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TOPOLOGICALLY TRIVIAL LEGENDRIAN KNOTS

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The first part of this paper contains a thorough exposition of the proof of the classification of topologically trivial Legendrian knots (i.e., Legendrian knots bounding embedded 2-disks) in tight contact 3-manifolds. These techniques were never published in detail when the classification result was announced over ten years ago. The final part of the present paper contains a systematic discussion of Legendrian knots in overtwisted contact manifolds, and in particular, gives the coarse classification (i.e., classification up to a global contactomorphism) of topologically trivial exceptional Legendrian knots in overtwisted contact S^3 according to the values of the invariants tb, r. We show, moreover, that such knots only occur for one of the infinitely many overtwisted contact structures on S^3. We remark that our tight classification result also implies that any topologically trivial loose Legendrian knots with same value of (tb, r) in an overtwisted contact 3-manifold are in fact Legendrian isotopic if $tb < 0$.

This paper deals with topologically trivial Legendrian knots in tight and overtwisted contact 3-manifolds. The first parts (Sections 1–3) contain a thorough exposition of the proof of the classification of topologically trivial Legendrian knots (i.e., Legendrian knots bounding embedded 2-disks) in tight contact 3-manifolds (Theorem 1.5), and, in particular, in the standard contact S^3. These parts were essentially written more than ten years ago, but only a short version [**27**], without the detailed proofs, was published. In that paper we also briefly discussed Legendrian knots in overtwisted contact 3-manifolds. The final part of the present paper (Section 4) contains a more systematic discussion of the overtwisted case. In [**27**], Legendrian knots in overtwisted manifolds were divided into two classes: *exceptional*, i.e., those with tight complement, and the complementary class of *loose* ones. Loose knots can be *coarsely*, i.e., up to a global coorientation-preserving contact diffeomorphism, classified (see Section 4 and also [**15**]) using the classification of overtwisted contact structures from [**22**]. On the other hand, Giroux-Honda's classification of tight contact structures on solid tori (see [**41**, **48**, **49**]) allows us to completely coarsely classify topologically trivial exceptional knots in S^3. In particular, we show the latter exist for only one overtwisted contact structure on S^3.

Since the paper [**27**], several new techniques (notably the Giroux–Honda method of convex surfaces, dividing curves and Legendrian bypasses) were developed which provide some shortcuts to the results of that paper, at least for knots, in S^3. However, we think that our explicit geometric methods may still have more than just a historic interest. Theorem 1.5 asserts that topologically trivial Legendrian knots, in a tight contact manifold, which are isotopic as framed knots are Legendrian isotopic. A special case of this result was proved in [**18**] (for topologically trivial Legendrian knots with maximal possible value of the Thurston-Bennequin invariant, i.e., $(tb, r) = (-1, 0)$; see Section 1.5 below). The result itself was then stated in [**19**], where an analogous theorem for transversal knots was proved. The proof in the present paper follows the general scheme of the proof first given in [**35**]. The same classification result does not hold for non-trivial knot type, in other words the so-called *regular isotopy* of topologically non-trivial Legendrian knots does not imply their Legendrian isotopy. The first example of this kind was established by Chekanov in [**9**] using a differential graded algebra of a Legendrian knot, a new invariant inspired by the theory of holomorphic curves [**45**], which Chekanov defined combinatorially.

Since the late 90's, many new invariants of Legendrian isotopy have been defined, starting with the Legendrian contact homology algebra (see [**9**, **10**] and [**26**, **29**]) and several variations of it (see [**17**, **34**, **55**, **56**, **61**]. Following some ideas from [**24**], a different type "decomposition invariant" was defined by Chekanov and Pushkar in [**11**] for their proof of Arnold's four-cusp conjecture [**4**]. More recently Ozsváth, Szabó and Thurston defined a new invariant for Legendrian knots in S^3 using a combinatorial version of knot Floer homology [**57**]. Besides the result on the unknot, Legendrian knots are now classified in some other topological isotopy classes of knots in tight contact 3-manifolds. Namely, Etnyre and Honda in [**31**] classified Legendrian torus knots and figure-eight knots. Ding and Geiges in [**12**] extended Etnyre–Honda results for other special classes of Legendrian knots and links.

The problem of classification of Legendrian (and Lagrangian) knots was first explicitly formulated by Arnold in [**1**]. However, problems related to Legendrian isotopy were studied earlier (see, for instance, [**2**, **3**, **5**, **6**, **21**, **44**], and also papers [**8**, **24**, **25**], which were written much earlier than they appeared). Legendrian knots nowadays is a very active subject, and we will not even attempt here to list all relevant results. We refer the reader to

Etnyre's paper [**30**] for a survey of the status of Legendrian isotopy problem back in 2003. The latest developments of the theory include, as we already mentioned above, applications of Heegaard Floer homology theory, as well as new SFT-inspired Legendrian isotopy invariants.

Contents

1. Preliminaries

Today there exist several good references for the basic facts and notions about contact 3-manifolds (see, for instance, the book of Geiges [**38**]). However, to fix the notation and terminology, we review the necessary introductory information in this section.

1.1. Contact structures. Let us recall that a *contact structure* on a connected 3-manifold M is a completely non-integrable tangent plane field ξ. Such a pair (M, ξ) is called a *contact manifold*. The contact structure ξ is said to be *coorientable* when the bundle ξ is coorientable in TM. In this case, ξ can be defined (as $\xi = \ker(\alpha)$) by a global 1-form α which is called a *contact form*. In the present paper, *we will henceforth assume all contact* *structures considered to be coorientable, unless otherwise specified.* The main results of the paper can be easily reformulated for the non-cooriented case.

Note that the orientation of M defined by the volume form $\alpha \wedge d\alpha$ depends only on the contact structure $\xi = \{\alpha = 0\}$, and not on the choice of the contact form α. When M's orientation has been fixed a priori, and happens to agree with that determined by ξ, ξ is said to be *positive*; when it disagrees, ξ is said to be *negative*. In the present paper, *we will consider only positive contact structures*; i.e., we will always endow M with the orientation determined by ξ.

Let us denote by $\mathrm{Diff}_0(M, \xi)$ the connected component of the identity of the group of *contactomorphisms* (i.e., group of ξ-preserving diffeomorphisms) of (M, ξ). This group acts transitively on points of any connected contact manifold (Darboux' theorem). Moreover, coordinates can always be given locally so that $\xi = \{dz = ydx\}$. The space $\mathbb{R}^3$ endowed with the contact structure $\xi = \{dz = ydx\}$ is called the *standard contact* 3*-space*. The contact form $dz + \rho^2 d\varpi$ in cylindrical coordinates defines on $\mathbb{R}^3$ a contact structure equivalent to the standard contact structure ξ. On a closed manifold, there are no deformations of contact structures (Gray's theorem [**43**]): *two contact structures on a closed contact manifold which are homotopic in the class of contact structures are isotopic.*

It has proven useful (see [**19**, **20**, **39**]) to divide contact structures on 3-manifolds into two complementary classes: *tight* and *overtwisted*. A contact structure ξ is called *overtwisted* if there exists an embedded 2-disk $D \subset M$ such that the boundary ∂D is tangent to ξ while the disk D is transversal to ξ along ∂D. Equivalently, one can request that D be everywhere tangent to ξ along ∂D. As it is shown in [**20**], one can always find a disk in an overtwisted contact manifold such that the characteristic foliation on it (see Section 1.2) is given by Figure 1.

Non-overtwisted contact structures are called *tight*. Most of this paper deals only with tight contact structures; however, in the final portion (Section 4) we discuss the situation for Legendrian knots in overtwisted contact manifolds.

1.2. Legendrian curves and characteristic foliation. A curve L in a contact 3-manifold (M, ξ) is called *Legendrian* if it is tangent to ξ. An oriented, closed, smooth Legendrian curve will be called a *Legendrian knot*. In this paper, we will study *topologically trivial* Legendrian knots, by which we mean those Legendrian knots which bound embedded disks in M. We will sometimes use the expression *Legendrian segment* to refer to a Legendrian embedding of a closed segment of $\mathbf{R}$.

According to a theorem of Rashevskii-Chow (see [**7**, **58**]), *any embedded curve* Γ *can be made Legendrian by a* C^0*-small isotopy.* This statement also holds for families of curves, but not in a relative form: two isotopic Legendrian curves are not, necessarily, Legendrian isotopic. According to the

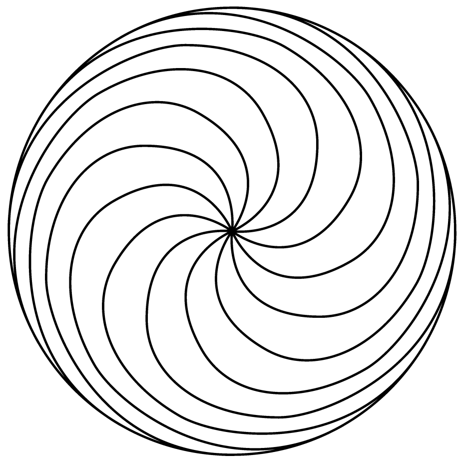


Figure 1. An overtwisted disk.

Darboux–Weinstein theorem, *diffeomorphic Legendrian submanifolds have contactomorphic neighborhoods.*

Let us mention the two standard ways of visualizing Legendrian curves in the standard contact $\mathbb{R}^3$. Let $L \subset \mathbb{R}^3$ be a Legendrian curve. Its orthogonal projection to the (x, y)-coordinate plane is called the *Lagrangian projection.*[1] We will denote the projection by p_{Lag} and denote the image $p_{\mathrm{Lag}}(L)$ by L_{Lag}. The projection to the (x, z)-coordinate plane is called the *front projection.* We denote this projection by p_{Front} and call the image $L_{\mathrm{Front}} = p_{\mathrm{Front}}(L)$ the *wavefront* of L.

Let us observe that L_{Lag} is an immersed curve, because the contact planes are transversal to the z-direction. The Legendrian curve L can be reconstructed from L_{Lag} by adding the coordinate $z = \int_{L_{\mathrm{Lag}}} y dx$. In particular, for a closed L, we should have $\oint_{L_{\mathrm{Lag}}} y\, dx = 0$, i.e., the algebraic area bounded by the immersed curve L_{Lag} should be equal to 0. Notice also that the Legendrian curve L is embedded iff each self-intersection point of L_{Lag} divides this curve into two parts of non-zero area.

On the other hand, the wavefront L_{Front} may have singularities. If the wavefront is smooth then it has to be a graph of a function $z = \alpha(x)$ defined on an interval I of the x-axis. In the general case, the front can be viewed as the graph of a multivalued function. Generically, different branches join pairwise in cusp points, so that the wavefront near a singular point is diffeomorphic to the semi-cubic parabola. In this generic case, therefore, the Legendrian curve L can be reconstructed from its wavefront by adding the

[1]Alternatively, if instead of standard coordinates we are using cylindrical coordinates (ρ, φ, z) on $\mathbf{R}^3$, with contact structure defined by the form $dz + \rho^2 d\varphi$, then the orthogonal projection to the $z = 0$ plane would likewise be called Lagrangian projection.

y-coordinate equal to the slope of this multivalued function. Notice that the Legendrian curve L is embedded iff each self-intersection point of L_{Front} is transversal.

An important way in which Legendrian curves arise is as follows. Let $F \subset M$ be a 2-surface. If M is transverse to ξ, then ξ intersects $T(F)$ along a line field $K \subset T(F)$ which integrates to a 1-dimensional foliation called the *characteristic foliation* of F and denoted F_ξ. The leaves of F_ξ are, of course, Legendrian curves.

Note that the foliation F_ξ may have singularities. A generic surface F will be transversal to ξ except possibly at a discrete set of points where it is tangent. The characteristic foliation that results in this generic case is a singular foliation with singularities occurring exactly at these isolated points of tangency (see Section 1.3).

The characteristic foliation F_ξ determines a germ of contact structure ξ along F (see [**40**]): a diffeomorphism between characteristic foliations on surfaces F and $\tilde{F}$ extends as a contactomorphisms between the neighborhoods of these surfaces.[2]

On the other hand, if two characteristic foliations F_ξ and $\tilde{F}_\xi$ are *homeomorphic* then there exists (see [**22**]) a C^0-perturbation $\hat{F}$ of the surface $\tilde{F}$ such that the characteristic foliations F_ξ and $\hat{F}_\xi$ are *diffeomorphic.*

1.3. Singularities of the characteristic foliation.

1.3.1. Elliptic and hyperbolic singularities. The index of the line field $K = \xi \cap T(F)$ at isolated singularities is well defined. Generically, it is equal to $+1$ or -1 and the singular point is called respectively *elliptic* or *hyperbolic* in these cases (see Figures 2(a) and (b) for elliptic, and Figure 2(c) for hyperbolic).

The two elliptic points shown in Figures 2(a) and (b) are topologically indistinguishable. Moreover, a surface can always be C^1-perturbed near an elliptic point to achieve characteristic foliation of the (b)-type. Near an elliptic point of (b)-type, one can choose Darboux cylindrical coordinates (ρ, φ, z) with the origin at the singular point, so that the contact structure is defined by the contact form $dz + \rho^2\, d\varphi$ and the surface F is given by the equation $z = 0$. We will always assume in this paper that elliptic singularities have this form.

We also assume that characteristic foliations we consider do not have separatrix connections between hyperbolic points. This generic assumption can be achieved by a C^∞-small perturbation of the surface.

1.3.2. The sign of a singular point. Suppose the surface F is oriented. Recall that ξ is assumed to be cooriented and M oriented. So F is cooriented in M. Also, F_ξ is cooriented in F and so has a canonical orientation

[2]One should be more accurate in what the diffeomorphism means near singular points of the characteristic foliation, see [**40**] for the details.

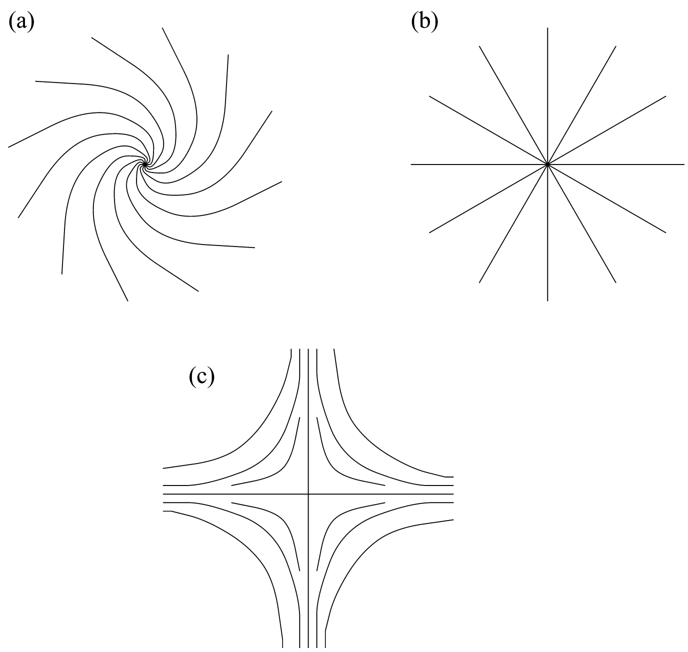


Figure 2. Elliptic singularities (top) and a hyperbolic singularity (bottom).

determined by that of F. The singularities of F_ξ will be called *positive* or *negative* when the coorientations of F and ξ, respectively, agree or disagree at these points. Since we are moreover assuming ξ to be positive (i.e., $\alpha \wedge d\alpha > 0$), it follows that the signs of elliptic singularities and the orientation of F_ξ are related:

Lemma 1.1. *Positive elliptic points are sources and negative elliptic points sinks of the characteristic foliation F_ξ of an oriented surface F in (M, ξ) (based on the assumption that ξ is positive).*

Notice that the sign of a hyperbolic point cannot be seen from the C^0-topology of the oriented characteristic foliation, because it is a C^1-, and not a C^0-invariant (see [**40**]).

1.4. Legendrian isotopy.

1.4.1. Legendrian isotopy vs. ambient contact isotopy. The following result is easily shown using the relative version of Darboux–Gray's theorem.

Lemma 1.2. *Let $L_t, t \in [0, 1]$ be a Legendrian isotopy. Then there exists an ambient contact isotopy $f_t, t \in [0, 1]$, in the space of contactomorphisms of (M, ξ) to itself such that $f_0 = \mathrm{id}$ and $f_t(L_0) = L_t$ for $t \in [0, 1]$.*

The classification of Legendrian knots up to Legendrian isotopy is thus equivalent to their classification up to ambient contact isotopy; i.e., $L \sim$

$L' \Leftrightarrow L' = \phi(L)$ for some global contactomorphism ϕ which is contactly isotopic to the identity.

