_k+1)$ may be written similarly

$$T^{-1}T_{qq} + T^{-1}T_{ss} - (T^{-1}T_q)^2 - (T^{-1}T_s)^2 + \sqrt{-1}[T^{-1}T_q, T^{-1}T_s] = 0.$$

Torus invariance then corresponds to restriction to diagonal matrices and again the last term vanishes and the equation reduces to the harmonic map equation. The curvature of $\mathcal{H}$ comes from the Poisson bracket and when we restrict to the flat subspace $\mathcal{H}(T)$ the noncommutativity disappears.

In the case of the unit disc, the normal derivative of the Green kernel is the Poisson kernel, whose restriction to $D\times\partial D$ takes the form

$$P(re^{\sqrt{-1}\theta}, e^{\sqrt{-1}\gamma}) = P_r(\theta-\gamma) = -\frac{1}{2\pi}\frac{1-r^2}{1-2r\cos(\theta-\gamma)+r^2}$$

(our convention is that the Green function be nonnegative, as explained in Section 3). Then we have the following more explicit statement of Theorem 1.1:

Corollary 1.1. *Let (M, L, ω) be a polarized toric Kähler manifold. Let φ be a solution of the HCMA equation* (1.13)–(1.15) *with $\psi : S^1 \to \mathcal{H}(T)$ a smooth map, and let $\varphi_k : D \to \mathcal{B}_k(T)$ be given on the open orbit by*

$$\varphi_k\left(re^{\sqrt{-1}\gamma}, z\right) = \frac{1}{k}\log \sum_{\alpha\in kP\cap\mathbb{Z}^m} |\chi_\alpha(z)|^2_{h_0^k} \times \exp\left(\int_{\partial D} P_r(\theta-\gamma)\log \|\chi_\alpha\|^2_{h^k_{\psi(\theta)}}\, d\theta\right).$$

Then $\lim_{k\to\infty}\varphi_k = \varphi$ in the $C^2(D\times M)$ topology.

The proof of Theorem 1.1 builds upon the machinery developed by Song–Zelditch for the study of geodesics in $\mathcal{H}(T)$. In the geodesic case, i.e., $N = [0,1]$, the approximating Bergman Kähler potentials take the form

$$\varphi_k(t,z) = \frac{1}{k}\log \sum_{\alpha\in kP\cap\mathbb{Z}^m} |\chi_\alpha(z)|^2_{h_0^k}\, e^{-(1-t)\log\|\chi_\alpha\|^2_{\mathrm{Hilb}_k(h_0)} - t\log\|\chi_\alpha\|^2_{\mathrm{Hilb}_k(h_1)}}. \tag{1.17}$$

We see that the straight line segment in the case $N = [0,1]$ is replaced by the harmonic extension of the boundary L^2 norming constants in the general case. Aside from justifying the general formula, we need to modify the estimates to apply to harmonic functions on N rather than linear functions on $[0,1]$. Using the localization lemma and the asymptotics of the peak values proved in [**26**] (which we recall in Section 2), the uniform convergence in C^2 reduces to a verification of orders of amplitudes where the analysis is carried out separately in the interior of the polytope and near its boundary. Since many details are similar to the geodesic case, we concentrate here only on the novel features and the reader of Section 5 would benefit from some familiarity with [**26**]. The reader is also referred to [**21**, Chapter 3], where the results of the present article appear with greater detail.

Let us make note of one more relation between the geodesic segment problem and the harmonic mapping problem. In both cases a key aspect of the toric situation is that the Legendre transform linearizes the harmonic map equation. This was known previously for geodesics [**10**, **15**, **23**] (see also [**26**] for a simple proof), but is observed for the first time here for general harmonic maps. We refer the reader to Section 3 where we also observe a generalization (3.16) of a well-known formula from convex analysis and show that the Eells–Sampson harmonic map flow is Legendre transformed to the usual heat flow. This fact is quite a remarkable property of the toric situation and does not hold for general variational problems. It follows that one can explicitly solve the WZW and harmonic map equations in terms of the associated symplectic potentials. We make crucial use of this in proving the convergence of the Bergman harmonic maps, and it is the reason why we do not founder amid regularity problems as in the general projective case. Our results give the first proof of convergence for higher dimensional harmonic maps into spaces of Kähler metrics. We hope to discuss in a separate article convergence results for WZW maps on general projective manifolds.

2. Background results

We begin by recalling some basic facts regarding toric varieties relevant to our setting. We refer the reader to [**6**, **21**, **24**, **26**] for more details.

We will work with coordinates on the open dense orbit of the complex torus given by $z = e^{\rho/2+\sqrt{-1}\theta}$, with $\rho, \theta \in \mathbb{R}^m \times (S^1)^m$. The real torus $T \cong (S^1)^m$ acts in a Hamiltonian fashion with respect to ω. The image of its moment map μ is a Delzant polytope $P \subset \mathbb{R}^m$. Let x denote the Euclidean coordinate on P. The polytope is given by $P = \{x \in \mathbb{R}^m : l_r(x) := \langle x, v_r\rangle - \lambda_r \leq 0, r = 1, \ldots, d\}$ where v_r is an outward pointing normal to the rth $(m-1)$-dimensional face of P (also called a facet) and is a primitive element of the lattice $\mathbb{Z}^m$.

The toric monomials $\{\chi_\alpha(z) := z^\alpha\}_{\alpha \in kP\cap\mathbb{Z}^m}$ are an orthogonal basis of $H^0(M, L^k)$ with respect to any element of $\mathcal{B}_k(T)$. Hence a toric inner product, equivalently a point in $\mathcal{B}_k(T)$, is completely determined by the L^2 norms (up to k^n/V), or *norming constants*, of the toric monomials:

$$\mathcal{Q}_{h^k}(\alpha) := \|\chi_\alpha\|^2_{h^k} = \int_{(\mathbb{C}^*)^m} |z^\alpha|^2 e^{-k\varphi} dV_h.$$

Unlike in the Introduction here we let $h = \mathrm{e}^{-\varphi}$ with φ a local Kähler potential on the open orbit (that does not extend globally). Define the normalized norms of the monomials

$$\mathcal{P}_{h^k}(\alpha, z) := \frac{|\chi_\alpha(z)|^2_{h^k}}{\|\chi_\alpha\|^2_{h^k}},$$

and their peak values

$$\mathcal{P}_{h^k}(\alpha) := \frac{|\chi_\alpha\left(\mu^{-1}_{h^k}\left(\frac{\alpha}{k}\right)\right)|^2_{h^k}}{\|\chi_\alpha\|^2_{h^k}}. \tag{2.1}$$

In order to complete the proof of Theorem 1.1 we will need some of the tools developed by Song–Zelditch, that we now recall. First, an asymptotic expression for $\mathcal{P}_{h^k}(\alpha)$ for families of toric Bergman metrics. This expression is sensitive to the distance of α/k to the boundary of the polytope. Recall that the kth Bargmann–Fock model on $(\mathbb{C}, \sqrt{-1}dz \wedge d\bar{z})$ is given by the holomorphic functions that are L^2 with respect to the Hermitian metric $h^k_{\mathrm{BF}} = e^{-k|z|^2}$ and a basis is given by all monomials z^α with $\alpha \in \mathbb{Z}_+$. One may compute that

$$\mathcal{P}_{h^k_{\mathrm{BF}}}(\alpha) = ke^{-\alpha}\frac{\alpha^\alpha}{\alpha!}. \tag{2.2}$$

Let $\delta_k = k^{-2/3}$. Denote by $\mathcal{F}_{\delta_k}(x) = \{r : l_r(x) < \delta_k\}$ the index set for those facets to which x is $k^{-2/3}$-close, and let $\delta^\sharp_k(x)$ denote the cardinality of this set. Set

$$\mathcal{G}_\varphi(x) := \left(\delta_\varphi(x) \cdot \Pi_{j \notin \mathcal{F}_{\delta_k(x)}} l_j(x)\right)^{-1},$$

where $\delta_\varphi(x)$ is defined in (3.6) below and put

$$\mathcal{P}_{\mathrm{BF},\delta_k}(\alpha) := \Pi_{j \in \mathcal{F}_{\delta_k(x)}} \mathcal{P}_{h^k_{\mathrm{BF}}}(\alpha_j).$$

These two terms are the far and near contributions to the asymptotics of the peak values $\mathcal{P}_{h^k}(\alpha)$:

