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CORRIGENDUM: GROMOV–WITTEN INVARIANTS OF SYMPLECTIC QUOTIENTS AND ADIABATIC LIMITS

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We correct the definition of the local equivariant action in [**1**].

Let (M,ω) be a symplectic manifold equipped with a Hamiltonian action by a compact Lie group G. Identify the Lie algebra $\mathfrak{g} = \mathrm{Lie}(\mathrm{G})$ with its dual by an invariant inner product and let $\mu : M \to \mathfrak{g}$ be a moment map for the action. We assume that μ is proper and G acts freely on $\mu^{-1}(0)$. Identify $S^1 \cong \mathbb{R}/2\pi\mathbb{Z}$ and let $(x,\eta) : S^1 \to M \times \mathfrak{g}$ be a smooth loop. The *equivariant length* of the loop (x,η) is defined by

$$\ell(x,\eta) := \int_0^{2\pi} |\dot{x} + X_\eta(x)| \, d\theta,$$

where $\mathfrak{g} \to \mathrm{Vect}(M) : \xi \mapsto X_\xi$ denotes the infinitesimal action. Fix a neighborhood U of $\mu^{-1}(0)$ with compact closure. In [**1**, Lemma 11.2] it is proved that, if U is sufficiently small, then there is a constant $c > 0$ such that, for every loop $(x,\eta) : S^1 \to U \times \mathfrak{g}$ there is a loop $g_0 : S^1 \to \mathrm{G}$ and an element $x_0 \in \mu^{-1}(0)$ satisfying $g_0(0) = \mathbb{1}$ and

$$(1)\quad \sup_{S^1} \left|\eta + \dot{g}_0 g_0^{-1}\right| \le c\ell(x,\eta), \quad d(x(\theta), g_0(\theta)x_0) \le c\left(|\mu(x(\theta))| + \ell(x,\eta)\right).$$

We shall define the *local equivariant symplectic action* $\mathcal{A}(x,\eta)$ under the assumption that

$$(2)\qquad \sup_{\theta\in S^1} |\mu(x(\theta))| + \ell(x,\eta) < \delta,$$

where δ is sufficiently small.

The mistake in [**1**] is that δ is chosen such that $\mu^{-1}((-\delta,\delta)) \subset U$ and $2c\delta$ is smaller than the injectivity radius of M. Apart from the fact that M might be noncompact and its injectivity radius could be zero, a counterexample (due to Fabian Ziltener, with injectivity radius equal to infinity) shows that this choice of δ does not suffice to obtain uniqueness of the pair (x_0, g_0) up to homotopy. Instead we must choose δ as follows.

Choose U to have compact closure and fix c as in (1). First, we can choose $\delta > 0$ so small that, if (x, θ) is a loop satisfying (2) then any two pairs (x_0, g_0) and (x_1, g_1) satisfying $g_0(0) = g_1(0) = \mathbb{1}$ and (1) can be connected by a homotopy (x_λ, g_λ) satisfying the same inequality with c replaced by a suitable larger constant C. (More precisely, let $\lambda \mapsto x_\lambda$ be the geodesic in $\mu^{-1}(0)$ from x_0 to x_1, estimate the distance of g_0 and g_1 by $c'\ell(x, \eta)$, choose $\zeta : S^1 \to \mathfrak{g}$ such that $\zeta(0) = 0$ and $g_1(\theta) = g_0(\theta)\exp(\zeta(\theta))$, and define $g_\lambda(\theta) := g_0(\theta)\exp(\lambda\zeta(\theta))$.) Second, we can choose δ so small that, for all $x \in M$, we have

$$|\mu(x)| < \delta \implies B_{C\delta}(x) \subset U.$$

Third, we can choose δ so small that $C\delta$ is smaller than the injectivity radius of M at all elements of U. Then, for every pair (x, η) satisfying (2) and every pair (x_0, g_0) satisfying (1) we have $g_0(\theta)x_0 \in U$ and $x(\theta) \in B_{C\delta}(g_0(\theta)x_0)$ for all θ. Hence there is a unique loop $\xi_0(\theta) \in T_{g_0(\theta)x_0}M$ such that

$$x(\theta) = \exp_{g_0(\theta)x_0}(\xi_0(\theta)), \quad |\xi_0(\theta)| < C\delta$$

for all θ. We define $u_0 : [0, 1] \times S^1 \to M$ by

$$u_0(\tau, \theta) := \exp_{g_0(\theta)x_0}(\tau\xi_0(\theta)).$$

If (x_1, g_1) is another pair satisfying (1) then there is a homotopy (x_λ, g_λ) from (x_0, g_0) to (x_1, g_1) satisfying (1) with c replaced by C. Hence the resulting maps $u_\lambda : [0, 1] \times S^1 \to M$ form a homotopy satisfying $u_\lambda(1, \theta) = x(\theta)$ and $\mu(u_\lambda(0, 0)) = 0$, $u_\lambda(0, \theta) \in \mathrm{G}u_\lambda(0, 0)$. This shows that the *local equivariant symplectic action* defined by

$$\mathcal{A}(x, \eta) := -\int u_0^*\omega + \int_0^{2\pi} \langle \mu(x(\theta)), \eta(\theta) \rangle \, d\theta$$

is independent of the choice of the pair (x_0, g_0), used to define it, provided (x, η) satisfies (2). With this choice of δ and this definition of the local equivariant symplectic action Lemma 11.3 and the proof of Proposition 11.1 in [**1**] remain valid as they stand.

Recently Fabian Ziltener extended the definition of the local equivariant symplectic action to general loops of small equivariant length, dropping the hypothesis that they are close to the zero set of the moment map [**2**]. Moreover, he proved an isoperimetric inequality with a sharp constant (compare with [**1**, Lemma 11.3]).

References

[1] A. R. Gaio and D. A. Salamon, *Gromov–Witten invariants of symplectic quotients and adiabatic limits*, J. Symplectic Geom. **1** (2005), 55–159.

[2] F. Ziltener, *The invariant symplectic action and decay for vortices*, J. Symplectic Geom. **7** (2009).

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