1.4.2. Legendrian trees. A *Legendrian graph* (in particular, a *tree*) in a contact 3-manifold (M, ξ) is a graph (tree) L embedded in M in such a way that all its edges are Legendrian segments, non-tangent to each other at the vertices. We call two graphs *diffeomorphic* if there exists a diffeomorphism between their neighborhoods which moves one of the trees into the other. Thus, diffeomorphic graphs not only have isomorphic structure as abstract trees but also have the same infinitesimal structure at the corresponding vertices. For instance, for a pair of corresponding vertices $p \in L$ and $p' \in L'$, the configurations of vectors in ξ_p and $\xi_{p'}$, which are tangent to the Legendrian segments beginning at these points, should be linearly isomorphic.

If a characteristic foliation F_ξ has no closed leaves, then separatrices of hyperbolic points form a Legendrian graph. In Section 2, we will consider certain Legendrian graphs (which will turn out to be trees) formed by leaves of a characteristic foliation.

Lemma 1.3. *If two Legendrian trees L and L' in M are diffeomorphic, then they can be deformed one into the other via an ambient contact isotopy.*

Proof. Any two Legendrian segments are obviously Legendrian isotopic, and according to Lemma 1.2, they can be moved one into the other via an ambient contact isotopy. In fact, this remains true even if the curves coincide near one of the ends and the isotopy is required to be fixed near this end. Let us enumerate the Legendrian segments forming the trees

$$L = \bigcup_1^N L_i, \quad L' = \bigcup_1^N L_i'$$

in such a way that the diffeomorphism of our hypothesis takes ${L_i}'$ to L_i, for all $i = 1, \ldots, N$, and moreover there is a vertex p and some $r \leq N$, such that $L_1 \cap L_i = \{p\} \forall i = 2, \ldots, r$ while $L_1 \cap L_j = \emptyset \forall j > r$. We begin by deforming L_1' into L_1 via an ambient isotopy. Next we want, keeping a neighborhood U of L_1' fixed, to deform L' into L in a neighborhood of the vertex p. The contact structure in a small neighborhood $U \ni p_2$ can be defined in Darboux coordinates, $\xi = \{dz = y\,dx\}$, so that the plane $\Pi = \{z = 0\}$ can be identified with the contact plane ξ_p. Let us project $L \cap U$ and $L' \cap U$ to this plane along the z-axis, and denote by $\tilde{L}$ and $\tilde{L}'$ their images. By the assumption there is a (germ of a) diffeomorphism $h : \Pi \to \Pi$ which sends $\tilde{L}'$ into $\tilde{L}$. One can assume that h is fixed at the points of the projection of L_1', preserves the area form $dx \wedge dy$, and is isotopic to the identity in the space of diffeomorphisms with these properties. This h can be canonically lifted to a local contactomorphism $H : U \to U$, defined by the formula $H(x, y, z) = (h(x, y), z + f(x, y))$, where f is determined by the conditions that $h^*(y\,dx) = df$ and that f vanish at the origin. The contactomorphism H sends $L' \cap U$ into $L \cap U$, and it is contactly isotopic to the identity. Thus we can use H to make L and L' coincide along L_1, and in the neighborhood of the vertex p. Continuing this process inductively (at each stage of induction using in the role of U a neighborhood that includes all edges arranged in previous levels of induction), we construct the required isotopy between the Legendrian trees L' and L. □

1.4.3. Elliptic pivot lemma. As was mentioned in Section 1.3.1, we assume in this paper that a (C^1-small) perturbation of a surface near an elliptic point, if needed, has already been performed so that the elliptic point is of so-called (b)-type; in which case, we can choose Darboux cylindrical coordinates (ρ, φ, z) with the origin at the singular point, so that the contact structure is defined by the contact form $dz + \rho^2\, d\varphi$ while the surface F is given by the equation $z = 0$.

Let us denote by L_c the piecewise-smooth Legendrian curve in F which consists of two rays $\varphi = 0$ and $\varphi = c$, where $c \in (0, \pi]$. In particular, L_π is just a Legendrian line. Notice that for any smooth curve $\Gamma \subset F$ there is a Legendrian curve $\Lambda \subset \mathbb{R}^3$ whose Lagrangian projection equals Γ. Indeed, given the smooth curve $\Gamma \subset F$, we add the coordinate $z = \int_\Gamma \rho^2\, d\varphi$ and consider the Legendrian lift of Γ. Suppose that a smooth embedded curve $\tilde{L}_c \subset F$ approximates L_c and coincides with L_c outside a small neighborhood $D_\varepsilon = \{\rho \le \varepsilon\} \subset F$. Suppose also that $\int_{\tilde{L}_c \cap D_\varepsilon} \rho^2 d\varphi = 0$. Then $\tilde{L}_c$ lifts to an embedded Legendrian curve $\hat{L}_c$ which approximates L_c and which coincides with L_c outside an ε-neighborhood of the origin. Thus, retaining the definition of L_c above, we have:

Lemma 1.4. *For any $\varepsilon > 0$, there exists a Legendrian isotopy $\hat{L}_c$, $c \in (0, \pi]$, such that $\hat{L}_\pi = L_\pi$ and for all $c \in (0, \pi]$, the curve $\hat{L}_c$ coincides with L_c outside of the ε-neighborhood of the origin.*

1.5. The invariants tb and r and main theorem.

1.5.1. The invariants tb and r. The two classical invariants of Legendrian knots are defined as follows. Let L be a Legendrian knot in (M, ξ) which is homological to 0. Suppose L' is the result of slightly pushing L along some vector field transversal to ξ. The intersection number $\mathrm{tb}(L)$ of L' with a spanning surface S for L is independent of the choice of this vector field and of the surface S. It is called the *Thurston–Bennequin invariant*[3] of L. Equivalently, $\mathrm{tb}(L)$ is the number of clockwise (positive) 2π twists of ξ with respect to the natural framing along L induced by a spanning surface for L.

[3]The invariant tb is closely related to Arnold's invariant J^+ and the Legendrian linking polynomial defined in $S^1 \times \mathbb{R}^2$; let us point out that for topologically trivial Legendrian knots, this Legendrian linking polynomial provides no additional information beyond tb.

Let $\beta \in H_2(M, L)$ be a relative homology class and F a surface in the class of β. Suppose τ is a positive tangent vector to the oriented curve L. The degree $r(L|\beta)$ of τ with respect to a trivialization of the bundle $\xi|_F$ does not depend on the choice of trivialization. Nor does it depend on the choice of a representative F of the class β. It is called the *rotation number* (or *Maslov index*) of L computed with respect to β. If $\tilde{\beta}$ is another class from $H_2(M, L)$, then we have

$$r(L|\beta) - r(L|\tilde{\beta}) = e(\xi)[\beta - \tilde{\beta}].$$

Here $e(\xi) \in H^2(M)$ denotes the Euler class of the 2-dimensional oriented bundle ξ, and the difference $\beta - \tilde{\beta}$ is considered as an absolute class from $H_2(M)$. For homotopically trivial knots, one can always choose F to be a (not necessarily embedded) disk. As it is proven in [**20**], the Euler class $e(\xi)$ of a tight contact structure ξ vanishes on spherical classes and thus $r(L|\beta)$ in this case is independent of β. In this case (or if β is clear from context), we write simply $r(L)$ instead of $r(L|\beta)$. Notice that $r(L)$ changes sign when the orientation of the knot L is reversed, while $\mathrm{tb}(L)$ is independent of the orientation of L.

1.5.2. Main theorem. Our main result is the following:

Theorem 1.5. *Let L and L' be two topologically trivial Legendrian knots in a tight contact 3-manifold. If $\mathrm{tb}(L) = \mathrm{tb}(L')$ and $r(L) = r(L')$ then L and L' are Legendrian isotopic.*

Theorem 1.5 is proved in Section 3.4.

1.5.3. The range of the invariants tb and r. The following inequality was proved by Bennequin (see [**6**]) for the standard contact structure on S^3 and was generalized in [**20**] to general tight contact manifolds:

Theorem 1.6. *Let (M, ξ) be a tight contact manifold and let L be a Legendrian curve which is homological to 0 in M. Then for any homology class $\beta \in H_2(M, L)$ and surface $F \in \beta$, we have*

$$\mathrm{tb}(L) \leq -\chi(F) - |r(L|\beta)|.$$

We remark that the numbers tb and r also satisfy the following congruence relation (follows, for instance, from Lemma 3.6 below).

Proposition 1.7. *Let (M, ξ) and L be as above. Then*

$$\mathrm{tb}(L) + r(L) \equiv 1 (\mathrm{mod}\, 2).$$

Let $[\mathcal{E}]$ denote the set of isotopy classes of (conventional) knots in M and let $[\mathcal{L}] = \pi_1(\mathcal{L})$ denote the set of Legendrian isotopy classes of Legendrian knots.

Suppose, for simplicity, that $e(\xi) = 0$. Then we have a map

$$\lambda : [\mathcal{L}] \to [\mathcal{E}] \times \mathbb{Z} \times \mathbb{Z},$$

where the first factor gives the topological class associated to a given Legendrian class and the second and the third factors give the values of the invariants r and tb on this class.

The inequality (Theorem 1.6) and the congruence (Proposition 1.7) impose restrictions on the image of the map λ. However, these are not the only restrictions. Additional restrictions have also been found by Kanda (see [**50**]), Fuchs and Tabachnikov (see [**37**]), Lisca and Matić (see [**51**, **52**]), Rudolph (see [**59**, **60**]) and for the analogous map in the case of Legendrian links Mohnke [**54**].

Our main Theorem 1.5 states that the map λ is injective when restricted to $\lambda^{-1}([0] \times \mathbb{Z} \times \mathbb{Z})$, where $[0]$ denotes the topological class of the unknot. The map is not however injective in general (see [**9**, **29**]).

Let $\mathcal{D} \subset \mathbb{Z} \times \mathbb{Z}$ be the range of the invariants (r, tb) on the space of topologically trivial Legendrian knots. Even though Theorem 1.6 and Proposition 1.7 preclude surjectivity of $\lambda_0 = \lambda|_{[\mathcal{L}]_0}$, where $[\mathcal{L}]_0 = \lambda^{-1}([0] \times \mathbb{Z} \times \mathbb{Z})$, the domain $\mathcal{D}$ of λ_0 is as large as possible (see Figure 3). Indeed:

Lemma 1.8. $\mathcal{D} = \{(m, -|m| - 2k - 1) \mid k \geq 0\}$.

Proof. The inequality of Theorem 1.6 and the congruence of Proposition 1.7 show that

$$\mathcal{D} \subset \{(m, -|m| - 2k - 1) \mid k \geq 0\}.$$

On the other hand, for any pair $(m, n) \in \mathcal{D}$, the catalog given in Figure 3 provides a wavefront of a Legendrian knot L in $\mathbb{R}^3$ with $\mathrm{tb}(L) = n, r(L) = m$. Since any contact manifold contains the standard contact $\mathbb{R}^3$ (by Darboux's theorem), all the examples of the catalog can be constructed in a general (M, ξ). □

1.5.4. Catalog of Legendrian unknots. Figure 3 provides a list of Legendrian wavefronts each lifting to a Legendrian unknot in standard $\mathbb{R}^3$ with specified values of tb, r.

In general, the values of tb, r can be read from a wavefront projection as follows:

Given a Legendrian knot L in $\mathbb{R}^3$ (or in $S^1 \times \mathbb{R}^2 = ST^*(\mathbb{R}^2)$), let Ω be the set of all points of self-intersection of the front $L_{\mathrm{Front}} = p_{\mathrm{Front}}(L)$, and let K be the set of all cusps of the front. Note that for the standard $(\mathbb{R}^3, dz - y\,dx)$, the projection p_{Front} is projection to the (x, z)-plane. Now, for each $p \in \Omega$, define[4] $or(p)$ to be $+1$ or -1 depending on whether the two rays of L_{Front} emanating *from* p lie on opposite sides of a vertical line through p or on the same side. These conventions are indicated in Figure 4. Also, for each $p \in K$, define $\kappa(p)$ to be $+1$ or -1 depending on whether the ray emanating from p lies above or below the ray entering p (i.e., has z-value higher or lower

[4]This is equivalent to defining $or(p)$ as the orientation determined by the pair of emanating rays from p written in the order (ray with greater slope, ray with lower slope).

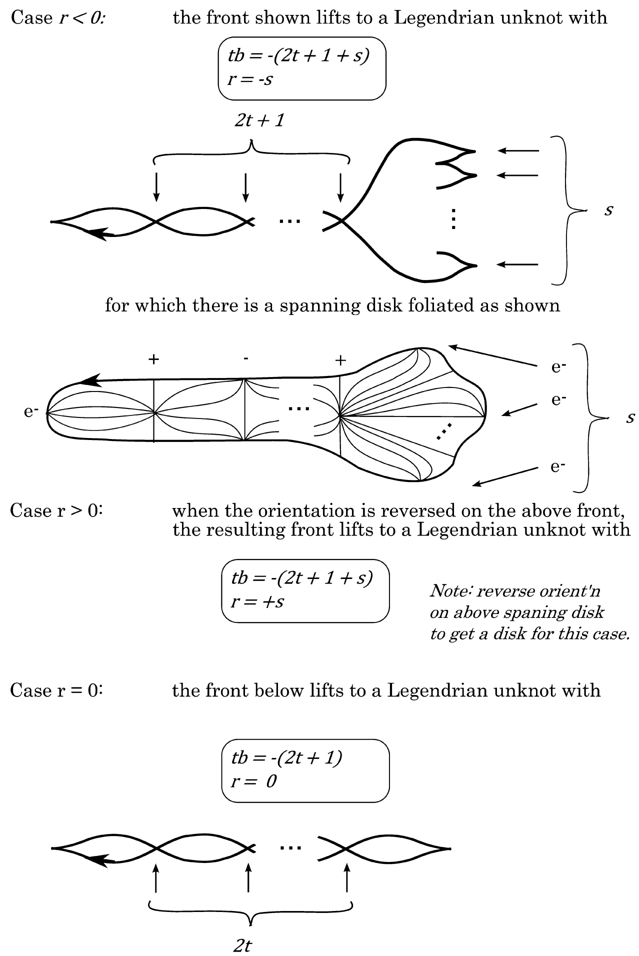


Figure 3. Catalog of wavefronts.

than the other ray for given x-value). Thus κ indicates whether the original curve L was rising ($\kappa(p) > 0$) or falling ($\kappa(p) < 0$) as it passed through the preimage of p.

Then,

$$tb(L) = -\sum_{p\in\Omega} or(p) - \frac{1}{2}\sum_{p\in K} |\kappa(p)|,$$
$$r(L) = \frac{1}{2}\sum_{p\in K} \kappa(p).$$

In the catalog of Figure 3, each (tb, r) case also includes a description or a sketch of a singular foliation on a disk D. It is easily checked that the

$$tb(L) = -\sum_{p\in\Omega} or(p) - \frac{1}{2}\sum_{p\in K} |\kappa(p)|$$
$$r(L) = \frac{1}{2}\sum_{p\in K} \kappa(p)$$

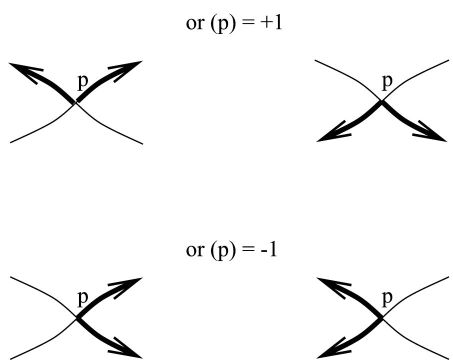


Figure 4. Conventions for crossings of fronts.

Legendrian lift for each front has a spanning disk with foliation as shown (see Section 3.3.2 for details).

2. Manipulation of characteristic foliation

The goal of this section is Lemma 2.7 below which, given a topologically trivial Legendrian knot L, establishes existence of a spanning disk whose characteristic foliation has a special form, which we call *elliptic*. We will achieve this by taking an arbitrary spanning disk for L and then altering its characteristic foliation via gradual deformation of the surface.

2.1. Birth, death and conversion. The basic tools which will allow us to effect the desired alterations of characteristic foliation are the controlled birth and death of elliptic–hyperbolic singularity pairs. The controlled death of such a pair, i.e., its killing, or *elimination*, is the subject of Lemma 2.1. It was first proved by Giroux in [**40**], and in a form improved by Fuchs, was presented in [**19**]. The creation of an elliptic–hyperbolic pair of singularities is more local and is relatively straightforward; in its most basic form (Lemma 2.2) it is often stated without proof. In the present paper, another form of pair-creation (called *elliptic-hyperbolic conversion*, given by Lemma 2.2) will also be used. All three types (elimination, conversion, creation) rely on the same basic idea: twisting a strip of surface along a Legendrian curve in a manner dictated by the twisting of ξ along the curve.