Lemma 2.1. (See [**26**, Propositions 6.1, 6.6].) *Let $\delta_k = k^{-2/3}$. Let $\{h_t\}_{t\in K}$ be a family of metrics with K compact. Then there exist $C > 0$ independent of t such that*

$$\mathcal{P}_{h^k_t}(\alpha) = Ck^{\frac{1}{2}\left(m - \delta^\sharp_k\left(\frac{\alpha}{k}\right)\right)}\sqrt{\mathcal{G}_\varphi\left(\frac{\alpha}{k}\right)}\mathcal{P}_{\mathrm{BF},\delta_k}\left(\frac{\alpha}{k}\right)\left(1 + R_k\left(\frac{\alpha}{k}, h_t\right)\right), \tag{2.3}$$

where $R_k = O\left(k^{-\frac{1}{3}}\right)$. This expansion is uniform in t and may be differentiated twice to give for $j = 1, 2$ and for some amplitudes S_j of order zero the expansion

$$\begin{aligned}\left(\frac{\partial}{\partial t}\right)^j \mathcal{P}_{h^k_t}(\alpha) = C_m k^{\frac{1}{2}\left(m - \delta^\sharp_k\left(\frac{\alpha}{k}\right)\right)}&\sqrt{\mathcal{G}_\varphi\left(\frac{\alpha}{k}\right)}\mathcal{P}_{\mathrm{BF},\delta_k}\left(\frac{\alpha}{k}\right)\\ &\times\left(S_j(t,\alpha,k) + R_k\left(\frac{\alpha}{k}, h_t\right)\right).\end{aligned} \tag{2.4}$$

Second, recall the following asymptotic localization of sums result:

Lemma 2.2. (See [**26**, Lemma 1.1].) *Let $B_k(y,\alpha) : kP \cap \mathbb{Z}^m \to \mathbb{C}$ be a family of lattice point functions satisfying $|B_k(y,\alpha)| \leq C_0 k^M$ for some $C_0, M \geq 0$. Fix $\delta \in (0, 1/2)$. Then there exists $C > 0$ such that*

$$\sum_{kP\cap\mathbb{Z}^m} B_k(y,\alpha)\mathcal{P}_{h^k_y}(\alpha, z) = \sum_{\alpha : |\frac{\alpha}{k} - \mu_y(z)| \leq k^{\delta - \frac{1}{2}}} B_k(y,\alpha)\mathcal{P}_{h^k_y}(\alpha, z) + O(k^{-C}).$$

3. Legendre transform to harmonic functions and Legendre duality of geometric flows

Over the open orbit of M, a toric Kähler potential may be identified with a convex function on $\mathbb{R}^m$, $\varphi = \varphi(\rho)$ in the logarithmic coordinates of Section 2. By abuse of notation we will frequently identify $\mathcal{H}(T)$ with the local torus-invariant Kähler potentials defined on the open orbit. The gradient of $\varphi(\rho) = \varphi(e^\rho)$ is a one-to-one map, identified with the moment map μ, whose image closure is P. The Legendre transform appeared in the context of symplectic toric manifolds in the work of Guillemin [**16**] and this tool lies at the heart of this section (some of our work in this section is also directly related to work of Semmes [**22**]). It takes Kähler potentials φ on the open orbit to symplectic potentials $\mathcal{L}\varphi = u_\varphi$, that are defined as convex functions on P with logarithmic singularities on ∂P, and relates the moment map, local symplectic potential $u_\varphi(x) = u_\varphi(\mu(\rho))$ and local Kähler potential as follows:

$$\begin{aligned} u_\varphi(x) &= \langle x, 2\log\mu^{-1}(x)\rangle - \varphi(\mu^{-1}(x)) \\ &= \langle x, (\nabla\varphi)^{-1}(x)\rangle - \varphi\big((\nabla\varphi)^{-1}(x)\big). \end{aligned} \tag{3.1}$$

In addition

$$(\nabla\varphi)^{-1}(x) = \nabla u(x) \tag{3.2}$$

and

$$(\nabla^2\varphi)^{-1}|_{(\nabla\varphi)^{-1}(x)} = \nabla^2 u|_x. \tag{3.3}$$

Any symplectic potential u can be written as $u_0 + f$, with respect to the canonical potential

$$u_0(x) = \sum_{k=1}^{d} l_k(x)\log l_k(x) \tag{3.4}$$

introduced by Guillemin, with f smooth up to the boundary [**16**]. Just as for Kähler potentials we may define the space of global symplectic potentials:

$$\mathcal{L}\mathcal{H}(T) = \{\, F \in C^\infty(P) \,:\, u_0 + F = \mathcal{L}\varphi \;\text{ with }\; \varphi \in \mathcal{H}(T)\}. \tag{3.5}$$

We will sometimes, by abuse of notation, identify elements of this space with their local symplectic potentials in the same manner as with $\mathcal{H}(T)$ itself.

Letting $G_\varphi(x) = \nabla^2 u_\varphi(x)$ we have the following formula of Abreu:

$$\det G_\varphi^{-1} = \delta_\varphi(x)\cdot \Pi_{r=1}^d l_r(x), \tag{3.6}$$

for some positive smooth function δ_φ [**1**].

Let $n = \dim_{\mathbb{R}} N$ and denote by $y^1, \ldots, y^n$ local coordinates over some coordinate patch $U \subset N$. We assume that N is oriented as a manifold with boundary, i.e., that the orientation on ∂N is the one induced from N.

Recall that there always exists a Dirichlet Green function $G(y,q) \in C^\infty(N \times N \setminus \mathrm{diag}(N))$ for the Laplacian $\Delta_N := \Delta_{N,f}$ on such a manifold [**3**, **14**]. If $v \in C^\infty(\partial N)$, the equations

$$\begin{aligned} \Delta_N w &= 0 \text{ on } \; N \\ w &= v \text{ on } \; \partial N \end{aligned}$$

have a unique smooth solution

$$w(\,\cdot\,) = -\int_{\partial N} v(q)\partial_{\nu(q)} G(\,\cdot\,, q)\, dV_{\partial N, f}(q) \tag{3.7}$$

(our convention will be that $G(y,q)$ is positive in the interior and vanishes when q is in the boundary), with $\nu(q) = \nu^i(q)\frac{\partial}{\partial y^i}|_q$ an outward unit normal to ∂N in N (coming from a Riemannian splitting $TN|_{\partial N} = T\partial N \oplus N_{\partial N}$, where $N_{\partial N}$ is the normal bundle to ∂N in N), and where we let $\partial_{\nu(q)} G(y,q) := \nu(q)G(y,q) \le 0$ be the normal derivative with respect to the second argument.

Let Γ^c_{ab} denote the Christoffel symbols of (N,f) with respect to local coordinates $y^1, \ldots, y^n$. Recall the following expression for the Christoffel symbols of $(\mathcal{H}, g_{L^2})$. Our proof is a slight variation on those in [**5**, **10**].

Lemma 3.1. *For every $e, f \in T_\varphi \mathcal{H}$ we have $\Gamma(e,f)|_\varphi = -\frac{1}{2} g_\varphi(\nabla e, \nabla f)$.*

Proof. Recall that the proof of the Koszul formula for the Levi–Civita connection of a finite-dimensional manifold [**18**, p. 122], carries over to infinite dimensions to show that if a Levi–Civita connection exists it is unique. Regard the functions c, e, f as constant vector fields on $\mathcal{H}$ and let D denote the Levi–Civita connection of g_{L^2}. Therefore the corresponding brackets vanish and we have

$$2g_{L^2}(D_c e, f)|_\varphi = c\, g_{L^2}(e,f) - f\, g_{L^2}(e,c) + e\, g_{L^2}(f,c). \tag{3.8}$$

Since

$$c\, g_{L^2}(e,f) = \frac{d}{dt}\Big|_0 \frac{1}{V}\int_M ef(\omega + t\sqrt{-1}\partial\bar\partial c)^n = \frac{1}{V}\int_M ef\Delta_\varphi c\, \omega^n,$$

we have

$$\begin{aligned} 2g_{L^2}(D_c e, f)|_\varphi &= \frac{1}{V}\int_M (ef\Delta_\varphi c - ce\Delta_\varphi f + fc\Delta_\varphi e)\, \omega^n \\ &= \frac{1}{V}\int_M (ef\Delta_\varphi c - f\Delta_\varphi(ce) + e\Delta_\varphi(fc))\, \omega^n \\ &= -\frac{1}{V}\int_M g_\varphi(\nabla c, \nabla e) f\, \omega^n. \end{aligned} \tag{3.9}$$

It follows that

$$D_c e = -\frac{1}{2} g_\varphi(\nabla c, \nabla e).$$

Finally, this expression is symmetric hence D is torsion free, and it is also compatible with g_{L^2} since $f\, g_{L^2}(c,e) = g_{L^2}(D_f c, e) + g(c, D_f e)$ is just

$$\frac{1}{V}\int_M ec\Delta_{\omega_\varphi} f\omega^n = \frac{1}{V}\int_M ec\frac{1}{2}\Delta_{g_\varphi} f\omega^n = -\frac{1}{V}\int_M \frac{1}{2}(e\nabla f\cdot\nabla c + c\nabla f\cdot\nabla e)\omega^n.$$

□

Several authors observed previously that the Legendre transform linearizes the geodesic equation, so that a geodesic φ_t with endpoints φ_0, φ_1 is given by $\varphi_t = \mathcal{L}(\mathcal{L}\varphi_0 + t(\mathcal{L}\varphi_1 - \mathcal{L}\varphi_0))$ [**10**, **15**, **23**]. We now observe that under the Legendre transform, a harmonic map into $\mathcal{H}(T)$ is mapped to a family of symplectic potentials that are harmonic functions in the N variables. The key point is that the Legendre transform eliminates the Christoffel symbols in a variational sense.

Proposition 3.1. *Let $\psi : \partial N \to \mathcal{H}(T)$ be a smooth map. There exists a unique harmonic map φ from N to $\mathcal{H}(T)$ that agrees with ψ on ∂N. Moreover, $\varphi = \mathcal{L}u$ where $u \in C^\infty(N \times P \setminus \partial P)$ satisfies $\Delta_N u = 0$ and $u|_{\partial N} = \mathcal{L}\psi$.*

By $\Delta_N u = 0$ we mean that for each $p \in P\backslash\partial P$ the function $u(p) \in C^\infty(N)$ satisfies $\Delta_N u(p) = 0$ in $N\backslash\partial N$.

Proof. The proof of the one-dimensional case [**26**, Proposition 2.1] carries over with minor changes. Indeed, harmonic maps into $\mathcal{H}(T)$ are stationary points of the functional

$$E(\varphi) = \int_N |d\varphi|^2 dV_{N,f} = \int_{N\times M} f^{ab}\frac{\partial\varphi}{\partial y^a}\frac{\partial\varphi}{\partial y^b}\omega_\varphi^n \wedge dV_{N,f}. \tag{3.10}$$

First, considering a variation of (3.1) at $\rho = \rho(x) = (\nabla\varphi)^{-1}(x)$ yields

$$\frac{\partial u}{\partial y^a}\Big|_x = \frac{du}{dy^a}\Big|_x = \sum_{j=1}^n x_j\frac{\partial\rho_j}{\partial y^a} - \frac{\partial\varphi}{\partial y^a}\Big|_\rho - \sum_{j=1}^n \frac{\partial\varphi}{\partial\rho_j}\frac{\partial\rho_j}{\partial y^a} = -\frac{\partial\varphi}{\partial y^a}\Big|_\rho,$$

since $\nabla\varphi(\rho) = x$.

Next, torus invariance allows us to integrate instead over the polytope and we have using (3.2) and (3.3) that

$$\begin{aligned}(\nabla\varphi)_\star(\omega_\varphi^n) &= (\nabla\varphi)_\star\big((\det\nabla^2\varphi)\, d\rho_1\wedge d\theta_1\wedge\cdots\wedge d\rho_m\wedge d\theta_m\big)\\ &= (\nabla u)_\star^{-1}\big((\det\nabla^2\varphi)\, d\rho_1\wedge d\theta_1\wedge\cdots\wedge d\rho_m\wedge d\theta_m\big)\\ &= (\nabla u)^\star\big((\det\nabla^2\varphi)\, d\rho_1\wedge d\theta_1\wedge\cdots\wedge d\rho_m\wedge d\theta_m\big)\\ &= (\det\nabla^2\varphi)(\det\nabla^2 u)\, dx^1\wedge\cdots\wedge dx^n = dx.\end{aligned} \tag{3.11}$$

This means that the metric g_{L^2} is pushed forward to the "Euclidean" metric on $\mathcal{LH}(T)$. Therefore the functional $E(\varphi)$ equals

$$\int_{N\times P} f^{ab}\frac{\partial u}{\partial y^a}\frac{\partial u}{\partial y^b}dx\wedge dV_{N,f}=\int_N |du|^2 dV_{N,f}, \tag{3.12}$$

which is equal to the energy of the map $u : N \to \mathcal{LH}(T)$. Since the target space is now flat with vanishing Christoffel symbols the Euler–Lagrange equation is $\Delta_N u = 0$.

Finally, note that since $u(q) = \mathcal{L}\psi(q)$ is convex for each $q \in \partial N$ it is also convex in the interior of N: Observe that from (3.7) it follows that the Hessian of u (in the P variables, namely x) for every $y \in N$ is given by

$$\nabla^2 u(y,x) = -\int_{\partial N}\nabla^2 u(q,x)\partial_{\nu(q)}G(y,q)dV_{\partial N,f}(q),$$

and since $-\partial_{\nu(q)}G(y,q) \geq 0$(see the paragraph after (3.7)) it follows that $\nabla^2 u(y,x)$ is therefore a positive-definite matrix. Therefore $\varphi := \mathcal{L}^{-1}u$ is in $\mathcal{H}(T)$ and solves the harmonic map equation with boundary values ψ, as required. □

The Eells–Sampson harmonic map heat flow [**13**] on the space of smooth maps from (N, f) to $(\mathcal{H}(T), g_{L^2})$ is given by

$$\partial_t\varphi = f^{ab}\partial_{y^a}\partial_{y^b}\varphi - f^{ab}\Gamma^c_{ab}\partial_{y^c}\varphi - \frac{1}{2}f^{ab}g(\nabla\partial_{y^a}\varphi, \nabla\partial_{y^b}\varphi), \tag{3.13}$$

while the heat flow on the space of symplectic potentials $\mathcal{LH}(T)$ is given by

$$\partial_t u = \Delta_N u. \tag{3.14}$$

Note that equations (3.13) and (3.14) hold without change for the global Kähler and symplectic potentials, respectively.

We record the following result although we will not make use of it for the proof of the main theorem.

Theorem 3.1. *Under the Legendre transform the Eells–Sampson harmonic map flow* (3.13) *on the space of Kähler potentials* $\mathcal{H}(T)$ *is mapped to the heat flow* (3.14) *on the space of symplectic potentials* $\mathcal{LH}(T)$.

Proof. As above, taking a variation of (3.1) yields

$$\left.\frac{\partial u}{\partial t}\right|_x = -\left.\frac{\partial \varphi}{\partial t}\right|_{\rho(x)}. \tag{3.15}$$

Intuitively, the equality of the energy functionals (3.10) and (3.12) then suggests that their Euler–Lagrange equations should coincide, however up to a sign, coming from the fact that an infinitesimal variation $\delta\psi(\rho)$ in one corresponds to an infinitesimal variation $-\delta\psi(x)$ in the second. More precisely, we have

$$-\Delta_N u = f^{ab}\partial_{y^a}\partial_{y^b}\varphi - f^{ab}\Gamma^c_{ab}\partial_{y^c}\varphi - \tfrac{1}{2}f^{ab}g(\nabla\partial_{y^a}\varphi, \nabla\partial_{y^b}\varphi). \tag{3.16}$$

To demonstrate (3.16), recall first the following formula for the second variation of the Legendre duals of a family of convex functions (parametrized by t, say) that have the same gradient image, that follows by taking a variation of (3.1) and using (3.2):

$$\begin{aligned}\frac{\partial^2 u}{\partial t^2}\Big|_x &= -\frac{\partial^2\varphi}{\partial t^2}\Big|_{(\nabla\varphi)^{-1}(x)} - \sum_{j=1}^n \frac{\partial^2\varphi}{\partial t\partial\rho_j}\frac{\partial((\nabla\varphi)^{-1}(x))_j}{\partial t}\\ &= -\frac{\partial^2\varphi}{\partial t^2}\Big|_{(\nabla\varphi)^{-1}(x)} - \sum_{j=1}^n \frac{\partial^2\varphi}{\partial t\partial\rho_j}\frac{\partial(\partial u/\partial x_j)}{\partial t}\\ &= -\frac{\partial^2\varphi}{\partial t^2}\Big|_{(\nabla\varphi)^{-1}(x)} - \langle\nabla(\partial\varphi/\partial t)|_{(\nabla\varphi)^{-1}(x)}, \nabla(\partial u/\partial t)|_x\rangle,\end{aligned} \tag{3.17}$$

or more succinctly

$$-\ddot\varphi = \ddot u + \langle\nabla\dot\varphi, \nabla\dot u\rangle. \tag{3.18}$$