Lemma 2.1 (Elimination). *Let S be an embedded surface in (M, ξ) such that S_ξ has exactly two singularities, p and q, which are respectively elliptic*

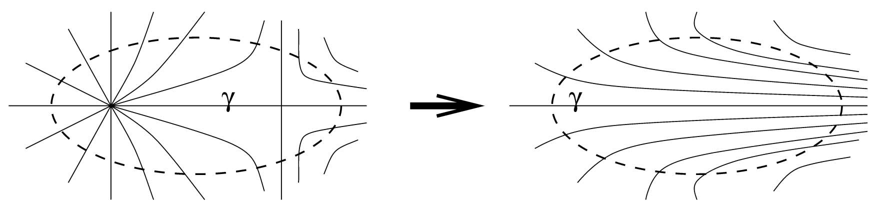


Figure 5. Elliptic-hyperbolic elimination.

and hyperbolic. Suppose that they are both of same sign and are connected by γ, one of q's separatrices. Then given any arbitrarily small neighborhood of γ, there exists a C^0-small isotopy of S supported in that neighborhood and fixing γ which results in a new surface having no singularities of its characteristic foliation (*see Figure* 5)*.*

Lemma 2.2 (Elliptic–hyperbolic conversion). *Let S be an embedded surface in (M, ξ) having just one singularity of its characteristic foliation called p. Suppose p is elliptic* (*hyperbolic*)*. Let γ and τ be two leaves of the characteristic foliation which pass through p, intersecting transversally and at this point only. Moreover, when p is hyperbolic, suppose the name γ is assigned to the stable* (*or unstable*) *separatrix of p for the case p negative* (*or positive*)*. Then given any arbitrarily small neighborhood of p, there exists a C^0-small isotopy of S supported in that neighborhood, fixing both γ and τ, which results in a new surface having a hyperbolic* (*elliptic*) *singularity at the intersection of γ and τ as well as two additional elliptic* (*hyperbolic*) *singularities, one on each side of p as shown in Figure* 6*, all three singularities being of the same sign as p and there being no further singularities.*

Note that cases (b),(c) in Figure 6 can be summarized by the observation that the newly created pair of elliptic sink (source) singularities will of necessity be created on whichever separatrix flowed into (out of) p; we have used the name γ for it. For case (a), we have the freedom to choose which of the leaves will be granted new singularities and we label it γ. For all three cases, the process can be viewed, on the one hand, as the *conversion* of the singularity-type of p together with the *creation* of two elliptic (or else two hyperbolic singularities) on either side or, on the other hand, as controlled birth of an elliptic–hyperbolic pair during which one of the two newly created singularities "displaces" p.

Lemma 2.3 (Basic form of pair-creation). *Let S be an embedded surface in (M, ξ) having no singularities in its characteristic foliation S_ξ. Choose a leaf γ of S_ξ. Then given any arbitrarily small neighborhood through which γ passes, there exists a C^0-small isotopy of S supported in that neighborhood and fixing γ which results in a new surface having exactly two singularities*

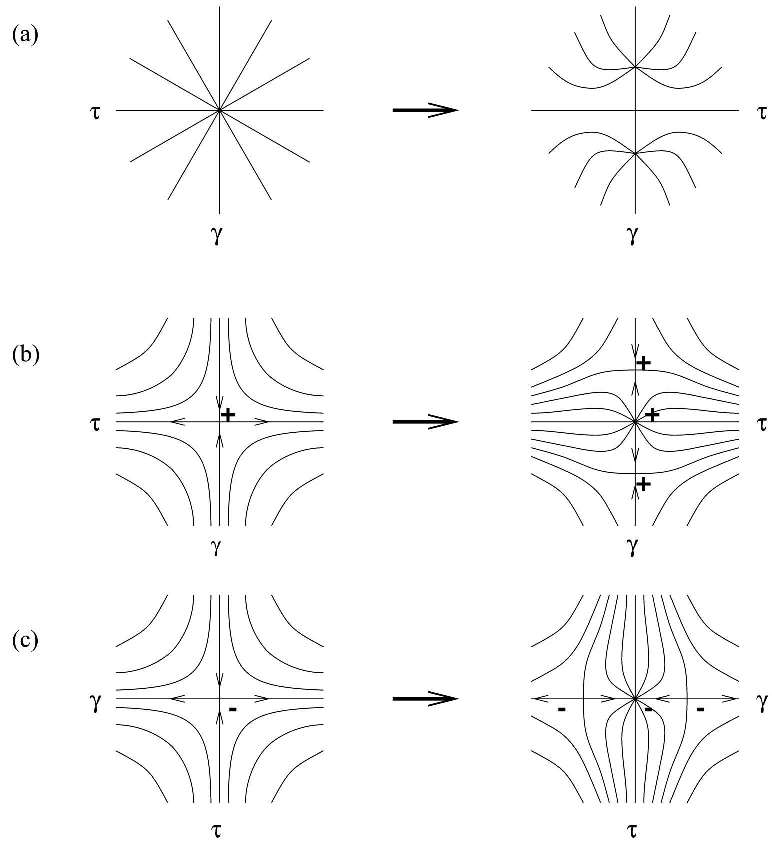


Figure 6. The cases of elliptic–hyperbolic conversion.

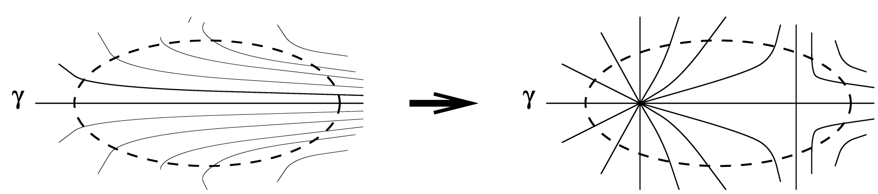


Figure 7. Elliptic–hyperbolic creation.

(*of same sign*) *which both lie on* γ*, one being elliptic, the other hyperbolic as shown in Figure* 7*.*

2.1.1. Singularity curves. Given an oriented embedded surface F, suppose we have the non-generic situation where there is a Legendrian curve $L \subset F$, consisting entirely of singularities of F_ξ. We then say L is a *singularity curve* on F. A model for such a situation is the x-axis (i.e., the line

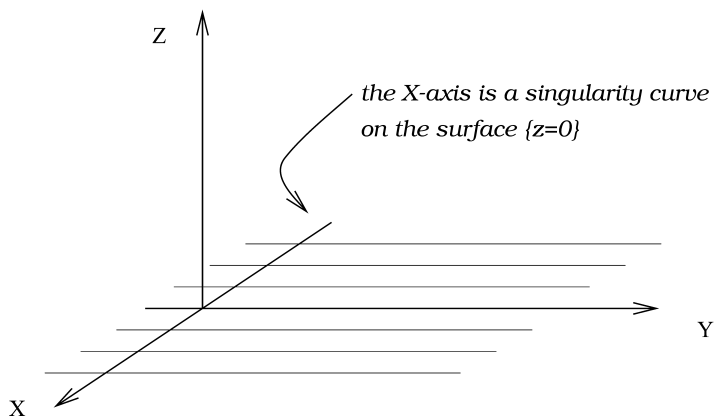


Figure 8. Model for singularity curve in standard $(\mathbf{R}^3, dz - ydx)$.

$\{y = z = 0\}$) on the surface $\{z = 0\}$ in the standard $(\mathbf{R}^3, dz - y\,dx)$. This is illustrated in Figure 8.

Note that on a given oriented embedded surface F, all the singularities in a singularity curve are of the same sign, and we can thus speak of positive or negative singularity curves on F.

2.1.2. Proof of manipulation lemmas.

Proof of Lemmas 2.1*,* 2.2*,* 2.3*.* First, consider a model Legendrian curve and its neighborhood. In the standard contact $(\mathbf{R}^3, dz - y\,dx)$, let L be the x-axis and let N be a small tubular neighborhood thereof.

Let Z be a cylinder of radius 1 with L as its core, parameterized by θ, x, where (r, θ) are polar coordinates in the (y, z)-plane. Note that the characteristic foliation of the surface Z has singularity curves along the top and along the bottom, and has no other singularities. Specifically, $\{y = 0, z = \pm 1\}$ are singularity curves and the characteristic foliation flows from the positive ($z = +1$) to the negative ($z = -1$) singularity curve, crossing $\{z = 0\}$ at an angle of $\frac{\partial\theta}{\partial x} = 1$.

Note that along any Legendrian curve, we have a dilating (in normal plane) contact flow and the tubular neighborhood this defines is standard, i.e., contactomorphic to such a tubular neighborhood constructed around any other Legendrian curve, in particular to the neighborhood $W = \{z^2 + y^2 \le r\}$ around L, with $W = \partial Z$. To prove each lemma, let N be a standard neighborhood of γ, i.e., one contactomorphic to W (with γ corresponding to L).

Now consider model surfaces in W, each of the form A_f for a smooth function $f : \mathbf{R} \to \mathbf{R}$, where A_f is the smooth surface swept out along L by a line normal to L whose angle with the horizontal at each point is $f(x)$.

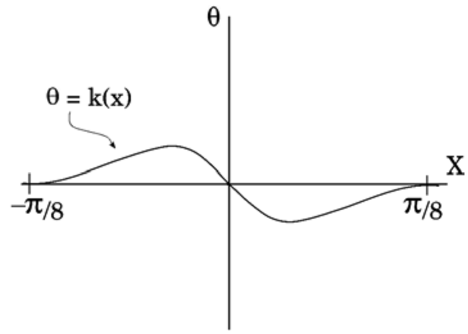


Figure 9. The function $k(x)$ used to perturb $g(x)$ in proof of Lemma 2.2.

Specifically, $A_f = \{(x, y, z) : y \sin f(x) = z \cos f(x)\} = \{(x, r, \theta) : \theta = f(x) \mod 2\pi\}$. These are so-called *staircase surfaces*.

For positive real ϵ, let $g(x) = \epsilon \cos x$ and $h(x) = -\epsilon$. It is easily verified that the characteristic foliation on A_h within W is singularity-free (as given for Lemma 2.3), and also that it meets Z in a transverse curve. One may also verify that for $\epsilon < 1$, the characteristic foliation on A_g for $-\pi < x < \pi$ is as given for Lemma 2.1; and finally, that A_g for $-\pi < x < -\frac{\pi}{4}$ has characteristic foliation like that on the left of part (a) in Lemma 2.2, while A_g for $\frac{\pi}{4} < x < \pi$ has foliation as on the left of parts (b),(c). In making this verification of characteristic foliation, one observes that the hyperbolic vs. elliptic nature of the singularity is a result of g' being respectively negative or positive,[5] where $g(x) = 0$, since A_g twists in the same direction as ξ along lines $\{z = 0, x = \pm\frac{\pi}{2}\}$ albeit more slowly (due to $\epsilon < 1$). Note that the curves $C = A_g \cap Z$ stay within an ϵ-neighborhood of the horizontal curves $H = \{z = 0\} \cap Z$ and that the slope $\frac{\partial\theta}{\partial x}$ of the curves C (in the cylinder Z) is $-\epsilon \sin x$, so the absolute slope is $\leq \epsilon$. Meanwhile, the leaves of Z_ξ have slope $\frac{\partial\theta}{\partial x} = 1$ along H and even higher slope elsewhere. Thus, in particular, C is transverse to ξ. In general, any staircase surface A_f, such that $|f'| < 1$, will meet Z in a transversal curve.

We now define two further functions that will be used to produce the model surfaces for the right hand sides of parts (a),(b),(c) in Lemma 2.2. Assume $\epsilon < \frac{1}{4}$. Let $k(x)$ be a function as shown in Figure 9, with the property that $|k'| \leq 2\epsilon$, that $k'(0) = -2\epsilon$, and that the absolute value of the function is bounded by ϵ. We use k to perturb g near $\pm\frac{\pi}{2}$. Specifically, let $g_-(x) = g(x)+k(x+\frac{\pi}{2})$ in the region $x < -\frac{\pi}{4}$ and let $g_+(x) = g(x)-k(x-\frac{\pi}{2})$ in the region $x > \frac{\pi}{4}$. Then $g'_-(-\frac{\pi}{2}) = -\epsilon$, while g had slope $+\epsilon$ there. And, $g'_+(\frac{\pi}{2}) = +\epsilon$, while g had slope $-\epsilon$ there.

[5]Cases will correspond to rate of rotation of surface compared to ξ having resp. different or same sign along two Legendrian axes through singularity.

Moreover, $|g'_-| \leq 3\epsilon < 1$ and likewise $|g'_+| < 1$, so the surfaces A_{g_-} and A_{g_+} (for the relevant x-regions) meet Z in transversal curves and it is easily checked that they exhibit the characteristic foliations respectively for the right hand sides of part (a) and of parts (b),(c) in Lemma 2.2 (with γ as the x-axis L).

We now have model surfaces for the left and right hand sides of all parts of the three lemmas. Each of these meets Z along curves transverse to ξ and stays within a small neighborhood of H. For each lemma statement, let B be the portion of Z between the two curves, and add it to the model for the left side, smoothing along the two curves where the joining is performed. This can be done without introducing singularities in the characteristic foliations since the curves are transverse to ξ. In each case, the new surface is a model for the right side (i.e., exhibits its foliation), contains γ, is C^0-close to the old surface and coincides with it away from L. We thus have the required C^0-small deformations. □

2.1.3. Manipulation lemma for singularity curves. With a modification of the above proof, one can also prove Lemma 2.4 below. One uses new model surfaces of the form A_f (once again with L in the role of γ), where f is defined piecewise as follows: for values of x on which we want a singularity curve, use $f(x) = 0$, where we want a half elliptic point (at $x = a$) use a function that is locally of the form $\epsilon(x-a)^3$, and where we want a half hyperbolic point (at $x = b$) one of the form $-\epsilon(x-b)^3$. Assume that the factor ϵ is chosen sufficiently small (as in previous proof) and the patching together of f is done reasonably, so that the resulting f has small absolute value and absolute slope and we can thereby ensure A_f meets Z along a transverse curve. The same argument as above then applies to construct the desired surface deformation.

Lemma 2.4. *Let S be an embedded surface in (M, ξ) and let $\gamma \in S$ be an embedded open interval. Suppose the characteristic foliation near γ is as shown on the left* (*resp. right*) *hand side of Figure* 10*, then there is a C^0-small isotopy of S supported in an arbitrarily small neighborhood of γ that fixes γ, ρ and* (*where applicable*) *τ, while resulting in a new surface with characteristic foliation as shown on the right* (*resp. left*) *side of this Figure.*

2.2. Characteristic foliation on the disk.

2.2.1. Program for standardizing the foliation. Given an (oriented) spanning surface S of a Legendrian knot L, the characteristic foliation S_ξ will be said to be in *normal absorbing form* (NAF) along L if the singularities on L alternate in sign and the positive ones are hyperbolic, the negative elliptic.

Once we have this form of boundary foliation, we know all flow between interior and boundary is directed towards the boundary. This makes it easier to control the interior foliation.

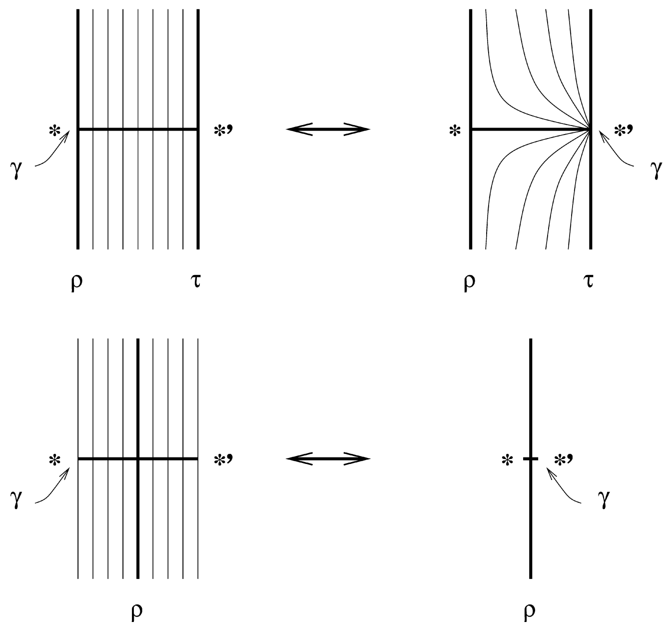


Figure 10. Manipulations of singularity curves (each *, $^{*\prime}$ means a fixed choice of elliptic or hyperbolic).

In particular, by obtaining NAF on the boundary, we are then able to eliminate all but positive elliptic and negative hyperbolic singularities on the interior; this process is described in the proof of Lemma 2.5. Interior characteristic foliation that has been reduced to this simple form, i.e., for which all singularities are either positive elliptic or negative hyperbolic, will be said to be *reduced*. Thus we will speak of a disk whose foliation is *reduced with NAF on the boundary*. In this state, we have control over the number and placement of the interior singularities as we shall see in Sections 2.2.3 and 2.2.7. Then, once this control is achieved, we can retain it while altering the types of some boundary and interior singularities so as to make the overall foliation more suited to the present proof. This final form of foliation will be called *elliptic form*; it is the subject of Section 2.2.5.

To summarize, the sequence of characteristic foliation types we will pass through is:

1) NAF on boundary:	just $h+$ and $e-$ on boundary.
2) Reduced with NAF on boundary:	just $h+$ and $e-$ on boundary, and, just $e+$ and $h-$ on interior.
3) Elliptic form:	mostly $h+$ and $h-$ on boundary,[6] just $e+$ and $e-$ on interior.

[6]For precise definition, see Section 2.2.5.

2.2.2. First steps of standardization. The following lemma deals with achieving steps 1 and 2 of the above standardization program.

Lemma 2.5. *In tight (M, ξ), let D be an oriented, embedded disk with Legendrian boundary L. Then there exists a C^0-small deformation of the disk which produces a new disk whose characteristic foliation is reduced with normal absorbing boundary.*

Proof. Given D and L as in the hypotheses, we will first arrange that signs alternate along L. By definition of $\mathrm{tb}(L)$, this invariant counts the net 2π-twisting along L of ξ with respect to the framing induced by D. This is the same as the net 2π-twisting along L of a normal to ξ with respect to a normal to D. Wherever these two normals are aligned along L, there will be a singularity of D_ξ. Let us assume the normals to ξ and D that we use are those induced by the respective coorientations of ξ and D, then this singularity will be positive (negative) when the normals coincide with (are negatives of) each other. Note that for each 2π-twist of ξ, there will thus be at least two singularities of D_ξ (one of each sign). Moreover, by twisting D along L, one can clearly attain the minimal situation where there are exactly $2|\mathrm{tb}(L)|$ singularities along L with alternating signs. Note that this deformation can be taken to be C^0-small. We now suppose it has been performed, and label the resulting disk once again D.

Using Lemma 2.2 allows us to keep L fixed and convert any positive elliptic singularities of L into positive hyperbolic singularities. Likewise, we can convert any negative hyperbolic singularities into positive hyperbolic singularities. If we let L correspond to τ of Lemma 2.2 in each case, then no new singularities will be introduced along L and we will eventually, after successive conversions, reach NAF. Note that the deformations of the disk which accomplish these conversions can be made C^0-small. Once again, label the resulting disk D.

Now that NAF has been achieved on D, there is an obvious transversal unknot Γ lying on D just inside L (see Figure 11). Because D_ξ was in NAF along L, the characteristic foliation flows outward across Γ and onto L. This flow exiting across a transversal unknot is the situation assumed in Proposition 2.3.1 and Section 4.4 of the paper [**19**] where an account is given of how to eliminate all interior singularities except for the positive elliptic and negative hyperbolic ones. The basic idea is as follows:

- First, destroy all hyperbolic-hyperbolic connections.[7]
- Then, suppose there exists a negative elliptic point p on the interior. All of the arcs flowing into p must originate at interior singularities, since the flow exits across Γ and there are no limit cycles in D_ξ

[7]Such connections are highly unstable; a C^1-small pushing upwards of the surface at a point along such a shared separatrix will slightly shift the characteristic flow so that the two separatrices no longer connect to each other.

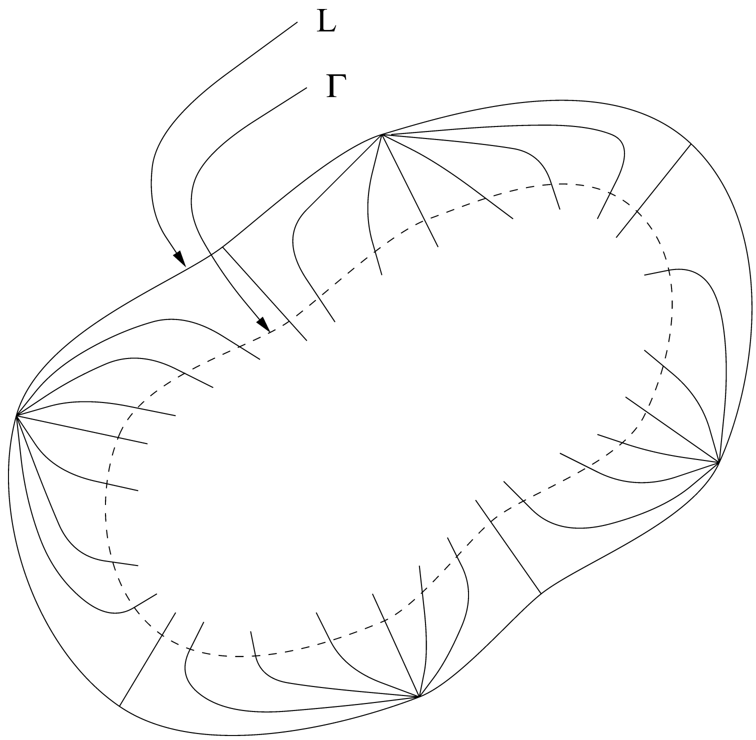


Figure 11. View of D near $\partial D = L$; Note NAF along L and resulting Γ just inside L.

(by tightness). Consider the *basin* of p, i.e., the closure of the set of all points connected to p by smooth arcs. It will be an immersed disk, with self-intersection only at (possibly) some subsegments of the boundary, and an embedding on the interior. Let B denote the boundary before immersion. No two hyperbolic points will be adjacent as we travel along B, since all h–h connections were destroyed. No two elliptic points can be adjacent either, since they must all be positive (sources flowing to p). Thus hyperbolic and (positive) elliptic singularities will alternate along B. Moreover, there must be at least two singularities in B. If all B's hyperbolic points were positive, we could kill them all, together with the positive elliptic points which must separate them, thus obtaining a limit cycle. So in fact, there must exist a negative hyperbolic point along B and we can use it to kill p. Thus we may assume there do not exist any negative elliptic points in the region enclosed by Γ.

- Finally, suppose there exists a positive hyperbolic point p on the interior. Look at its stable separatrices. These cannot come from a limit cycle, a point outside Γ or a hyperbolic point, so they must come from an interior positive elliptic point. Kill the pair. Thus we may assume there do not exist any positive hyperbolic points in the region enclosed by Γ. The characteristic foliation in this region is therefore now in reduced form.

- Besides the breaking of hyperbolic connections, all deformations were of the elimination type given in Lemma 2.1, and so we may assume the overall deformation required is C^0-small. □

2.2.3. Counting singularities. The following count of interior singularities holds as long as we have NAF on the boundary, whether or not the interior has been reduced.

Lemma 2.6. *Let F be an embedded disk spanning the Legendrian knot L and having NAF on L. Let $e_\pm$ and $h_\pm$ be the number of $\pm$-ve interior elliptic and hyperbolic singularities, respectively. Then,*

$$e_\pm - h_\pm = \frac{1}{2}(1 \mp \mathrm{tb}(L) \pm r(L)).$$

Proof. A similar fact is mentioned for transversal knots in [**19**]. Either version can be directly derived from the more general calculation in [**46**]. One may also easily derive the Legendrian version from the transversal version. Indeed, given a Legendrian knot $L = \partial F$ with NAF, and considering a small tubular neighborhood of L, one may assume standard coordinates and write down an explicit isotopy taking the usual (see [**19**]) positive transversalization $T_+(L)$ of L (whose intersection number with F is $\mathrm{tb}(L)$) to the positive transversal knot which lies just inside any NAF boundary on the spanning surface (this is Γ of Figure 11). In the present setting, let us also refer to this curve as Γ. Recall from [**19**] that $l(T_+(L)) = \mathrm{tb}(L) - r(L)$ and also the transversal version of the counting formula we are trying to establish, namely $e'_\pm - h'_\pm = \frac{1}{2}(1 \mp l(T_+(L))$, where $e'_\pm, h'_\pm$ refer to those singularities inside Γ. Since the small collar of surface separating Γ from L in Figure 11 contains no interior singularities of F_ξ, the number of interior singularities is the same for Γ and L, i.e., $e_\pm = e'_\pm, h_\pm = h'_\pm$. Thus we obtain the desired formula. □

Having the interior in reduced form as well would simply mean that in the equation of Lemma 2.6, $h+ = e- = 0$. In the spirit of this equation and also the elimination lemma (Lemma 2.1), pairs of opposite-type but same-sign singularities can be viewed as “non-essential”; since, in the equation, the contributions made by members of such a pair cancel each other, and by the elimination lemma (Lemma 2.1) the members themselves can sometimes be “canceled”. Therefore, particularly useful forms for the interior characteristic foliation on a surface are ones where no such non-essential pairs exist, i.e., for each sign only one type occurs, in particular, *reduced form* where we have only $e+, h-$ or *elliptic form* (to be defined in Section 2.2.5) where we have only $e+, e-$.

2.2.4. Legendrian tree of a reduced form disk. We will consider *Legendrian trees* (see Section 1.4) embedded into a spanning surface. We assume that vertices of such trees are located at singularities of the characteristic foliation, and each edge is either singularity-free, or contains exactly one hyperbolic singularity. Usually the trees we consider will have edges all of the same type, either all singularity-free or all hyperbolic-containing.

When the interior characteristic foliation of a disk is reduced (and the boundary is Legendrian in NAF or is transversal), it exhibits a Legendrian tree with elliptic singularities as vertices and hyperbolic-containing edges. To see this, note that the only interior singularities are positive elliptic and negative hyperbolic, and the latter have stable separatrices coming from the former, thus making up the hyperbolic-containing edges of our graph. In fact, the graph is a tree, as discussed in [**19**]. Indeed, the unstable separatrices of these interior hyperbolic points must go to the boundary which allows one to show that the graph is a deformation retract of the whole disk and is thus connected and simply-connected.

Also important for dealing with these trees is the following observation: each positive elliptic interior point (vertex of the tree) must be connected to *at least one* positive hyperbolic point on the boundary. This follows from the tightness of (M, ξ), because if a vertex has no connection to boundary hyperbolic points, then all the negative interior hyperbolic points to which it is connected[8] will have unstable separatrices flowing to boundary elliptic points in pairwise fashion (i.e., two hyperbolic separatrices flowing to each of these boundary elliptic points). With the help of the elliptic pivot lemma, we would thus obtain a closed Legendrian curve consisting only of negative elliptic points (from boundary) and negative hyperbolic points (from interior). This would violate the tightness of (M, ξ). Indeed, using the elimination lemma (Lemma 2.1), we could pairwise eliminate all singularities and thereby create a limit cycle. Figure 12 illustrates this for the case of a vertex with only one attached edge.

2.2.5. Elliptic form on the spanning disk. Let D be a spanning disk for the Legendrian unknot L. The characteristic foliation D_ξ will be said to be in *elliptic form* when the signs of boundary singularities alternate, all interior singularities are elliptic positive or elliptic negative and, besides the direct connection to its neighbor via a subsegment of L, each boundary point is connected only to interior points, and moreover each (elliptic) interior point is connected to at least two boundary hyperbolic points.[9] An example of an elliptic form disk foliation is given in Figure 13.

[8](whose separatrices form the edges adjacent to that vertex)

[9]This last condition is included to rule out the pathological situation with only one interior elliptic point and two boundary points (one elliptic, one hyperbolic); when we have more than one interior elliptic point, it is implied by the other conditions, and so redundant.

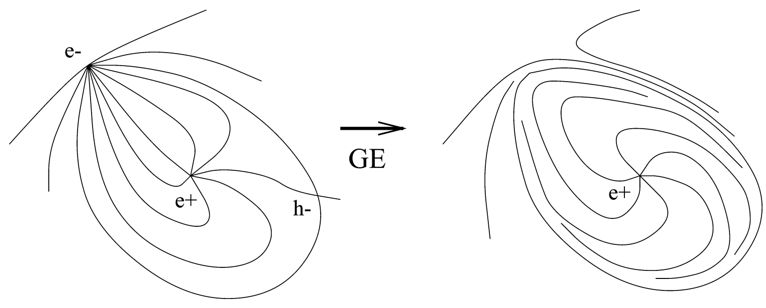


Figure 12. Tightness of $(M, \xi) \Rightarrow$ Each $e+$ inside attached to $h+$ on L.

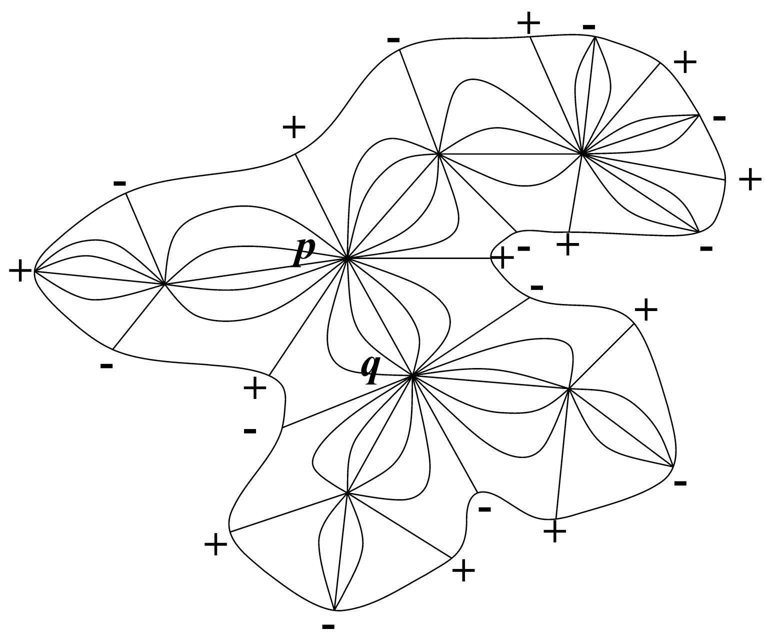


Figure 13. Example of an elliptic form disk foliation.

These conditions imply that boundary hyperbolic points of a given sign are connected to each other in groups of two or more via their separatrices which meet in an elliptic interior point of that same sign. These separatrices thus divide the surface into regions[10] of type(a) or type(b) on which there are no other interior singularities (see Figure 14). This is illustrated in Figure 15 for several vertices of a spanning disk. When we consider just half

[10]The word *region* will be informally used in this paper to refer to a topological embedding of the disk minus some segments of the boundary, such that the boundary is piecewise smooth.

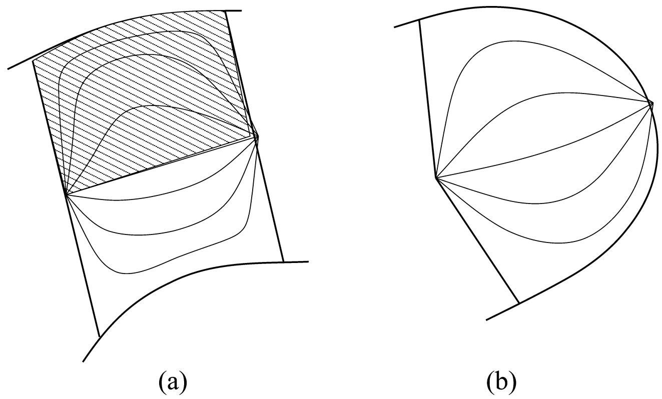


Figure 14. Types of region on disk in elliptic form.

of a type(a) region, as shaded in Figure 14, we will call it a semi-type(a) region. Note, on the other hand, that if a disk can be divided into type(a) and type(b) regions, then it automatically satisfies the properties of elliptic form. So for an embedded disk D, the existence of a decomposition of D_ξ into type(a) and type(b) regions is equivalent to D_ξ being in elliptic form.

It is also useful to note that if a disk with elliptic form foliation is oppositely oriented, it is still in elliptic form; i.e., the definition is invariant under reversal of sign of the singularities.

2.2.6. Final step of standardization: reduced form $\longrightarrow$ elliptic form. The following lemma shows we can realize step 3 of the standardization program for disk foliation, namely we can achieve elliptic form.

Lemma 2.7. *In tight (M, ξ), let D be an oriented embedded disk with Legendrian boundary L. Suppose D_ξ is in reduced form with normal absorbing boundary. Then there is a C^0-small isotopy of D fixing L which results in a new spanning disk having characteristic foliation in elliptic form.*

Proof. We are assuming D_ξ is reduced so it has a Legendrian tree T. Consider the singularities along L. Every positive singularity is hyperbolic, and so connected to only one interior singularity. The negative singularities, on the other hand, are elliptic and may be connected to several negative interior singularities (which are hyperbolic). We want to alter the foliation so that each boundary singularity is attached to only one interior singularity of its same sign.