Now, the terms that are linear in the first derivatives on each side of (3.16) are equal to each other by the first variation formula for the Legendre transform (3.15). Next, fix $y \in N$ and choose coordinates on N for which $f^{ab} = \delta^{ab}$ at $y \in N$. Then (3.18) gives

$$-\partial_{y^a}\partial_{y^a}u = \partial_{y^a}\partial_{y^b}\varphi + \langle\nabla\partial_{y^a}u, \nabla\partial_{y^a}\varphi\rangle.$$

But now

$$\partial_{x_j}(\partial_{y^a}u(x)) = -\partial_{x_j}(\partial_{y^a}\varphi(\rho)) = -\partial_{\rho_k}\partial_{y^a}\varphi\cdot\partial_{x_j}\rho_k = -\partial_{\rho_k}\partial_{y^a}\varphi\cdot\partial_{x_j}(\nabla u)_k.$$

Therefore using (3.3) and the fact that g_φ is represented in coordinates on the open orbit by $\nabla^2\varphi$ we see that (3.16) holds. Thus, the Legendre transform sends solutions of (3.13) to solutions of (3.14). □

In general, one does not expect the Euler–Lagrange equations of two equal functionals defined on two different spaces to transform to each other. In our situation this does happen and in essence is due to the fact that the Legendre transform eliminates the Christoffel symbols not only in a variational sense but pointwise.

Observe that equation (3.16) generalizes the well-known formula (3.18) from convex analysis for the second variation of a family of convex functions on $\mathbb{R}^m$ parametrized by $(\mathbb{R}, dx)$ that have the same gradient image. The factor $\frac{1}{2}$ in our formula comes from the conventions we used to relate the Riemannian and Kähler metrics.

It is not a priori clear that the nonlinear flow (3.13) exists for all time and converges, despite the fact that the target is formally nonpositively curved.

However this is certainly the case for the linear heat flow (3.14). As an immediate corollary of Theorem 3.1 we have:

Corollary 3.1. *The Eells–Sampson harmonic map flow (*3.13*) on the space of Kähler potentials $\mathcal{H}(T)$ exists for all time and converges exponentially fast to the harmonic map given by Proposition* 3.1*.*

4. The approximating sequence

Given a harmonic map $\varphi : N \to \mathcal{H}(T)$ we now define the purported approximating sequence of harmonic maps $\varphi_k : N \to \mathcal{B}_k(T)$. First, given a family of toric Kähler metrics ψ parametrized by ∂N we 'project' the family pointwise by $\mathrm{FS}_k \circ \mathrm{Hilb}_k$ onto $\mathcal{B}_k(T)$ to obtain a family of toric Bergman metrics parameterized by ∂N. Each of these metrics is determined by its L^2 norming constants, hence by the diagonal matrices

$$\mathrm{diag}(\mathcal{Q}_{h^k_{\psi(q)}}(\alpha))_{\alpha \in kP \cap \mathbb{Z}^m}, \quad q \in \partial N.$$

For each α, we solve the boundary problem

$$\begin{aligned} \Delta \lambda_\alpha(y) = 0, \quad y \in N, \\ \lambda_\alpha = \log \mathcal{Q}_{h^k_{\psi(q)}}(\alpha), \quad q \in \partial N. \end{aligned}$$

We then map back to $\mathcal{B}_k$ via FS_k to obtain the family

$$\varphi_k(y, z) = \frac{1}{k} \log \sum_{\alpha \in kP \cap \mathbb{Z}^m} e^{-\lambda_\alpha(y)} |\chi_\alpha(z)|^2_{h_0^k} \in \mathcal{B}_k(T)$$

of harmonic maps alluded to in Theorem 1.1. This may be written somewhat more explicitly in terms of the Green kernel:

$$\begin{aligned} \varphi_k(y, z) = \frac{1}{k} \log \sum_{\alpha \in kP \cap \mathbb{Z}^m} |\chi_\alpha(z)|^2_{h_0^k} \\ \times \exp\Big(\int_{\partial N} \partial_{\nu(q)} G(y, q) \log ||\chi_\alpha||^2_{h_{\psi(q)}} dV_{\partial N, f}(q) \Big). \end{aligned}$$

Our first aim is to prove the C^0 convergence by showing

$$\begin{aligned} \varphi_k(y, z) - \varphi(y, z) \\ = \frac{1}{k} \log \sum_{\alpha \in kP \cap \mathbb{Z}^m} |\chi_\alpha(z)|^2_{h^k_{\varphi(y)}} \\ \times \exp\Big(\int_{\partial N} \partial_{\nu(q)} G(y, q) \log \|\chi_\alpha\|^2_{h_{\psi(q)}} dV_{\partial N, f}(q) \Big) = O(\tfrac{\log k}{k}). \end{aligned} \tag{4.1}$$

We begin by rewriting the sum in a convenient way.

Put

$$\mathcal{R}_k(y,\alpha) := \exp\Big(-\int_{\partial N} \partial_{\nu(q)} G(y,q) \log \frac{\mathcal{Q}_{h^k_{\varphi(y)}}(\alpha)}{\mathcal{Q}_{h^k_{\psi(q)}}(\alpha)} dV_{\partial N,f}(q)\Big). \tag{4.2}$$

Then proving (4.1) is equivalent to proving

$$\frac{1}{k} \log \sum_{\alpha \in kP \cap \mathbb{Z}^m} \mathcal{R}_k(y,\alpha) \mathcal{P}_{h^k_{\varphi(y)}}(\alpha, z) = O(\log k/k). \tag{4.3}$$

Put $u_y := u_{\varphi(y)} = u(y,\cdot)$, for $y \in N$. In light of the results in the geodesic case [**26**] we expect the asymptote of $\mathcal{R}_k$ to be the following:

$$\mathcal{R}_\infty(y,x) := \exp\left(-\frac{1}{2}\int_{\partial N} \partial_{\nu(q)} G(y,q) \log \frac{\det \nabla^2 u_y(x)}{\det \nabla^2 u_q(x)} dV_{\partial N,f}(q)\right).$$

Note that by (3.6) we have

$$\mathcal{R}_\infty(y,x) = \exp\left(-\frac{1}{2}\int_{\partial N} \partial_{\nu(q)} G(y,q) \log \frac{\delta_{\psi(q)}(x)}{\delta_{\varphi(y)}(x)} dV_{\partial N,f}(q)\right),$$

and therefore $\mathcal{R}_\infty \in C^\infty(N \times P)$ (up to the boundary).

In light of Lemma 2.1 it will be useful to express the ratio $\mathcal{R}_k$ in terms of the functions $\mathcal{P}_{h^k}(\alpha)$ (2.1) in the following form:

Lemma 4.1. *One has*

$$\mathcal{R}_k(y,\alpha) = \exp\left(-\int_{\partial N} \partial_{\nu(q)} G(y,q) \log \frac{\mathcal{P}_{h^k_{\psi(q)}}(\alpha)}{\mathcal{P}_{h^k_{\varphi(y)}}(\alpha)} dV_{\partial N,f}(q)\right).$$

Proof. By definition,

$$\mathcal{R}_k(y,\alpha) = \exp\left(\int_{\partial N} \partial_{\nu(q)} G(y,q) \log \frac{\mathcal{Q}_{h^k_{\psi(q)}}(\alpha)}{\mathcal{Q}_{h^k_{\varphi(y)}}(\alpha)} dV_{\partial N,f}(q)\right).$$

Specializing (3.1) to the lattice point α we have

$$u_\varphi(\alpha) = \langle \alpha, 2\log \mu^{-1}(\alpha)\rangle - \varphi(\mu^{-1}(\alpha)),$$

implying that

$$\log \mathcal{Q}_{h^k}(\alpha)\mathcal{P}_{h^k}(\alpha) = ku(\alpha/k). \tag{4.4}$$