The strategy for accomplishing this will be to use the positive boundary singularities' unique connections to interior points in order to divide the disk into regions that are easier to deal with. Let $P \subset D_\xi$ consist of the stable separatrices of positive boundary hyperbolic points, together with those points, and all positive interior elliptic points. This divides D into regions:

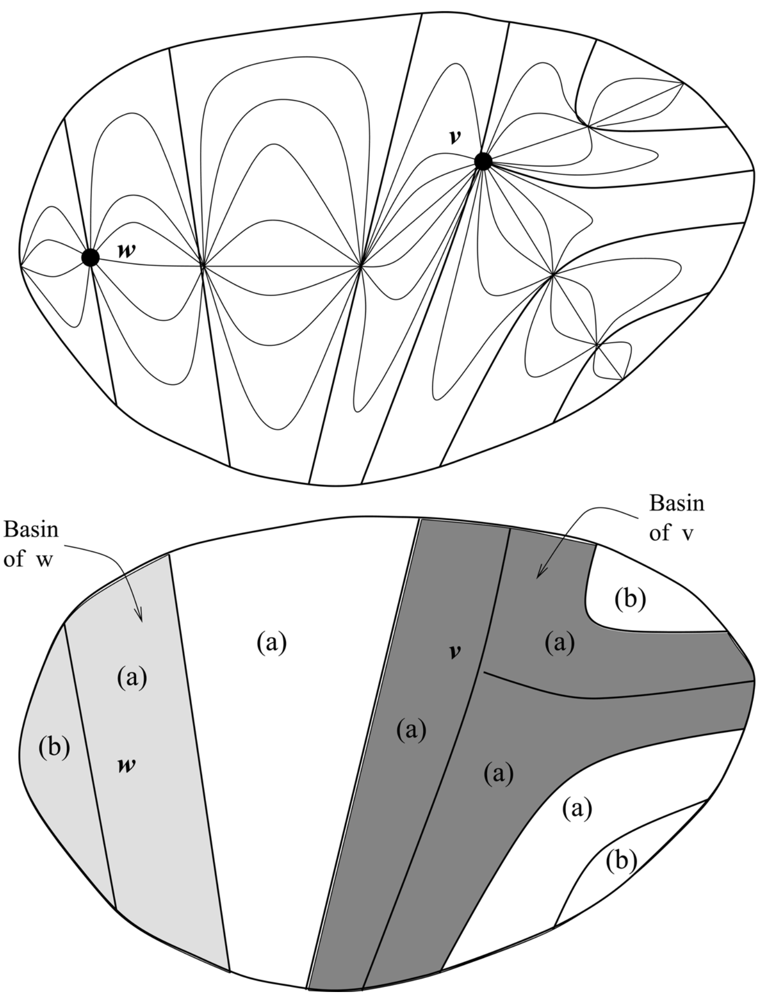


Figure 15. Example of how an elliptic form disk decomposes into type(a) and type(b) regions.

the connected components of $D \setminus P$. Every region meets L in a disjoint union of open Legendrian segments which contain exactly one singularity (negative elliptic). Those regions which meet L in a single segment are actually of type(b)[11], having a positive elliptic point on the interior and a negative elliptic point on the boundary. These we will not alter. Suppose, on the other hand, that R is a region meeting L in more than one segment, i.e., that $R \cap L$ has n components for $n \geq 2$. These are Legendrian segments. Note that on the interior of R there are $n-1$ negative hyperbolic points. Each of these is attached to exactly two of the segments just mentioned. Convert all $n-1$

[11](see Figure 14)

negative elliptic points on these segments to hyperbolic points (Lemma 2.2), and then cancel negative pairs (Lemma 2.1). One negative interior elliptic point remains, and no other interior points. Now consider $\partial R \backslash L$. It has n connected components (since $R \cap L$ did), and each component is either a single Legendrian segment from B or a union of two such segments. In the first case, the segment is homotopic to a point relative to L and consists of a single hyperbolic separatrix between a positive elliptic point p on the interior and a positive hyperbolic point on the boundary. Convert the boundary point (Lemma 2.2) to an elliptic point and then cancel the newly created interior hyperbolic point with p (Lemma 2.1). This is illustrated for a typical region R in Figure 16. Note that every time we carry out this procedure, we create a type(b) region, having a negative elliptic point on the boundary. All the rest of R is broken into type(a) regions by the hyperbolic separatrices that connect interior to boundary points. So R can be decomposed into type(a) and type(b) regions.

Assume we have performed the above procedure on all regions determined by P. Each is then decomposable into type(a) and type(b) regions, so this is true of the whole disk; i.e., the disk foliation is now in elliptic form. □

2.2.7. Legendrian trees of an elliptic form disk: skeletons and extended skeletons. Just as reduced form with normal absorbing boundary results in an obvious Legendrian tree on the interior, so too does elliptic form. Suppose D is an embedded disk with Legendrian boundary L whose characteristic foliation D_ξ is in elliptic form. Consider all (interior as well as boundary) elliptic singularities of D_ξ, and for each pair of these points which are connected by a smooth family of singularity-free Legendrian arcs, choose one representative of this family of arcs. Define the *skeleton* of D_ξ to be the Legendrian graph that has for vertices all interior elliptic points and for edges the representative arcs between these points. Let the *extended skeleton* be the Legendrian graph containing the skeleton but having as new vertices the elliptic boundary points and as new edges the representative arcs attaching these points to vertices of the skeleton. This is illustrated in Figure 17 below. We have defined the skeleton and extended skeleton as abstract graphs, as well as giving their corresponding Legendrian embedding. Note that as Legendrian graphs, they are well-defined by D_ξ up to self-diffeomorphism of D_ξ and choice of representative arcs. As abstract graphs, they are thus well-defined up to graph isomorphism given the topology of D_ξ.

Moreover, the skeleton and extended skeleton are both deformation retracts of the disk. This follows from the decomposition of an elliptic form disk into type(a) and type(b) regions as described in Section 2.2.5, because on each such region, we can clearly define an appropriate deformation retract

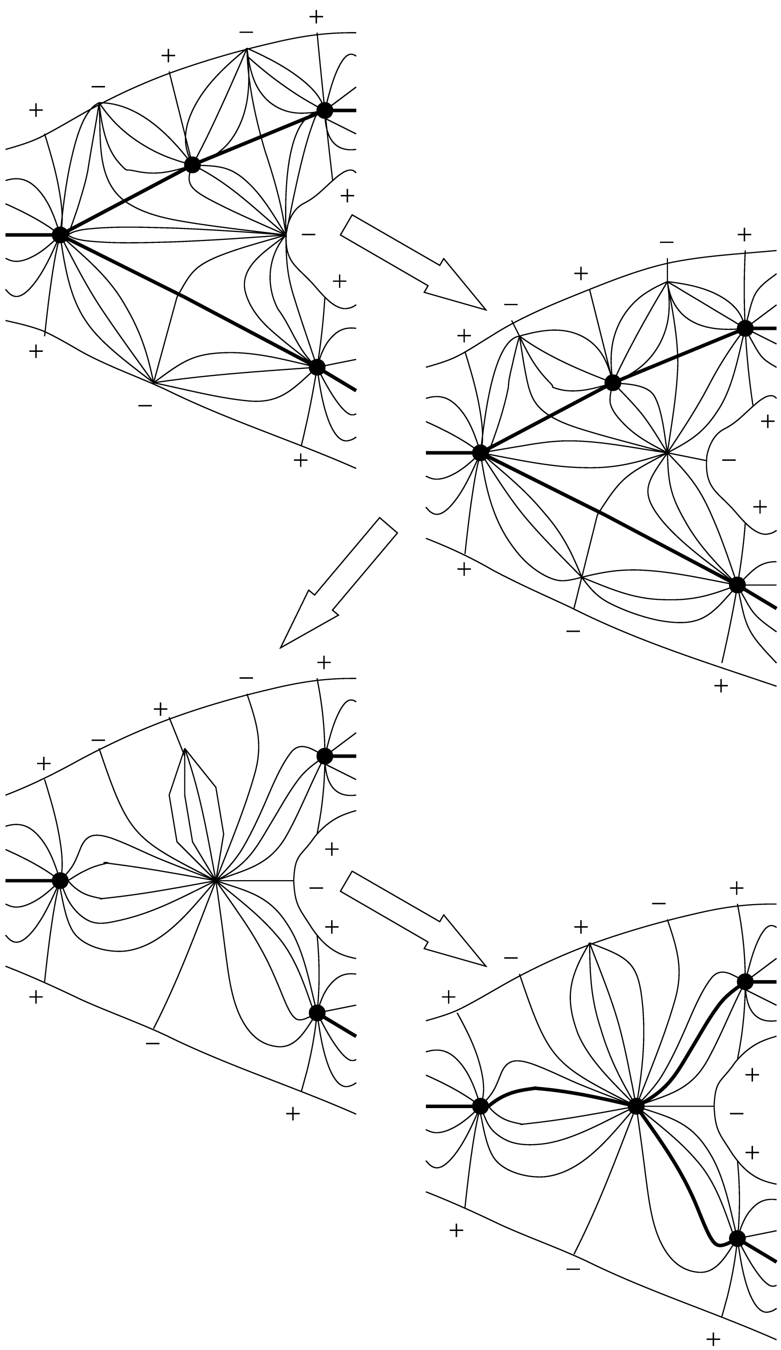


Figure 16. Example of procedure to alter region R .

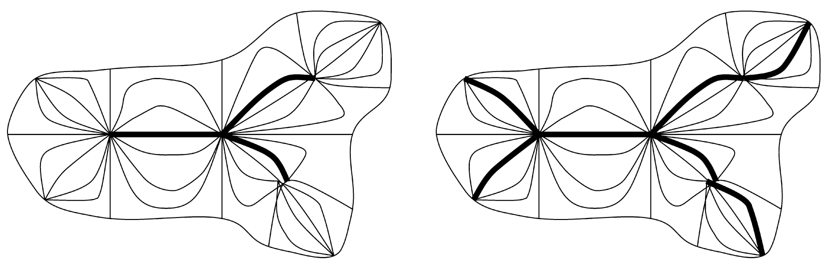


Figure 17. Extending the skeleton.

which is standard where such regions meet. As a result, we see that the skeleton and extended skeleton are in fact connected and simply-connected. They are thus Legendrian trees.

2.2.8. Graph terminology. In dealing with graphs, we will use the following terminology. The *valence* of a vertex in an abstract graph will be defined as the number of edges attached to that vertex. When a vertex of a tree has valence one, it will be called an *end vertex*. The edge to which it is attached will be called an *end edge*. Edges which share a common vertex are said to be *adjacent edges*. As well, vertices which are the endpoints of some edge are called *adjacent vertices*.

2.2.9. Signed tree exhibited by an elliptic form disk. We will say the embedding $\mathcal{T}$ of a tree in the plane is *signed* if there is a map from the set of vertices of $\mathcal{T}$ to $\{+,-\}$ such that adjacent vertices have opposite sign. Such a map will be called a *signing* of $\mathcal{T}$.

Suppose $\mathcal{T}$ and $\mathcal{T}'$ are signed embeddings of abstract trees in the plane. We will say they are *equivalent* if there is a homeomorphism of the plane which takes one tree to the other and respects the signings.

Recall that the skeleton of an elliptic form disk D was defined as a Legendrian embedding ϕ of an abstract tree T into D. By using an embedding φ such that D is the image under φ of the standard unit disk, we obtain an embedding $\varphi^{-1} \circ \phi$ of T into the plane. It has a natural signing: that given by the signs of singularities in D_ξ. Observe, that the equivalence class of this signed embedded tree is well-defined by the topology of D_ξ, and does not depend on the choice of φ. The same is true for the extended skeleton. Given a signed embedding of a tree in the plane, we will say it is *exhibited* by an elliptic form disk D if it is in the equivalence class determined by the *extended skeleton* of D_ξ, and we will call this equivalence class the *extended skeleton type* of D.

2.2.10. Acceptable tree embeddings. Let $\mathcal{T}$ be a planar embedding of an abstract tree. We will say $\mathcal{T}$ is *acceptable* if it has the following properties:

1) it has at least one edge,
2) all edges are straight segments with slope between $\pm\epsilon$,
3) at any given vertex, there is at most one edge attached on the left side of that vertex, all others are attached on the right (this implies there is a *left-most vertex*),
4) the left-most vertex is an end vertex (this implies there is a *left-most edge*).

3. Proof of the main theorem

3.1. General scheme of proof. We now outline our strategy for proving the main theorem. Overall, we will show any two Legendrian unknots with a given value of (tb, r) are isotopic to a lift of the (unique) catalog front with that value (and so are isotopic to each other). This will be broken into the following steps.

1) Construct from any signed acceptable tree embedding $\mathcal{T}$ a wavefront $\mathcal{W}_T$ with the property that its Legendrian lift has an elliptic form spanning disk exhibiting $\mathcal{T}$ (Sections 3.2.1, 3.2.2, 3.3.1 and 3.3.2).
2) Establish a transversal homotopy from any wavefront obtained by the above construction[12] to the (unique) catalog front with the same value of (tb, r) (Lemma 3.1).
3) Show that all Legendrian knots having elliptic form spanning disks exhibiting the same embedded tree are Legendrian isotopic; i.e., the extended skeleton type of elliptic form spanning disk determines the Legendrian isotopy class of the (Legendrian) boundary (Lemma 3.2).

It is possible to avoid the use of wavefronts by completely standardizing the disk foliation (see [**35**]), but then a more general version of step (2) is required to finish the proof; namely, that any two Legendrian knots with diffeomorphic spanning-disk foliation (not necessarily of elliptic form) are Legendrian isotopic. This is shown in [**35**] using a 1-parametric version of results from [**20**], which although formulated in [**20**] is not proved there. Thus we have decided instead to follow the method of proof outlined above. The use of wavefronts in the present proof is similar to, though more elaborate than, the use of wavefronts in [**19**] (transverse knot version of our theorem).

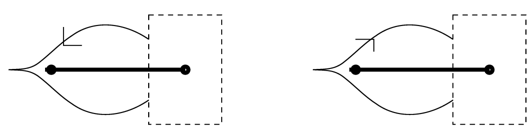
[12]These will be called *tree-based* wavefronts.

3.2. Wavefront arguments.

3.2.1. Front construction algorithm. Suppose we are given a signed acceptable tree embedding $\mathcal{T}$. The algorithm below constructs from $\mathcal{T}$ a wavefront that will be denoted $\mathcal{W}_T$.

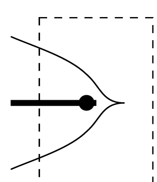
Choose a small neighborhood around each vertex, so that no neighborhoods overlap. In the diagrams below, the thick lines represent edges, the thin curves represent fronts and the dotted boxes represent these vertex neighborhoods. For each edge, we will refer to the subsegment between the two endpoint neighborhoods as the *open edge*. The algorithm below steps through all vertices, setting v equal to the current vertex and then defining a portion of front corresponding to v's neighborhood and any open edges attached to v on the right. We assume these portions of front are always connected to each other smoothly.

Anchor step. Let v be the left-most vertex of $\mathcal{T}$. If v is positive use the first front below and if v is negative use the second front to define the portion of $\mathcal{W}_T$ corresponding to v's neighborhood and to the (single) attached open edge. Now set v to be the right endpoint of this edge and *go to the induction step.*

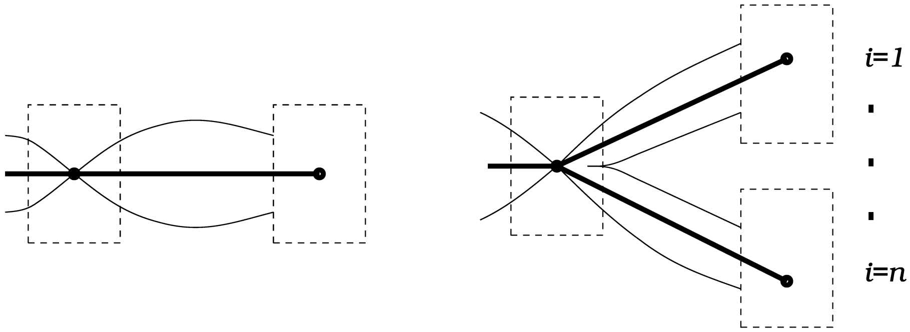


Induction step. Let n be the number of edges attached to v on the right side. We will define the portion of $\mathcal{W}_T$ corresponding to v's neighborhood and to these n open edges.

If $n = 0$, then v is an end vertex. Use the following front, and then *end the induction.*



If $n \neq 0$, then v is not an end vertex. Replace the subtree that is attached to the right of v with a new subtree that is obtained from the old by reflecting in the horizontal axis. Then, if $n = 1$ use the front at left, if $n > 1$ use the front at right. Now, number the n edges attached to v on the right sequentially from $1 \cdots n$ starting at the top. For each $i = 1 \cdots n$, set v to the right endpoint of the ith edge and *repeat the induction step* for this v.



It is useful to note that as a result of this construction: a cusp in $\mathcal{W}_T$ corresponding to a positive (end) vertex in $\mathcal{T}$ is oriented downward, as are both arcs of a self-intersection point (i.e., front crossing) corresponding to a positive (non-end) vertex. The opposite is true for negative vertices.