Since u is harmonic in y it follows that

$$\log \mathcal{Q}_{h^k_{\varphi(y)}}(\alpha)\mathcal{P}_{h^k_{\varphi(y)}}(\alpha) = -\int_{\partial N} \partial_{\nu(q)} G(y,q) \log \mathcal{Q}_{h^k_{\psi(q)}}(\alpha)\mathcal{P}_{h^k_{\psi(q)}}(\alpha) dV_{\partial N,f}(q),$$

which together with the definition concludes the proof. $\square$

5. Proof of Theorem 1.1

Recall that the standard elliptic regularity theory applies to the operator Δ_N [**3**, **14**]. Namely, if $w \in C^\infty(N)$ satisfies $\Delta_N w = 0$ on $N \backslash \partial N$, and $w = v$ on ∂N, there exists $C = C(N, f)$ such that the Schauder estimates hold:

$$\|w\|_{C^{2,1/2}(N)} \leq C(\|w\|_{C^0(N)} + \|v\|_{C^{2,1/2}(\partial N)}). \tag{5.1}$$

Moreover, the maximum principle implies that $\|w\|_{C^0(N)} \leq \|v\|_{C^0(\partial N)}$, and so the estimates are only in terms of $\|v\|_{C^{2,1/2}(\partial N)}$.

First, we need the following asymptotic regularity for the coefficients $\mathcal{R}_k(y, \alpha)$.

Lemma 5.1. *There exists a positive constant $C > 0$ such that for all k, y, α one has*

$$1/C < \mathcal{R}_k(y, \alpha) < C. \tag{5.2}$$

Moreover, one has

$$\mathcal{R}_k(y, \alpha) - \mathcal{R}_\infty(y, \alpha/k) = O(k^{-\frac{1}{3}}) \tag{5.3}$$

uniformly in (y, α), and $\log \mathcal{R}_k(y, \alpha)$ is uniformly bounded in $C^2(N)$.

Proof. The Bargmann–Fock terms in (2.3) depend only on the geometry of P and not on $y \in N$ and so are cancelled in the ratio $\mathcal{R}_k(y, \alpha)$. Therefore, by Lemma 4.1,

$$\begin{aligned}
\log \mathcal{R}_k(y, \alpha) &= \int_{\partial N} -\partial_{\nu(q)} G(y, q) \log \mathcal{P}_{h^k_{\psi(q)}} dV_{\partial N, f}(q) - \log \mathcal{P}_{h^k_{\varphi(y)}} \\
&= \frac{1}{2} \int_{\partial N} -\partial_{\nu(q)} G(y, q) \log \left(\mathcal{G}_{\psi(q)} \left(\frac{\alpha}{k} \right) \right) dV_{\partial N, f}(q) \\
&\quad - \frac{1}{2} \log \left(\mathcal{G}_{\varphi(y)} \left(\frac{\alpha}{k} \right) \right) \\
&\quad + \int_{\partial N} -\partial_{\nu(q)} G(y, q) \log \left(1 + R_k \left(\frac{\alpha}{k}, h_{\psi(q)} \right) \right) dV_{\partial N, f}(q) \\
&\quad - \log \left(1 + R_k \left(\frac{\alpha}{k}, h_{\varphi(y)} \right) \right).
\end{aligned}$$

The first two terms simplify to

$$\frac{1}{2} \int_{\partial N} -\partial_{\nu(q)} G(y, q) \log \frac{\delta_{\varphi(y)} \left(\frac{\alpha}{k} \right)}{\delta_{\psi(q)} \left(\frac{\alpha}{k} \right)} dV_{\partial N, f}(q) = \log \mathcal{R}_\infty(y, \alpha/k),$$

and so the Schauder estimates (5.1) together with Lemma 2.1 imply equation (5.3) and hence also the uniform estimate (5.2) (indeed, Lemma 2.1 shows the fourth term is $O\left(k^{-\frac{1}{3}}\right)$ while the third term is a harmonic function on N which is of order $O\left(k^{-\frac{1}{3}}\right)$ on ∂N).

We now turn to prove the higher derivative estimates. A first derivative of the fourth term yields, according to (2.4),

$$\frac{S_1(y,\alpha,k)+R_k\left(\frac{\alpha}{k},h_{\varphi(y)}\right)}{1+R_k\left(\frac{\alpha}{k},h_{\varphi(y)}\right)},$$

and this is uniformly bounded according to Lemma 2.1. In a similar fashion it follows that second derivatives are uniformly bounded as well. Finally, the Schauder estimates (5.1) may be invoked again for the third term and these will be uniform since the same argument as for the fourth term implies that $\|\log\left(1+R_k\left(\frac{\alpha}{k},h_{\psi(q)}\right)\right)\|_{C^2(\partial N)}$ is uniformly bounded. □

Note that estimate (5.2) immediately implies the C^0 convergence of φ_k to φ with remainder as in (4.1) since we have an asymptotic expansion for the Szegő kernel that to first order equals

$$\Pi_{h^k_{\varphi(y)}}(z,z)=\sum_{\alpha\in kP\cap\mathbb{Z}^m}\mathcal{P}_{h^k_{\varphi(y)}}(\alpha,z)=1+O(k^{-1}). \tag{5.4}$$

We now turn to showing C^1 and C^2 convergence. In other words, our aim is now to show that the $C^2(N\times M)$ norm of the left-hand side of (4.3) is still $O(k^{\delta-\frac{1}{3}})$. In order to prove these estimates it is crucial to make use of some cancellations. These can be understood as follows. When one replaces all the coefficients $\mathcal{R}_k(y,\alpha)$ by a constant, one reduces to the case of a zero-dimensional map, or equivalently to the known asymptotic expansion of the Szegő kernel that may be differentiated any number of times with a small error. Now there are two cases. When a coefficient $\mathcal{R}_k(y,\alpha)$ or a derivative thereof only multiplies a normalized monomial $\mathcal{P}_{h^k_{\varphi(y)}}(\alpha,z)$ it is enough to use the uniform estimates given by Lemma 5.1 and one does not need to keep track of error terms. However, as is usually the case, if the coefficient $\mathcal{R}_k(y,\alpha)$ or a derivative thereof multiplies another term that itself depends on k, one needs to keep track of the remainder of order $O\left(k^{-\frac{1}{3}}\right)$ given by Lemma 5.1. When such an error is introduced we simultaneously apply Lemma 2.2 to localize to those lattice points satisfying $|\alpha|\leq k^{\frac{1}{2}+\delta}$. Remembering the overall factor of $\frac{1}{k}$ one then estimates the remainders thus introduced.

Let us now consider derivatives solely in the M-directions. A derivative of (4.3) in the ρ_j directions amounts to multiplying each coefficient in the sum (4.3) by a factor of

$$k\left((\nabla\varphi_y)(z)-\frac{\alpha}{k}\right)_j=k\left(\mu_y(z)-\frac{\alpha}{k}\right)_j$$

(recall that the moment map μ_y is the gradient of the open orbit Kähler potential $\varphi(e^\rho)$).

Namely, in the interior of P one has

$$\frac{\partial}{\partial \rho_j}(\varphi_k - \varphi)(y,z) = \frac{1}{k}\frac{\sum_{\alpha \in kP\cap\mathbb{Z}^m} k(\mu_y(z) - \frac{\alpha}{k})\mathcal{R}_k(y,\alpha)\mathcal{P}_{h^k_{\varphi(y)}}(\alpha)}{\sum_{\alpha \in kP\cap\mathbb{Z}^m} \mathcal{R}_k(y,\alpha)\mathcal{P}_{h^k_{\varphi(y)}}(\alpha,z)}.$$

The factor of k is cancelled by the overall factor of $\frac{1}{k}$ and the coefficients $\mathcal{R}_k(y,\alpha)$ are uniformly bounded due to (5.2). Thus by Lemma 2.2 one may restrict to those α such that $|\mu_y(z) - \frac{\alpha}{k}| \leq k^{\delta - \frac{1}{2}}$ (introducing an error $O(k^{-M})$ for some large $M > 0$). It follows then that

$$\frac{\partial}{\partial \rho_j}(\varphi_k - \varphi)(y,z) = O\left(k^{\delta - \frac{1}{2}}\right). \tag{5.5}$$

Near the boundary of P one performs the same computation but with respect to the slice-orbit coordinates (the same remark applies to all the computations in this Section). Note that the argument reduced to the one in [**26**, Section 7.2], once we had (5.2).

Next, we consider second derivatives in the M-directions. Symmetrizing sums (see [**26**, Section 8]) one obtains in the interior of P,

$$\begin{aligned}\frac{\partial^2}{\partial \rho_i \partial \rho_j}(\varphi_k - \varphi)(y,z) = &-\frac{\partial^2 \varphi(y,z)}{\partial \rho_i \partial \rho_j} \\ &+ \frac{1}{k}\frac{\begin{array}{c}\frac{1}{2}\sum_{\alpha,\beta \in kP\cap\mathbb{Z}^m}(\alpha_i - \beta_i)(\alpha_j - \beta_j)\mathcal{R}_k(y,\alpha) \\ \times \mathcal{R}_k(y,\beta)\mathcal{P}_{h^k_{\varphi(y)}}(\alpha,z)\mathcal{P}_{h^k_{\varphi(y)}}(\beta,z)\end{array}}{\left(\sum_{\alpha \in kP\cap\mathbb{Z}^m}\mathcal{R}_k(y,\alpha)\mathcal{P}_{h^k_{\varphi(y)}}(\alpha,z)\right)^2}.\end{aligned}$$

Equation (5.3) allows to reduce the computations to those in the case $N = [0,1]$: After localizing (keeping only those α that are $O\left(k^{\frac{1}{2}+\delta}\right)$-close to $k\mu_y(z)$) we have the estimate $\frac{1}{k}(\alpha_i - \beta_i)(\alpha_j - \beta_j) = O(k^{2\delta})$. We then replace the coefficients $\mathcal{R}_k(y,\alpha)$ by the uniform constant (independent of α) $\mathcal{R}_\infty(y,\mu_y(z))$ at the price of an overall error $O(k^{2\delta - \frac{1}{3}})$. But now what is left is then precisely cancelled by $-\frac{\partial^2\varphi(y,z)}{\partial\rho_i\partial\rho_j}$ (up to an error of $O(k^{-2})$) due to the complete asymptotics of the Szegő kernel of a single metric. To prove this last claim, we consider the situation of a family of Szegő kernels parametrized by a compact manifold N, corresponding to the family of Hermitian metrics $h_y,\ y \in N$. In the toric situation this may be written explicitly as

$$\Pi_k(z,z) := \Pi_{h^k_y}(z,z) = \sum_{\alpha \in kP\cap\mathbb{Z}^m}\frac{|\chi_\alpha(z)|^2_{h^k_y}}{\mathcal{Q}_{h^k_y}(\alpha)} = \sum_{\alpha \in kP\cap\mathbb{Z}^m}\frac{e^{\langle\alpha,\rho\rangle - k\varphi_y}}{\mathcal{Q}_{h^k_y}(\alpha)}.$$

Set

$$\tilde{\Pi}_k(z,z) := \sum_{\alpha \in kP\cap\mathbb{Z}^m} \frac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_{h_y^k}(\alpha)}.$$

Then, $\frac{1}{k}\log \Pi_{h_y^k}(z,z) = O(k^{-2})$ has a complete asymptotic expansion, and a first space derivative gives

(5.6)

$$\frac{\partial\varphi(y,z)}{\partial\rho_j} = \frac{\partial}{\partial\rho_j}\frac{1}{k}\log\tilde{\Pi}_k(z,z) + O(k^{-2}) = \frac{1}{k}(\tilde{\Pi}_k(z,z))^{-1}\sum_{\alpha\in kP\cap\mathbb{Z}^m}\alpha_j\frac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_{h_y^k}(\alpha)}.$$

Similarly a second space derivative takes the form

$$\begin{aligned}(5.7)\qquad & \frac{\partial^2\varphi(y,z)}{\partial\rho_i\partial\rho_j} + O(k^{-2}) \\ &= \frac{1}{k}(\tilde{\Pi}_k(z,z))^{-2}\Bigg[\sum_{\alpha\in kP\cap\mathbb{Z}^m}\alpha_i\alpha_j\frac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_{h_y^k}(\alpha)}\sum_{\beta\in kP\cap\mathbb{Z}^m}\frac{e^{\langle\beta,\rho\rangle}}{\mathcal{Q}_{h_y^k}(\beta)} \\ &\quad - \sum_{\alpha\in kP\cap\mathbb{Z}^m}\alpha_i\frac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_{h_y^k}(\alpha)}\sum_{\beta\in kP\cap\mathbb{Z}^m}\beta_j\frac{e^{\langle\beta,\rho\rangle}}{\mathcal{Q}_{h_y^k}(\beta)}\Bigg] \\ &= \frac{(\tilde{\Pi}_k(z,z))^{-2}}{k}\frac{1}{2}\sum_{\alpha,\beta\in kP\cap\mathbb{Z}^m}(\alpha_i-\beta_i)(\alpha_j-\beta_j)\frac{e^{\langle\alpha+\beta,\rho\rangle}}{\mathcal{Q}_{h_y^k}(\alpha)\mathcal{Q}_{h_y^k}(\beta)}\end{aligned}$$

(by symmetrizing sums). But now multiplying numerators and denominators throughout by $e^{-k\varphi_y(z)}$ the right-hand side becomes

$$\frac{(\Pi_k(z,z))^{-2}}{k}\frac{1}{2}\sum_{\alpha,\beta\in kP\cap\mathbb{Z}^m}(\alpha_i-\beta_i)(\alpha_j-\beta_j)\mathcal{P}_{h^k_{\varphi(y)}}(\alpha,z)\mathcal{P}_{h^k_{\varphi(y)}}(\beta,z)$$

Since $\Pi_k(z,z) = \sum \mathcal{P}_{h^k_{\varphi(y)}}$, this proves our original claim, and hence the convergence of the second space derivatives:

$$\frac{\partial^2}{\partial\rho_i\partial\rho_j}(\varphi_k-\varphi)(y,z) = O(k^{2\delta-\frac{1}{3}}). \tag{5.8}$$

Therefore it remains to consider derivatives that also involve the N-directions.

We consider first one derivative in the N-directions. One has

$$\begin{aligned}(5.9)\qquad & \frac{\partial}{\partial y^a}(\varphi_k-\varphi)(y,z) \\ &= -\frac{\partial\varphi}{\partial y^a} + \frac{1}{k}\frac{\sum_{\alpha\in kP\cap\mathbb{Z}^m}\mathcal{R}_k(y,\alpha)\frac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_{h^k(y)}(\alpha)}\partial_{y^a}\log\frac{\mathcal{R}_k(y,\alpha)}{\mathcal{Q}_{h^k(y)}(\alpha)}}{\sum_{\alpha\in kP\cap\mathbb{Z}^m}\mathcal{R}_k(y,\alpha)\frac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_{h^k(y)}(\alpha)}}.\end{aligned}$$

Since the asymptotic expansion (5.4) can be differentiated and is uniform over compact families [**7**, **29**] we have

$$
\begin{aligned}
O(k^{-2}) &= \frac{1}{k}\frac{\partial}{\partial y^a}\log \Pi_{h^k_{\varphi(y)}}(z,z) \\
&= -\frac{\partial \varphi}{\partial y^a} + \frac{1}{k}\frac{\sum_{\alpha\in kP\cap\mathbb{Z}^m}\frac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_{h^k(y)}(\alpha)}\partial_{y^a}\log\frac{1}{\mathcal{Q}_{h^k(y)}(\alpha)}}{\sum_{\alpha\in kP\cap\mathbb{Z}^m}\frac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_{h^k(y)}(\alpha)}}.
\end{aligned}
\tag{5.10}
$$

First note that the term

$$
\frac{1}{k}\frac{\sum_{\alpha\in kP\cap\mathbb{Z}^m}\mathcal{R}_k(y,\alpha)\frac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_{h^k(y)}(\alpha)}\partial_{y^a}\log\mathcal{R}_k(y,\alpha)}{\sum_{\alpha\in kP\cap\mathbb{Z}^m}\frac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_{h^k(y)}(\alpha)}}
$$

is of order $O(k^{-1})$ by Lemma 5.1. Thus, we are left with the task of comparing the last term of (5.10) with

$$
\frac{1}{k}\frac{\sum_{\alpha\in kP\cap\mathbb{Z}^m}\mathcal{R}_k(y,\alpha)\frac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_{h^k(y)}(\alpha)}\partial_{y^a}\log\frac{1}{\mathcal{Q}_{h^k(y)}(\alpha)}}{\sum_{\alpha\in kP\cap\mathbb{Z}^m}\mathcal{R}_k(y,\alpha)\frac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_{h^k(y)}(\alpha)}}.
$$

We now localize the sums about the image of the moment map using Lemma 2.2 introducing negligible errors (of arbitrarily high order $O(k^{-M})$). Then we use Lemma 5.1 to replace each occurrence of $\mathcal{R}_k(y,\alpha)$ by $\mathcal{R}_\infty(y,\mu_y(z))$ plus an error of order $O(k^{-1/3})$ (the error comes both from (5.3) and the fact that we Taylor expand $\mathcal{R}_\infty(y,\mu_y(z))$ about $\mathcal{R}_\infty(y,\alpha/k)$ and use the fact that since we localized the sum we have $|\mu_y(z)-\alpha/k| = O(k^{\delta-1/2})$). In the terms involving $\mathcal{R}_\infty(y,\mu_y(z))$ the factors of $\mathcal{R}_\infty(y,\mu_y(z))$ actually cancel and so cancel with the last term of (5.10) after localizing the latter.