3.2.2. Tree-based wavefronts. Define a *tree-based front* to be any front that is obtained as $\mathcal{W}_T$ for some signed acceptable tree embedding $\mathcal{T}$.

Suppose T is an abstract tree with the property that at most one of its vertices has valence greater than two, and at most one of the edges attached to this vertex is a non-end edge. We will then say T is *almost linear.* Note that the class of catalog fronts is exactly the class of tree-based fronts of the form $\mathcal{W}_T$, such that $\mathcal{T}$ is a signed acceptable embedding of an almost linear tree.

3.2.3. Simplifying fronts. A transversal homotopy of fronts lifts to a Legendrian isotopy between the corresponding Legendrian knots. Indeed, this follows from the fact that we can canonically lift each transitional front in the homotopy of fronts and this lift (a Legendrian knot) will be embedded iff each self-intersection point of the corresponding front is transversal (recall from Section 1.2).

Lemma 3.1. *Any tree-based front can be transversally homotoped to the catalog front with that value of* (tb, r).

Proof. Consider two signed planar tree embeddings $\mathcal{T}$ and $\mathcal{T}'$. Suppose $\mathcal{T}'$ is obtained from $\mathcal{T}$ by moving an end edge from one positive (resp. negative) vertex to another positive (resp. negative) vertex. We claim there is a transversal homotopy of wavefronts taking $\mathcal{W}_T$ to $\mathcal{W}_{\mathcal{T}'}$. This proves the lemma, since any tree $\mathcal{T}$ can be reduced to an almost linear tree by a sequence of such moves[13]. To verify the claim, consider the subtree $\mathcal{S}$ which is the common part of $\mathcal{T}$ and $\mathcal{T}'$, let e be the edge appearing only in $\mathcal{T}$ and

[13]We establish (as stated) only movement of an edge between *same sign* vertices, and with such a tool can only reduce a signed tree to *almost* linearity — not necessarily linearity.

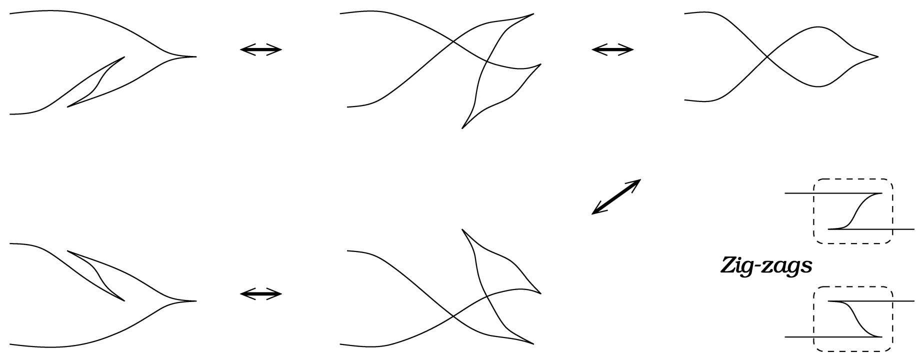


Figure 18. Transverse homotopy of fronts for proof of Lemma 3.1.

let e' be the edge only in $\mathcal{T}'$. Let us designate the two vertices v and w (in all three trees), such that e is attached to v and e' is attached to w. If neither of these is an end vertex in $\mathcal{S}$, then note that $\mathcal{W}_T$ and $\mathcal{W}_{T'}$ differ only by the placement of an upward (resp. downward) oriented *zig-zag*[14]. If, on the other hand, either of v and w is an end vertex in $\mathcal{S}$, then apply the transversal homotopy of fronts shown in Figure 18 — near v in $\mathcal{W}_T$, near w in $\mathcal{T}'$ (start at right and go to either top left or bottom left). The two resulting fronts then differ only in the placement of an upward (resp. downward) oriented zig-zag, (while $\mathcal{W}_S$ is identical to both fronts with the zig-zag removed). So to prove the claim, it suffices to show that we can move an upward (resp. downward) zig-zag from any position in a connected front to any other by a transversal front homotopy. Clearly, we can displace the zig-zag past any transversal crossing in this manner, so we need only check whether we can also displace it past cusps. For an upward (resp. downward) cusp, this is clear. The same transversal homotopy of fronts just given above shows we can do so for downward (resp. upward) cusps as well (start at left, go to right and then back to left). □

3.3. Spanning disks.

3.3.1. Exceptional spanning disks and elliptic form disks. Let D be an embedded disk in (M, ξ) with Legendrian boundary L. If the characteristic foliation D_ξ can be decomposed into regions of the types shown in Figure 19, allowing each singularity curve to have one interior corner (point of non-smoothness), then we will say D is an *exceptional spanning disk* for L. Note that if a spanning disk is in elliptic form, then we can convert it to an exceptional spanning disk, by application of Lemma 2.4 (using the decomposition into type(a) and type(b) sectors that we have for elliptic form foliations), and vice versa. We will also, by analogy, speak of the *extended skeleton exhibited by an exceptional spanning disk.*

[14](see Figure 18).

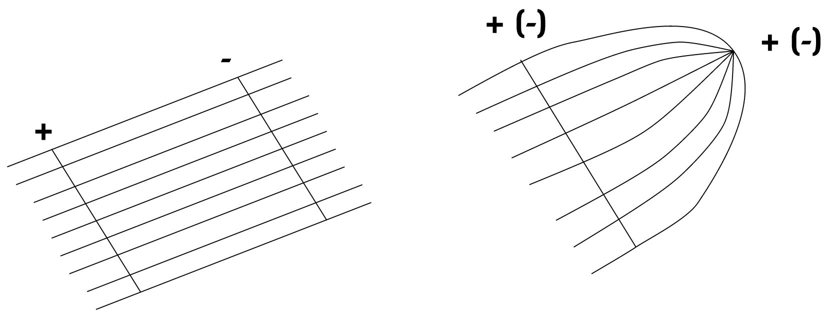


Figure 19. Types of region composing an exceptional spanning disk.

3.3.2. Exceptional spanning disk for the lift of $\mathcal{W}_{\mathcal{T}}$. Suppose we are given a tree-based front $\mathcal{W}_{\mathcal{T}}$ with Legendrian lift $\mathcal{L}$. We will define an exceptional spanning disk D for $\mathcal{L}$ by specifying D_ξ, the family of Legendrian curves that foliate D. This can be done by giving the corresponding family of wavefronts for these Legendrian curves. Since $\mathcal{W}_{\mathcal{T}}$ was defined (in the front construction algorithm of Section 3.2.1) portion by portion for the edges of $\mathcal{T}$, we specify the relevant wavefronts in this same manner, namely we do so for each type of wavefront portion created in that algorithm. This is given in Figure 20.

3.3.3. Spanning disks exhibiting the same extended skeleton.

Lemma 3.2. *Suppose Legendrian knots L and L' bound D and D' with diffeomorphic characteristic foliations in elliptic form. Then L and L' are Legendrian isotopic.*

Proof. First, convert the elliptic form spanning disks to exceptional spanning disks (as mentioned in Section 3.3.1). Call these new disks also D and D'. Because the extended skeletons T and T' exhibited by D and D' are diffeomorphic there exists, according to Lemma 1.3 a global contact isotopy taking T' to T. We may assume the surfaces coincide in a small neighborhood R of this common extended skeleton. Written right on each surface, there is an isotopy supported in the complement of small neighborhoods of the end vertices which brings the portion of the Legendrian knot outside these neighborhoods arbitrarily close to the skeleton. This is indicated in Figure 21.

Suppose that the relevant portion of L and also of L' are each isotoped far enough that they reach the region R (around the common extended skeleton) where D and D' coincide. By applying the elliptic pivot lemma (i.e., Lemma 1.4), we may extend each of these isotopies to the neighborhoods of end vertices. We thus obtain Legendrian isotopies taking L and L' to the same Legendrian knot. Thus L and L' are Legendrian isotopic. □

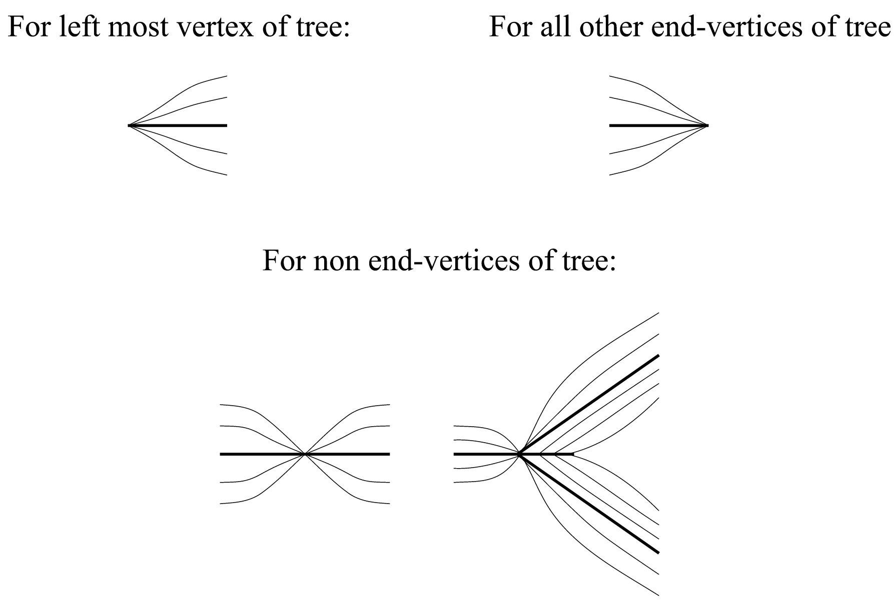


Figure 20. Construction of an exceptional spanning disk D for the lift of $\mathcal{W}_{\mathcal{T}}$ (by specifying wavefront projection of D_ξ on each portion of $\mathcal{W}_{\mathcal{T}}$ created in the algorithm of Section 3.2.1).

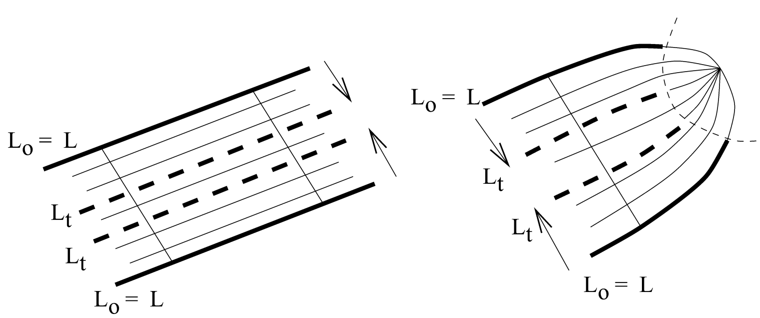


Figure 21. The isotopy written on an exceptional spanning disk.

3.4. Proof of main theorem.

Proof of Theorem 1.5. Given a Legendrian unknot L in tight (M, ξ), let D be an oriented embedded disk with $\partial D = L$. By applying Lemma 2.5 followed by Lemma 2.7, we obtain a spanning disk D' for L that is in elliptic form. Let $\mathcal{T}$ be an acceptable signed planar tree embedding that is in the extended skeleton type of D'. Let $\mathcal{W}_T$ be the tree-based front constructed from $\mathcal{T}$ by the algorithm of Section 3.2.1. Let $\mathcal{L}$ be a Legendrian lift of $\mathcal{W}_T$

to a Darboux neighborhood in (M,ξ). Then L and $\mathcal{L}$ are Legendrian isotopic by Lemma 3.2 since $\mathcal{L}$ has an elliptic form spanning disk that exhibits the same $\mathcal{T}$ as does the disk D'. Now, by Lemma 3.1, all lifts like $\mathcal{L}$ having a given value of (tb, r) are Legendrian isotopic to any lift of the (unique) catalog front with that value of (tb, r). So, any Legendrian unknot having the same value of (tb, r) as L has will be Legendrian isotopic to L. □

4. Legendrian knots in overtwisted contact 3-manifolds

Knots can be classified up to isotopy or up to global diffeomorphism. In the classical case of knots in $\mathbb{R}^3$, these two problems are equivalent because the group of compactly supported diffeomorphisms of $\mathbb{R}^3$ is connected (in fact even contractible, see [**47**]). The same is true for Legendrian knots in tight (i.e., standard) contact $\mathbb{R}^3$. Indeed, according to [**20**], the group of compactly supported contact diffeomorphisms of the standard contact $\mathbb{R}^3$ is connected as well. However, as it was observed by Dymara (see [**14**, **16**] and Corollary 4.15 below), the group of coorientation-preserving contactomorphisms of any closed overtwisted contact manifold is disconnected.

We consider in the next sections the problem of *coarse* classification of Legendrian knots in an overtwisted contact manifold, i.e., the problem of classification of Legendrian knots up to a global, coorientation-preserving contact diffeomorphism. The status of the Legendrian isotopy problem is discussed in Section 4.3 below.

4.1. Coarse classification of loose knots. Let (M,ξ) be an overtwisted contact manifold. A Legendrian knot (or link) $L \subset M$ is called *loose* if the restriction of the contact structure ξ to the complement $L^c = M \setminus L$ is still overtwisted. Otherwise, i.e., if $\xi|_{M\setminus L}$ is tight, we will call L *exceptional*.

One can immediately observe the following examples of loose knots.

Lemma 4.1. a) *Suppose that (M,ξ) is a non-compact manifold overtwisted at infinity,*[15] *then any Legendrian link $L \subset M$ is loose.*

b) *Let (M,ξ) be any overtwisted contact manifold and L a topologically trivial Legendrian knot with* $\mathrm{tb}(L) \le 0$, *then L is loose.*[16]

Proof. Part (a) is clear. If $\mathrm{tb}(L) = 0$, then the image of L under Reeb flow for a short time has linking number 0 with L, so it spans an overtwisted disk in L^c. Now suppose $\mathrm{tb}(L) < 0$. Then L has a spanning disk D with NAF boundary. If L^c were tight, then we could standardize the disk interior too, and thus obtain standard foliation on the whole closed disk. This would imply a neighborhood B of D is isomorphic to a neighborhood of a standard

[15] i.e., overtwisted outside of any compact set. It is proven in [**23**] that $\mathbb{R}^3$, for instance, has a unique overtwisted at infinity contact structure.

[16] More constraints on exceptional knots follow from Theorem 4.7 below.

disk (as listed in the catalog), and hence is tight. But D^c is also a tight ball, so then all of M would be tight. □

Remark 4.2. In our earlier paper [**27**], we claimed without a proof a stronger version of Lemma 4.1, that any homological to 0 Legendrian knot in any contact manifold is loose if it violates the Bennequin inequality 1.6. This is wrong even for topologically trivial knots, as our Theorem 4.7 below shows.

The problem of coarse classification of loose Legendrian knots is of pure homotopical nature:

Proposition 4.3. *Let $L_1, L_2 \subset (M, \xi)$ be two loose Legendrian knots. Suppose that there exists a diffeomorphism $f : M \to M$ which sends L_1 to L_2, $\xi_1|_{L_1}$ to $\xi_2|_{L_2}$ and such that the plane fields ξ_2 and $df(\xi_1)$ are homotopic on $M \setminus L_2$ relative to the boundary. Then L_1 and L_2 are coarsely equivalent.*

Corollary 4.4. *Two topologically trivial loose Legendrian knots in an overtwisted contact manifold (M, ξ) are coarsely equivalent if and only if they have the same values of* tb *and r.*

Remark 4.5. If the Euler class $e(\xi)$ does not vanish, then, unlike in the tight case, the definition of $r(L)$ is ambiguous and may depend on a choice of a disk D spanning the knot L, $r(L) = r(L|D)$, see Section 1.5.1. The above corollary should be understood in the sense that if two knots have the same tb, and there are spanning disks such that $r(L_1|D_1) = r(L_1|D_2)$, then there exists a coarse equivalence which sends D_1 onto D_2.

Proof of Proposition 4.3. By the Darboux–Weinstein theorem (see Section 1.1 above) we can assume that the diffeomorphism f sends ξ_1 to ξ_2 on a neighborhood $U_1 \supset L_1$ and $U_2 = f(U_1) \supset L_2$. Then the contact structures ξ_2 and $\widetilde{\xi_1} = df(\xi_1)$ coincide on the boundary of U_2 and are homotopic as plane fields via a homotopy fixed on U_2. Hence, according to the classification of overtwisted contact structures in [**22**] (see also Theorem 4.13 below), there exists an isotopy $h_t : M \to M, t \in [0, 1]$, which is fixed on U_2 and such that $h_0 = \mathrm{Id}$, $dh_1(\widetilde{\xi_1}) = \xi_2$. Then the composition $h_1 \circ f$ is a contactomorphism $(M, \xi_1) \to (M, \xi_2)$ as required which sends L_1 to L_2. □

Remark 4.6. Fuchs and Tabachnikov informed us that they independently observed a similar result. A related result in the context of Legendrian isotopy classification of loose knots is proved by Dymara in [**14**] under the additional assumption that there exists an overtwisted disk not meeting either $L_i, i = 1, 2$, see Corollary 4.15.2 below.