It remains to show that the error term overall contributes $O(k^{-1/3})$ to the sum (5.9) (here we assume that $\delta\in(0,\frac{1}{6})$). To that end, because of the factor of $1/k$ it is enough to show that there exists a uniform constant $C>0$ independent of k such that

$$
|\partial_{y^a}\log\mathcal{Q}_{h^k(y)}(\alpha)| \le Ck. \tag{5.11}
$$

Recall that the duality (4.4) of Lemma 4.1 implies

$$
\partial_{y^a}\log\mathcal{Q}_{h^k_{\varphi(y)}}(\alpha) = k\partial_{y^a}u_y(\alpha/k) - \partial_{y^a}\log\mathcal{P}_{h^k_{\varphi(y)}}(\alpha). \tag{5.12}
$$

The second term of the right-hand side is uniformly bounded by applying (2.4). To evaluate the first term recall that

$$u_y = -\int_{\partial N} u_q \partial_{\nu(q)} G(y,q) dV_{\partial N,f}(q), \quad y \in N. \tag{5.13}$$

In terms of canonical symplectic potential u_0 of (3.4) one may write $u_y = u_0 + f_y$ for some globally smooth function $f_y \in \mathcal{LH}(T)$ on P and thus we have

$$\partial_{y^a} u_y = \partial_{y^a} f_y = -\partial_{y^a} \int_{\partial N} f_q \partial_{\nu(q)} G(y,q) dV_{\partial N,f}(q), \quad y \in N. \tag{5.14}$$

This is uniformly bounded according to the Schauder estimates. Combining the above estimate (5.11) follows.

In sum we have shown that (assuming $\delta \in (0, \frac{1}{6})$)

$$\frac{\partial}{\partial y^a}(\varphi_k - \varphi)(y,z) = O(k^{-1/3}), \tag{5.15}$$

which concludes the case of a single N-derivative.

We now consider the case of mixed second derivatives. We will always assume $\alpha, \beta \in kP \cap \mathbb{Z}^n$ and so omit that from the summation notation in what follows. To simplify the notation further we will fix a point $(y,z) \in N \times M$ and use the following abbreviations:

$$\partial_a := \partial_{y^a} = \frac{\partial}{\partial y^a},\ \partial_{ab} := \partial_{y^a}\partial_{y^b} = \frac{\partial^2}{\partial y^a \partial y^b},$$

$$\mathcal{R}_\alpha := \mathcal{R}_k(y,\alpha),\ \mathcal{Q}_\alpha := \mathcal{Q}_{h^k_{\varphi(y)}}(\alpha),\ \mathcal{P}_\alpha := \mathcal{P}_{h^k_{\varphi(y)}}(\alpha),\ \widetilde{\mathcal{P}}_\alpha := \frac{e^{\langle \alpha, \rho\rangle}}{\mathcal{Q}_{h^k(y)}(\alpha)}.$$

Symmetrizing sums again, it follows that

$$\begin{aligned} &\frac{\partial^2}{\partial y^a \partial \rho^j}(\varphi_k - \varphi)(y,z) \\ &= -\frac{\partial^2 \varphi(y,z)}{\partial y^a \partial \rho^j} + \frac{1}{k} \frac{\frac{1}{2}\sum_{\alpha,\beta}(\alpha_j - \beta_j)\frac{\partial}{\partial y^a} \log\left(\frac{\mathcal{R}_\alpha}{\mathcal{Q}_\alpha}\frac{\mathcal{Q}_\beta}{\mathcal{R}_\beta}\right) \mathcal{R}_\alpha \mathcal{R}_\beta \widetilde{\mathcal{P}}_\alpha \widetilde{\mathcal{P}}_\beta}{\left(\sum_\alpha \mathcal{R}_\alpha \widetilde{\mathcal{P}}_\alpha\right)^2}. \end{aligned} \tag{5.16}$$

Localizing sums exchanges the term $(\alpha_j - \beta_j)$ for $O(k^{\frac{1}{2}+\delta})$ up to an error of $O(k^{-M})$ for some large $M > 0$. By applying Lemma 5.1 the term

$$\frac{1}{k} \frac{\frac{1}{2}\sum_{\alpha,\beta}(\alpha_j - \beta_j)\frac{\partial}{\partial y^a} \log\left(\frac{\mathcal{R}_\alpha}{\mathcal{R}_\beta}\right) \mathcal{R}_\alpha \mathcal{R}_\beta \widetilde{\mathcal{P}}_\alpha \widetilde{\mathcal{P}}_\beta}{\left(\sum_\alpha \mathcal{R}_\alpha \widetilde{\mathcal{P}}_\alpha\right)^2}$$

is $O(k^{\delta-1/2})$. As before, we now replace the coefficients $\mathcal{R}_k$ in (5.16) for the lattice points α that remain (near $\mu_y(z)$) by the uniform constant $\mathcal{R}_\infty(y,\mu_y(z))$ plus an error of order $O(k^{-1/3})$. Any term that does not multiply $\log\frac{\mathcal{Q}_\beta}{\mathcal{Q}_\alpha}$ is of order $O(k^{\delta-1/2})$, or smaller, by using Lemma 5.1. Now there are two kinds of terms left to estimate. The first involve $\partial_a \log\frac{\mathcal{Q}_\beta}{\mathcal{Q}_\alpha}$ with all $\mathcal{R}_k$ replaced by $\mathcal{R}_\infty(y,\mu_y(z))$. Then the coefficients $\mathcal{R}_\infty(y,\mu_y(z))$ cancel out and we are left with the Szegő kernel approximation of $\frac{\partial^2\varphi(y,z)}{\partial y^a\partial\rho^j}$ (up to $O(k^{-2})$) and this cancels with $-\frac{\partial^2\varphi(y,z)}{\partial y^a\partial\rho^j}$ appearing in (5.16). The second type of contributions comes from error terms of order $O(k^{-1/3})$ multiplying $\partial_a\log\frac{\mathcal{Q}_\beta}{\mathcal{Q}_\alpha}$. By using (5.12) we may express $\log\frac{\mathcal{Q}_\beta}{\mathcal{Q}_\alpha}$ in terms of the global symplectic potentials and using in addition the fact that α and β are localized to a neighborhood of size comparable to $k^{1/2+\delta}$ about $k\mu_y(z)$ we obtain

$$\begin{aligned}(5.17)\qquad \left|\partial_a\log\left(\frac{\mathcal{Q}_\beta}{\mathcal{Q}_\alpha}\right)\right| &\leq C + k|\partial_a f_y(\beta/k)-\partial_a f_y(\alpha/k)|\\ &\leq C + kC_1|(\alpha-\beta)/k| = O(k^{\frac{1}{2}+\delta}),\end{aligned}$$

where C_1 is the Lipschitz constant of the smooth function $\partial_a f_y$ (as a function of N). By the maximum principle C_1 is uniformly bounded in terms of the boundary data (i.e., $\mathrm{FS}_k\circ\mathrm{Hilb}_k(\psi)$) since $\partial_a f_y$ is in fact harmonic in the N-variables. Taking into account the terms $\alpha_j-\beta_j = O(k^{\delta-1/2})$ and the overall factor of $1/k$ this then implies that the second type of contributions are of order $O(k^{2\delta-1/3})$.

Note that the symmetrization of the sums was crucial here. In sum,

$$(5.18)\qquad \frac{\partial^2}{\partial y^a\partial\rho^j}(\varphi_k-\varphi)(y,z) = O(k^{2\delta-\frac{1}{3}}).$$

Finally, we consider the case of two derivatives in the N-directions. This case is somewhat more involved than the previous ones and unlike in the case $N=[0,1]$ we also need to consider mixed N-derivatives. We have

$$\begin{aligned}(5.19)\qquad \partial_{ab}(\varphi_k-\varphi)(y,z) &= -\partial_{ab}\varphi\\ &+\frac{1}{k}\frac{\sum_{\alpha,\beta}\mathcal{R}_\alpha\widetilde{\mathcal{P}}_\alpha\mathcal{R}_\beta\widetilde{\mathcal{P}}_\beta\cdot\left[\partial_{ab}\log\left(\dfrac{\mathcal{R}_\alpha}{\mathcal{Q}_\alpha}\right) +\frac{1}{2}\partial_a\log\left(\frac{\mathcal{R}_\alpha}{\mathcal{Q}_\alpha}\frac{\mathcal{Q}_\beta}{\mathcal{R}_\beta}\right)\partial_b\log\left(\frac{\mathcal{R}_\alpha}{\mathcal{Q}_\alpha}\frac{\mathcal{Q}_\beta}{\mathcal{R}_\beta}\right)\right]}{\left(\sum_\alpha\mathcal{R}_\alpha\widetilde{\mathcal{P}}_\alpha\right)^{-2}}.\end{aligned}$$

We rewrite this as

$$\partial_{ab}(\varphi_k-\varphi)(y,z) = -\partial_{ab}\varphi + A + B + C,$$

where

(5.20)

$$A = \frac{1}{k}\frac{\sum_{\alpha,\beta}\mathcal{R}_\alpha\widetilde{\mathcal{P}}_\alpha\mathcal{R}_\beta\widetilde{\mathcal{P}}_\beta\cdot\left[\partial_{ab}\log\left(\frac{1}{\mathcal{Q}_\alpha}\right)+\frac{1}{2}\partial_a\log\left(\frac{\mathcal{Q}_\beta}{\mathcal{Q}_\alpha}\right)\partial_b\log\left(\frac{\mathcal{Q}_\beta}{\mathcal{Q}_\alpha}\right)\right]}{\left(\sum_\alpha\mathcal{R}_\alpha\widetilde{\mathcal{P}}_\alpha\right)^{-2}},$$

(5.21)

$$B = \frac{1}{k}\frac{\sum_{\alpha,\beta}\mathcal{R}_\alpha\widetilde{\mathcal{P}}_\alpha\mathcal{R}_\beta\widetilde{\mathcal{P}}_\beta\cdot\left[\partial_{ab}\log\left(\mathcal{R}_\alpha\right)+\frac{1}{2}\partial_a\log\left(\frac{\mathcal{R}_\alpha}{\mathcal{R}_\beta}\right)\partial_b\log\left(\frac{\mathcal{R}_\alpha}{\mathcal{R}_\beta}\right)\right]}{\left(\sum_\alpha\mathcal{R}_\alpha\widetilde{\mathcal{P}}_\alpha\right)^{-2}},$$

(5.22)

$$C = \frac{1}{k}\frac{\sum_{\alpha,\beta}\mathcal{R}_\alpha\widetilde{\mathcal{P}}_\alpha\mathcal{R}_\beta\widetilde{\mathcal{P}}_\beta\cdot\left[\partial_a\log\frac{\mathcal{R}_\alpha}{\mathcal{R}_\beta}\partial_b\log\frac{\mathcal{Q}_\beta}{\mathcal{Q}_\alpha}\right]}{\left(\sum_\alpha\mathcal{R}_\alpha\widetilde{\mathcal{P}}_\alpha\right)^{-2}}.$$

By differentiating (5.4) we obtain similarly (analogously to the computations leading to (5.7)),

(5.23)

$$O(k^{-2}) = -\partial_{ab}\varphi + \frac{1}{k}\frac{\sum_{\alpha,\beta}\widetilde{\mathcal{P}}_\alpha\widetilde{\mathcal{P}}_\beta\cdot\left[\partial_{ab}\log\left(\frac{1}{\mathcal{Q}_\alpha}\right)+\frac{1}{2}\partial_a\log\left(\frac{\mathcal{Q}_\beta}{\mathcal{Q}_\alpha}\right)\partial_b\log\left(\frac{\mathcal{Q}_\beta}{\mathcal{Q}_\alpha}\right)\right]}{\left(\sum_\alpha\widetilde{\mathcal{P}}_\alpha\right)^{-2}}.$$

We now localize the sums to a ball of radius $k^{\frac{1}{2}+\delta}$ about $k\mu_y(z)$ introducing a negligible remainder/error and replace the two occurrences of $\mathcal{R}_\gamma$ outside of the square brackets in (5.19) as well as in the denominator by the lattice-point-independent constant $\mathcal{R}_\infty(y,\mu_y(z))$ (in what follows, we refer to this operation as "replacement") introducing an error of order $O(k^{-\frac{1}{3}})$ for each replacement. First, observe that the replacement in the denominator is negligible. What remains to be checked is that the overall error introduced by replacements elsewhere is of order $O(k^{2\delta-\frac{1}{3}})$.

In the term A these replacements introduce a term that is cancelled by substituting the expression for $-\partial_{ab}\varphi$ given by (5.23) into (5.19). In addition we introduce error terms. Let $\epsilon_\gamma := \mathcal{R}_\gamma - \mathcal{R}_\infty(\mu_y(z)) = O(k^{-\frac{1}{3}})$ for each lattice point γ in a localized sum. The highest order remainders are terms of the form $\frac{1}{k}\epsilon_\gamma \cdot (B_1 + B_2 + B_3)$ where

$$B_1 := B_{1,1} + B_{1,2} := \partial_{ab} \log\left(\frac{1}{\mathcal{Q}_\alpha}\right) + \frac{1}{2}\partial_a \log\left(\frac{\mathcal{Q}_\beta}{\mathcal{Q}_\alpha}\right)\partial_b \log\left(\frac{\mathcal{Q}_\beta}{\mathcal{Q}_\alpha}\right),$$
$$B_2 := \partial_{ab} \log \mathcal{R}_\alpha + \frac{1}{2}\partial_a \log\left(\frac{\mathcal{R}_\alpha}{\mathcal{R}_\beta}\right)\partial_b \log\left(\frac{\mathcal{R}_\alpha}{\mathcal{R}_\beta}\right),$$
$$B_3 := \partial_a \log\left(\frac{\mathcal{R}_\alpha}{\mathcal{R}_\beta}\right)\partial_b \log\left(\frac{\mathcal{Q}_\beta}{\mathcal{Q}_\alpha}\right).$$

The errors introduced in the replacements in the term A are of the form $\frac{1}{k}\epsilon_\gamma B_1$. The errors introduced in the replacements of the terms B and C are $\frac{1}{k}\epsilon_\gamma B_2$ and $\frac{1}{k}\epsilon_\gamma B_3$, respectively.

First, $\frac{1}{k}\epsilon_\gamma B_2 = O(k^{-\frac{4}{3}})$ by Lemma 5.1. To bound B_1 and B_3 we will use estimate (5.17) for the derivatives of the norming constants $\mathcal{Q}_\alpha$. First, it gives directly that $\frac{1}{k}\epsilon_\gamma B_3 = O(k^{\delta-5/6})$. Second, by squaring (5.17) we also obtain $\frac{1}{k}\epsilon_\gamma B_{1,2} = O(k^{2\delta-\frac{1}{3}})$. Finally, the maximum principle also gives, similarly to the argument proving (5.11), that

$$\left|\partial_{ab} \log \mathcal{Q}_\alpha\right| \leq C + k|\partial_{ab} f_y(\alpha/k)| \leq C_2 k,$$

and thus $\frac{1}{k}\epsilon_\gamma B_{1,1} = O(k^{-\frac{1}{3}})$. Altogether we have shown that all the remainders introduced by the replacements are of order $O(k^{2\delta-\frac{1}{3}})$ for some $\delta \in (0, 1/6)$.

Hence we have shown that

$$\partial_{ab}(\varphi_k - \varphi)(y, z) = O(k^{2\delta-\frac{1}{3}}),$$

and this, together with estimates (5.5), (5.8), (5.15), and (5.18), concludes the proof of Theorem 1.1.

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