4.2. Coarse classification of exceptional knots. Topologically trivial *exceptional* knots in an overtwisted contact S^3 can be completely coarsely classified using the theorem of Giroux–Honda which classifies tight contact structures on solid tori. Let us recall that according to [**22**], (positive) overtwisted contact structures on S^3 are classified by the homotopy class of the corresponding cooriented plane field, which is in turn defined by its *Hopf invariant*. Namely, let us fix a trivialization of TS^3, say by the frame $ip, jp, kp \in T_pS^3, p \in S^3$, where we view S^3 as the unit sphere in $\mathbb{R}^4$ identified with the quaternion space $\mathbb{H}$. Then any cooriented plane field gives a map $S^3 \to S^2$, and we denote by $h(\xi)$ its Hopf invariant, i.e., the linking number of properly oriented preimages of two regular points in S^2. In particular, the Hopf invariant of the standard contact structure ζ orthogonal to the Hopf fibration is equal to 0. We will denote by ξ_h the unique positive overtwisted contact structure with the Hopf invariant h.[17]

Let us recall that according to Lutz (see [**53**]), any plane homotopy class of contact structures can be obtained from a given one by Lutz twists. Let us compute the Hopf invariant of the overtwisted contact structure ξ obtained from ζ by the π-Lutz twist along one of the fibers, say $F = \{e^{it}, t \in \mathbb{R}/2\pi\mathbb{Z}\}$, of the Hopf fibration. Note that the linking number of two Hopf fibers is equal to $+1$. Take a trivialization $U = D^2 \times S^1$ of the Hopf fibration near one of the fibers. Then the normal vector field $jp, p \in F$ rotates -2 times with respect to the constant framing given by the trivialization (this is why the transversal unknot represented by the fiber has self-linking number equal to -1). Suppose we perform the π-Lutz twist of the contact structure ζ along F. Given any unit vector field $\mathbf{v} = a(ip)+b(jp)+c(kp)$, $p \in F$, with constant coefficients in the basis $ip, jp, kp \in T_pS^3$, we denote by $\Gamma_{\mathbf{v}}$ the set of points in S^3 where the normal vector field to the contact structure points in the direction of $\mathbf{v}$. Then $\Gamma_{\mathbf{v}}$ is the preimage of the point $(a, b, c) \in S^2 \subset \mathbb{R}^3$ under the Gauss map $S^3 \to S^2$ which corresponds to the contact structure. Clearly, if $b^2 + c^2 \neq 0$, then $\Gamma_{\mathbf{v}}$ is a circle spiraling around F minus 2 times, exactly as does the normal vector field $jp, p \in F$. The Hopf invariant $h(\xi)$ is by definition the linking number of $\mathrm{lk}(\Gamma_{\mathbf{v}}, \Gamma_{\mathbf{v}'})$ for two different vector fields $\mathbf{v}$ and $\mathbf{v}'$. Note that by a continuity argument all these spirals should be coherently oriented. On the other hand, $\Gamma_{\mathbf{v}'}$ is isotopic to F in the complement of $\Gamma_{\mathbf{v}}$, and hence

$$h(\xi) = \mathrm{lk}(\Gamma_{\mathbf{v}}, \Gamma_{\mathbf{v}'}) = \mathrm{lk}(\Gamma_{\mathbf{v}}, F) = 1 - 2 = -1,$$

i.e., $\xi = \xi_{-1}$.

Note that given a k-component link $L = L_1 \cup \cdots L_k \subset S^3$ of transversal knots in ζ with self-linking numbers l_i, $i = 1, \ldots, k$, simultaneous π-Lutz twists along all components of L produce an overtwisted contact structure

[17] It is also customary in the contact geometric literature to use, instead of the Hopf invariant, the so-called d_3-invariant, introduced by Gompf in [**42**], see also [**13**]. For S^3, the invariants d_3 and h are related by the formula $d_3 = -h - \frac{1}{2}$.

ξ with

$$h(\xi)=\sum_{1}^{k} l_i+2\sum_{1\le i<j\le k} l_{ij},$$

where $l_{ij}=\mathrm{lk}(L_i,L_j)$ is the linking number of positively oriented transversal curves L_i and L_j. In particular, one can observe[18] that the Lutz twist along k fibers of the Hopf fibration produces a contact structure with the Hopf invariant $k(k-2)$.

Theorem 4.7. *The contact manifold* (S^3,ξ_h) *admits an exceptional unknot if and only if* $h=-1$. *Moreover, exceptional unknots are classified up to coarse equivalence by the invariants* tb *and* r *and the following is a complete list of equivalence classes:* $(\mathrm{tb},r)=(1,0)$, $(\mathrm{tb},r)=(n,\pm(n-1))$ *for positive integer* n.

This confirms Conjecture 41 of [EtN], which predicted ξ_{-1} to be the only contact structure on S^3 for which exceptional unknots exist. It also implies a corrected version of Conjecture 42 of [EtN]. This conjecture claimed a reduced set of possible values of (tb,r). John Etnyre informed us that jointly with T. Vogel they independently proved Theorem 4.7.

Before proving Theorem 4.7, we need to develop some necessary preliminary information and recall some known facts.

4.2.1. Neighborhoods of Legendrian knots. Let ξ be a contact structure in a contact manifold (M,ξ), and $L\subset M$ a Legendrian knot. Let (x,y,z), $y,z\in\mathbb{R}$, $x\in\mathbb{R}/2\pi\mathbb{Z}$, be the canonical Darboux coordinates in a neighborhood of L, so that the contact structure ξ on this neighborhood is given by the contact 1-form $dz-y\,dx$ and L is given by equations $y=z=0$. Passing to cylindrical coordinates

$$(x,r,\theta)\mapsto(x,r\cos\theta,r\sin\theta),\quad x,\theta\in\mathbb{R}/2\pi\mathbb{Z},r\in[0,\infty),$$

we get

$$dz-y\,dx=r\cos\theta\,d\theta+\sin\theta\,dr-r\cos\theta\,dx,$$

and $L=\{r=0\}$. Note that the characteristic foliation on the torus $T_\varepsilon=\{r=\varepsilon\}$ is given by the form $\cos\theta(d\theta-dx)$, and thus T_ε is ruled by Legendrian curves $\theta=x+\mathrm{const}$, and has two *singularity curves* $\theta=\pm\frac{\pi}{2}$, where the contact structure is tangent to the torus (see Section 2.1.1 above for the definition of singularity curves).[19]

[18] This was pointed out to us by Giroux.

[19] Note that T_ε is convex in the sense of [**28**], i.e., admits a transversal contact vector field (e.g., the field $Y=z\frac{\partial}{\partial z}+y\frac{\partial}{\partial y}$). The *dividing curves* of a convex surface, defined by Giroux (see [**40**]), are the sets of points where the contact vector field is tangent to the contact plane field. They are, up to isotopy, independent of a choice of the contact vector field (and for Y are given by the equation $\theta=0,\pi$). Hence the number of dividing curves and their slopes are respectively the same as those of the singularity curves.

We say that a torus T with a fixed coordinate system $x, \theta \in \mathbb{R}/2\pi\mathbb{Z}$ has a *Legendrian ruling with the slope* λ if it has a foliation by Legendrian curves which is isotopic to the linear foliation with the slope λ. For a rational number $\lambda = \frac{p}{q}$, the torus T is ruled by closed Legendrian curves in the class of the curve $qx = p\theta$. In particular, the torus T_ε is ruled by Legendrian curves with the slope 1.

Lemma 4.8. 1) *For any function* $\phi : \mathbb{R}/2\pi \to (0,\infty)$ *such that* $\phi'(\pm\frac{\pi}{2}) = 0$, *the torus* $T_\phi = \{r = \phi(\theta)\}$ *has two singularity curves* $\theta = \pm\frac{\pi}{2}$ *and admits a Legendrian foliation with a slope* $\lambda = \lambda(\phi)$.

2) *For any number* μ, *there exists a function* $\phi : \mathbb{R}/2\pi \to (0,\infty)$, *arbitrarily* C^0*-close to* ε *which satisfies* $\phi'(\pm\frac{\pi}{2}) = 0$ *and such that* $\lambda(\phi) = \mu$.

Proof. The characteristic foliation defined in coordinates (θ, x) by the equation

$$(\phi\cos\theta + \phi'\sin\theta)d\theta - \phi\cos\theta\, dx = 0, \tag{4.1}$$

or equivalently

$$\frac{dx}{d\theta} = 1 + \psi'\tan\theta,$$

where $\psi(\theta) = \ln\phi(\theta)$. The condition $\phi'(\pm\frac{\pi}{2}) = 0$ ensures that T_ϕ has two singularity curves $\theta = \pm\frac{\pi}{2}$ and that the Legendrian foliation transversely intersect these curves. To prove the second part of the lemma, it remains to find a periodic ψ, which is C^0-close to the constant $\ln\varepsilon$ which satisfies the condition

$$\frac{1}{2\pi}\int_0^{2\pi}(1 + \psi'\tan\theta)d\theta = \mu. \tag{4.2}$$

Let us consider the function $\alpha_{\delta,\sigma}(u) = \delta\left(-1 + \left(\frac{u}{\sigma}\right)^4\right)$. Then $-\delta \le \alpha_{\delta,\sigma}(u) \le 0$ for $u \in [-\sigma, \sigma]$ and $\alpha_{\delta,\sigma}(\pm\sigma) = 0$. On the other hand,

$$\int_{-\sigma}^{\sigma}\frac{\alpha'_{\delta,\sigma}}{u}du = \frac{8\delta}{3\sigma}\underset{\sigma\to 0}{\to}\infty.$$

Note that for small u we have $\tan(\frac{\pi}{2} + u) \simeq -u$. Take now a continuous piecewise-smooth 2π-periodic function

$$\psi(u) = \begin{cases} \ln\varepsilon + \alpha_{\delta,\sigma_1}\left(u - \dfrac{\pi}{2}\right), & u \in \left[\dfrac{\pi}{2} - \sigma_1, \dfrac{\pi}{2} + \sigma_1\right], \\ \ln\varepsilon - \alpha_{\delta,\sigma_2}\left(u - \dfrac{3\pi}{2}\right), & u \in \left[\dfrac{3\pi}{2} - \sigma_2, \dfrac{3\pi}{2} + \sigma_2\right], \\ \ln\varepsilon, & \text{elsewhere}, \end{cases}$$

and smooth its corners. Then choosing appropriate sufficiently small σ_1, σ_2 and δ, we can arrange that the integral (4.2) takes an arbitrary value. while the function ψ is C^0-close to $\ln\varepsilon$ and satisfies the condition $\phi'(\pm\frac{\pi}{2}) = 0$. $\square$

Note that the Legendrian foliation on T_ϕ is transversal to the vector field ∂_x, and hence we can orient it by the coordinate θ. We will call this orientation *canonical.*

Lemma 4.9. *Let ϕ be a function as in Lemma* 4.7.1. *Consider a vector field X tangent to the Legendrian foliation on T_ϕ, and which defines the canonical orientation of T_ϕ. Then X extends to the solid torus $U_\phi = \{r \le \phi\}$ as a non-vanishing vector field tangent to the contact structure ξ and the core Legendrian curve L, and defines the given orientation of L.*

Proof. For a positive number $\sigma < \min \phi$, we denote $\phi_t := \sigma + t(\phi - \sigma)$. Then tori T_{ϕ_t} foliate the domain $U_{\sigma,\phi} = \{\sigma \le r \le \phi\}$. Each of these tori admits a Legendrian foliation as in Lemma 4.8.1. Hence the vector field X extends to $U_{\sigma,\phi}$ as the vector field tangent to the Legendrian ruling on each of the tori T_{ϕ_t}. It further extends to the solid torus $U_\sigma = \{r \le \sigma\}$ as the vector field tangent to the Legendrian ruling on round tori T_r, $0 \le r \le \sigma$. □

For what follows, we will need only tori with rational slopes of the form $\lambda = -\frac{1}{n}$, $n > 0$, and will denote by U_n the neighborhood $\{r \le \phi(\theta)\}$ with $\lambda(\phi) - \frac{1}{n}$. Any two such neighborhoods are related by a contact isotopy which fixes L.[20]

Consider a solid torus $Q = D^2 \times S^1$ with the fixed coordinates $t, u \in \mathbb{R}/2\pi\mathbb{Z}$ the boundary torus $\partial D^2 \times S^1$, where $t \in \partial D^2$ and $u \in S^1$. We say that a contact structure ξ on N is in the *standard form* near T with the *boundary slope* $\frac{p}{q} \ne 0$ if it is ruled by Legendrian curves in the class of the meridian, and has two singularity curves with the slope $\frac{p}{q}$. The contact structure ξ near T can be defined by a contact form λ such that $\lambda|_T = \cos(u - \frac{p}{q}t)du$.

Lemma 4.10. *Let L be a topologically trivial Legendrian knot in S^3 with a contact structure ξ. Suppose* $\mathrm{tb}(L) = n$. *Let U_n be a standard neighborhood defined above.*

1) *Then the complementary torus $N^c = S^3 \setminus \mathrm{Int}\, U_n$ is in the standard form near $T = \partial N^c = \partial U_n$, and the singular curves on its boundary have the slope n with respect to the coordinate system given by the meridians of the solid tori N^c and U_n.*
2) *Let L' be any of Legendrian curves which form the Legendrian ruling. Let us orient L' so it represents the same homology class of U_n as L. Then $r(L') = -r(L)$.*

Proof. 1. Let us consider the lattice $\Lambda = \mathbb{Z} \oplus \mathbb{Z}$ on the plane $\mathbb{R}^2$ with coordinates (θ, x). By construction, the boundary torus $T = \partial U_n$ is identified with $\mathbb{R}^2/2\pi\Lambda$. Let $e_\theta = (1,0), e_x = (0,1)$ be the vectors of the basis. The vector

[20] It is interesting to notice that the tori with different slopes μ_1 and μ_2 are contactomorphic (via a Dehn twist along the singularity curve) if and only if $\mu_1 - \mu_2 \in \mathbb{Z}$. In particular, the boundary tori ∂U_n and ∂U_m are not contactomorphic if $m \ne n$.

e_θ corresponds to the meridian of U_n, while e_x directs the singularity curve. Let μ denote a vector corresponding to the meridian of the complementary torus N^c. We choose μ in such a way that the pair (e_θ, μ) be a positive basis of the lattice. Hence $e_\theta \wedge \mu = 1$. On the other hand, by the definition of the Thurston–Bennequin invariant, we must have $e_x \wedge \mu = n$. Hence, μ is the vector $-ne_\theta + e_x$, which is tangent to the ruling direction, as required. Vectors μ, e_τ define a positive basis with respect to the orientation of T as the boundary of N^c, and we have $e_x = \mu + ne_\theta$, i.e., the singularity curve has the slope n.

2. First note that the standard orientation of the ruling is opposite to the homological orientation as in the statement of the lemma. Let us recall that the rotation number $r(L)$ is the obstruction for extending the tangent vector field to L as a non-vanishing vector field tangent to ξ along a surface spanning L in S^3. By assumption, L' spans a disk in the complement of U_n. On the other hand, the vector field tangent to L and L', but defining opposite homological orientations of L and L', extends to U_n as the vector field tangent to ξ. Hence, $r(L') = -r(L)$. □

4.2.2. Special contact structure on the solid torus. We explicitly describe here special tight contact structures on the solid torus which we will need in the proof of Theorem 4.7.

Given $n \in \mathbb{N}$, let $\mathcal{F}_n$ be a characteristic foliation on the disk $D = D^2$ such that ∂D is Legendrian with $2n$ hyperbolic singularities of alternating signs and there are one central negative elliptic singularity and n positive elliptic singularities only on the disk away from the boundary (see Figure 22). Moreover, assume that positive hyperbolic singularities on ∂D are sources of the foliation along ∂D and that polar coordinates (ρ, α) are given on the disk such that the foliation has rotational symmetry group $\mathbb{Z}_n$ and only the $2n$ Legendrian curves drawn radially, attaching boundary hyperbolic points to the center, are actually ever radial. Note this disk may be realized as a disk with Legendrian boundary having $(\mathrm{tb}, r) = (-n, n-1)$.[21] In fact, its characteristic foliation is up to orientation the only possible $\mathbb{Z}_n$ rotationally symmetric foliation of a disk with $\mathrm{tb} = -n$ boundary that can occur in a tight manifold. This disk is convex, according to [**40**], i.e., it admits a transversal contact vector field. It then follows that there exists a vertically invariant tight contact structure $\tilde{\xi}$ on $Z = D \times \mathbb{R}$ such that $D_{\tilde{\xi}} = \mathcal{F}_n$. Note that we must have $\partial_t \in \tilde{\xi}_p$ at some point p along each Legendrian arc between opposite sign boundary hyperbolic points. In constructing $\tilde{\xi}$, we may begin with a germ of contact structure on D such that this verticality of the contact structure occurs exactly once on each such arc. This means the

[21] Alternatively the tightness of the germ of the contact structure inducing this characteristic foliation follows from Giroux's criterion: that the dividing multicurve of the surface has no closed components, see [**40**].

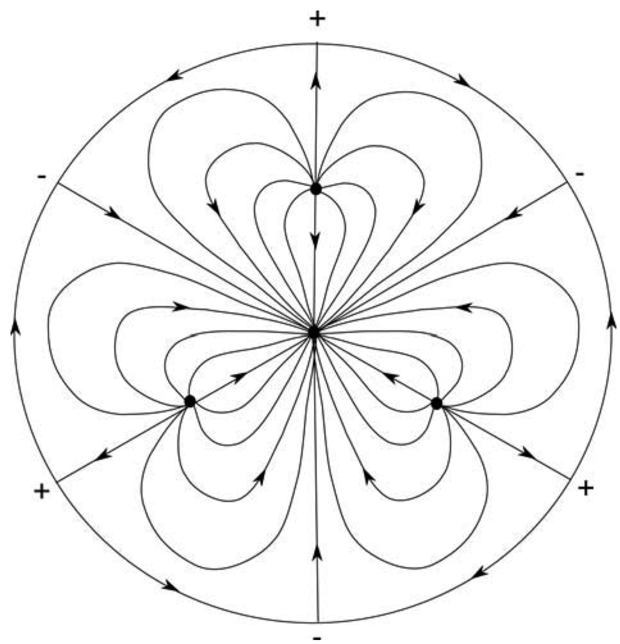


Figure 22. $\mathbb{Z}_n$ rotationally symmetric characteristic foliation $\mathcal{F}_n$ of a disk D, for the case $n = 3$.

characteristic foliation on ∂Z has singularity curves over these points and no other singularities. We may further assume the germ of contact structure on D to have $\mathbb{Z}_n$ rotational symmetry, so that all of $\tilde{\xi}$ does as well.

Now let ϕ be the composition of translation in the vertical direction by 1 unit and rotation of the disk by $\frac{2\pi}{n}$. Note that $\tilde{\xi}$ is invariant under ϕ. Let ζ_n^- be the natural contact structure induced by $\tilde{\xi}$ on the quotient $Q = Z/_\phi$. Then (Q, ζ_n^-) is tight since it is covered by $(Z, \tilde{\xi})$. Note that the foliation on its boundary still has exactly two singularity curves.

We can do a similar construction, reversing the orientation of the disk D, i.e., beginning with the disk with one central positive elliptic singularity and n negative elliptic singularities, and thus with $(\mathrm{tb}, r) = (n, 1 - n)$. We denote that resulting tight contact structure on Q by ζ_n^+.

In either case, we denote by $\mathbf{t}$ the contact vector field in Q which is the image of the vertical vector field ∂_t under the quotient map $D \times \mathbb{R} \to Q$.

4.2.3. Giroux–Honda's classification of tight contact structures on the solid torus. The following theorem is an extract from Giroux–Honda's classification of tight contact structures on the solid torus (see [**41**] and [**48**]).

Theorem 4.11 (Giroux, Honda). 1) *Let ξ and ξ' be tight contact structure on a solid torus N which are in the standard form near the boundary with the boundary slope $k \neq 0$. Suppose that the Legendrian meridian μ has* *the same rotation number for both contact structures. Then ξ and ξ' are diffeomorphic by a diffeomorphism fixed on the boundary $T = \partial N$.*

2) *Let ξ be a tight contact structure on a solid torus N which is in the standard form near the boundary with integer boundary slope $n \neq 0$. Then the rotation number $r(\mu)$ of the Legendrian meridian on its boundary is equal to $\pm(n-1)$. Moreover, both these values of r (the unique value 0 if $n = 1$) are realizable by tight contact structures.*

Note that the contact structures whose existence is claimed in Theorem 4.11.2 must by Theorem 4.11.1 be the contact structures $\zeta_n^\pm$ explicitly constructed in step 4 above.

4.2.4. Proof of Theorem 4.7. Let L be an exceptional Legendrian knot in a contact (S^3, ξ). Then, according to Lemma 4.1(b), we have $\mathrm{tb}(L) = n > 0$. Let $r = r(L)$. Consider the standard neighborhood $U_n \supset L$. Then, according to Lemma 4.10.1, the boundary $T = \partial U_n$ is ruled by Legendrian curves in the class of the meridian of the complementary torus $N^c = S^3 \setminus \operatorname{Int} U_n$, and using Lemma 4.10.2, we conclude that the rotation number of any Legendrian meridian is equal to $-r(L) = -r$. Hence, Giroux–Honda's Theorem 4.11 implies that $r = \pm(n-1)$ and that there exists a contactomorphism $(Q, \zeta_n^\pm) \to (N^c, \xi|_{N^c})$, where the sign is the *same* as the sign of r. It remains to compute the Hopf invariant of the constructed contact structure

$$(S^3, \widetilde{\xi}_n^\pm) = (U_n, \xi|_{U_n}) \underset{f}{\cup} (Q, \zeta_n^\pm).$$

We do below the computation for the case of $r = 1 - n$. The case of the positive rotation number is similar.[22]

Let us choose a reference vector field $\mathbf{v} \in TS^3$ as follows. On U_n we take $\mathbf{v}$ to be equal to the vector field $X \in \xi$ constructed in Lemma 4.9. Note that on the boundary $\partial Q = \partial D \times S^1$, we have $X = -\mu$ (we continue to use the notation introduced in the proof of Lemma 4.10.1), i.e., X is tangent to the meridians $\partial D \times x$, $x \in S^1$, of Q, but determines their opposite orientation. Next, we extend $\mathbf{v}$ to a small neighborhood $\Omega = \{1 - \sigma \leq \rho \leq 1\}$ of the boundary $T = \partial Q = \{\rho = 1\}$, where $\rho \in [0,1]$ is the radial coordinate on the disk D. We extend $\mathbf{v}$ to Ω as tangent to tori $\{\rho = \mathrm{const}\}$ and rotating clockwise (with respect to the orientation of $T = \partial Q$) from $-\mu$ to the vector field e_x directing the singularity curves, as ρ decreases from 1 to $1 - \sigma$. Note that the vector field e_x on T coincides with the contact vector field $\mathbf{t}$ constructed in Step 2 on Q. Hence, we can extend $\mathbf{v}$ to the rest of Q as equal to $\mathbf{t}$ elsewhere. It is straightforward to check that the vector field $\mathbf{v}$ thus defined is homotopic to the basic vector field $ip \in TS^3$ of the chosen framing (ip, jp, kp) of TS^3.

[22] In fact, the positive rotation case formally follows from the negative: if L is a Legendrian unknot with $(\mathrm{tb}, r) = (n, 1-n)$, then the same knot with the opposite orientation has $(\mathrm{tb}, r) = (n, n-1)$.

Choose now a Riemannian metric on S^3 in the following special way. Let us recall that Q is the quotient of $D \times \mathbb{R}$ by a map $\phi : D \times \mathbb{R}$ which is the composition of translation by 1 and rotation by $\frac{2\pi}{n}$. The map ϕ is an isometry of the standard Euclidean product metric on $D \times S^1$, and hence Q inherits the quotient metric. We extend this metric arbitrarily to the complement $U_n = S^3 \setminus Q$. Let us denote by $\mathbf{w}$ the vector field normal to the contact plane field $\widetilde{\xi}$. By definition, the Hopf invariant $h(\widetilde{\xi})$ is the linking number of appropriately oriented curves $\Gamma_\pm = \{\mathbf{w} = \pm\mathbf{v}\}$. First, note that $\Gamma_\pm \cap U_n = \varnothing$. Indeed, in U_n, the contact vector field $\mathbf{v}$ is tangent to ξ while $\mathbf{w}$ is orthogonal to ξ. We also have $\Gamma_\pm \cap \Omega = \varnothing$. Indeed, let us lift all the objects to the universal cover $D^2 \times S^1$. The vector field bt is vertical, the background metric is the Euclidean product metric and the contact structure, as well as the lifted vector fields $\mathbf{v}$ and $\mathbf{w}$, is invariant with respect to translations along the vertical axis. The only possible points of Ω where we could have $\mathbf{v} = \pm\mathbf{w}$ are along separatrices, which connect radially (i.e., in the ∂_ρ-direction) to hyperbolic boundary points. Indeed, only along the separatrices both vector fields are tangent to concentric tori $\{\rho = \text{const}\}$. However, as ρ decreases from 1 to $1-\sigma$, both vector fields rotate clockwise, $\mathbf{v}$ rotates $-\frac{\pi}{2}$ from $-\mu$ to e_x, while $\mathbf{w}$ rotates by a small angle away from $\pm e_x$. In both cases, we cannot have $\mathbf{v} = \pm\mathbf{w}$ anywhere in Ω. Finally, inside $Q \setminus \Omega$, the curves Γ_+ and Γ_- coincide with the locus of respectively positive or negative elliptic singular points of characteristic foliations on disks $D \times x, x \in S^1$. Hence, Γ_+ is the core circle of Q, while Γ_- is isotopic in the complement of Γ_+ to the core circle L of U_n. The analysis of the Hopf map near an elliptic point shows that the orientation induced by the Hopf map on $\Gamma_\pm$ is the same as defined by the coorientation of the contact structure. In other words, Γ_+ is oriented by $\mathbf{v}$, while Γ_- is oriented by $-\mathbf{v}$, Therefore, $\text{lk}(\Gamma_+, \Gamma_-) = \text{lk}(-L, \Gamma_+) = -1$.

The exceptional knots $K_n^\pm$ with $(\text{tb}, r) = (n, \pm(n-1))$ in ξ_{-1} can be explicitly exhibited in the contact (S^3, ξ_{-1}), see [**14**]. Namely, let us view S^3 as a quotient space of the solid torus $D^2 \times S^1$ where each longitude $x \times S^1$ $x \in \partial D^2$ on its boundary is collapsed to a point. Consider a contact structure ξ given in cylindrical coordinates on T by a 1-form $\cos f(r)dz + \sin f(r)d\theta$, where the function $f(r)$ has the following properties: f is monotone function with $f(0) = f'(0) = 0, f'(1) = 0$ and $f' > 0$ on $(0,1)$. Then, if $f(1) = \frac{\pi}{2}$, then the contact structure is tight, and if $f(1) = \frac{3\pi}{2}$, the contact structure ξ is isomorphic to ξ_{-1}. Consider the latter case. There are exactly two values $r_0, r_1 \in (0,1)$, $r_0 < r_1$, such that $\tan f(r_0) = -n, \tan f(r_1) = -\frac{1}{n}$. Then the tori $T_{r_0} = \{r = r_0\}$ and $T_{r_1} = \{r = r_1\}$ are foliated by Legendrian knots with $\text{tb} = n$ and $r = \pm(n-1)$, where the sign of the rotation number depends on the orientation of the knots. Dymara explicitly verified in [**14**] that these knots are exceptional. She also conjectured that two Legendrian exceptional Legendrian knots with the same (tb, r), one on T_{r_0} and the other on T_{r_1}, are not Legendrian isotopic. By Theorem 4.7 they are, of course, coarsely equivalent.

4.3. Coarse classification vs. Legendrian isotopy.

4.3.1. Legendrian knots with negative tb.

Proposition 4.12. *Let L_1, L_2 be two topologically trivial Legendrian knots in an overtwisted contact manifold (M, ξ) with the same values of* tb, r*. Suppose that* $\mathrm{tb}(L_1)(=\mathrm{tb}(L_2)) < 0$*. Then L_1 and L_2 are Legendrian isotopic.*

Proof. According to Lemma 4.1(a), the knots L_1 and L_2 are loose. There exists a Legendrian knot L_0 with the same tb and r which is contained in a small Darboux ball. Proposition 4.3 then implies that all three knots, L_0, L_1 and L_2, are coarsely equivalent. Hence, L_1 and L_2 are contained in neighborhoods contactomorphic to the standard tight contact 3-ball. The space of embeddings of the tight 3-ball in any contact manifold is connected, and hence the claim follows from Theorem 1.5 in the tight case. □

4.3.2. Topology of the contactomorphism group of an overtwisted contact manifold. Let (M, ξ) be a connected contact manifold. M can be either closed, or non-compact. In the latter case, all diffeomorphisms we consider are assumed to be with compact support. Similarly, homotopies of contact structures and plane fields are always assumed to be fixed at infinity. Let us denote by

- $\mathrm{Diff}_0(M)$ the identity component of the group of compactly supported diffeomorphisms;
- $\mathrm{Diff}_0(M, \xi)$ the subgroup of $\mathrm{Diff}_0(M)$ consisting of those diffeomorphisms preserving the contact structure ξ and its coorientation;
- $\mathrm{Distr}(M|\xi)$ and $\mathrm{Cont}(M|\xi)$ the spaces of, respectively, plane fields and contact structures on M which coincide with ξ at infinity if M is non-compact;
- $\mathrm{Distr}_0(M|\xi)$ and $\mathrm{Cont}_0(M|\xi)$ the connected components of ξ in $\mathrm{Distr}(M|\xi)$ and $\mathrm{Cont}(M|\xi)$, respectively.

We recall:

Theorem 4.13 (See [22]). 1) *If ξ is overtwisted, then the group* $\mathrm{Diff}_0(M)$ *acts transitively on* $\mathrm{Cont}_0(M, \xi)$*.*

2) *If M is non-compact and ξ is overtwisted at infinity, then the inclusion*

$$j : \mathrm{Cont}_0(M|\xi) \hookrightarrow \mathrm{Distr}_0(M|\xi)$$

is a homotopy equivalence.

Remark 4.14. Though the homotopy equivalence is stated in [**22**], the proof there is given only for 1-parametric families, which only implies an isomorphism on π_0 and an epimorphism on π_1.

Thus, for any overtwisted ξ, the evaluation map $f \mapsto f_*\xi$ defines a Serre fibration $\pi : \mathrm{Diff}_0(M) \to \mathrm{Cont}_0(M|\xi)$, with the fiber $\mathrm{Diff}_0(M,\xi)$. If M is not compact and ξ is overtwisted at infinity,[23] then the base $\mathrm{Cont}_0(M|\xi)$ of this fibration is homotopy equivalent to $\mathrm{Distr}_0(M|\xi)$.

Corollary 4.15 (comp. Dymara, [15]). 1) *The classifying space* $\mathrm{BDiff}_0(\mathbb{R}^3,\xi)$ *is homotopy equivalent to* $\mathrm{Cont}_0(\mathbb{R}^3|\xi) \simeq \mathrm{Distr}_0(\mathbb{R}^3|\xi) \simeq \mathrm{Map}(S^3,S^2)$ *for any overtwisted at infinity contact structure* ξ *on* $\mathbb{R}^3$. *Here we denote by* $\mathrm{Map}(S^3,S^2)$ *the space of based maps* $S^3 \to S^2$. *In particular,*

$$\pi_0(\mathrm{Diff}_0(\mathbb{R}^3,\xi)) = \pi_1(\mathrm{BDiff}_0(\mathbb{R}^3,\xi)) = \pi_4(S^2) = \mathbb{Z}_2.$$

2) *The coarse classification of Legendrian knots in* $(\mathbb{R}^3,\xi)$ *coincides with their classification up to Legendrian isotopy.*

Proof. The statement 4.15.1 follows from the fact that the homotopy classes of plane fields on a 3-manifold M coincide with homotopy classes of maps $M \to S^2$, and from Hatcher's theorem [**47**] that the group $\mathrm{Diff}(\mathbb{R}^3)$ is contractible.

To prove Corollary 4.15.2, let us consider a contactomorphism $f \in \mathrm{Cont}_0(\mathbb{R}_3,\xi)$ which maps one of two coarsely equivalent Legendrian knots L_1 and L_2 onto the other one. According to Corollary 4.15.1, there exists exactly two connected components of $\mathrm{Cont}_0(\mathbb{R}_3,\xi)$. Suppose that f does not belong to the identity component. Take a ball $B \subset \mathbb{R}^3 \setminus \operatorname{supp} f$ such that the contact structure $\xi|_{\operatorname{Int} B}$ is overtwisted at infinity. Then again applying Corollary 4.15.1, we construct a contactomorphism $g \in \mathrm{Diff}_0(\operatorname{Int} B,\xi)$ which is not isotopic to the identity in $\mathrm{Diff}_0(\operatorname{Int} B,\xi)$. Consider a diffeomorphism $\widetilde{f} \in \mathrm{Diff}_0(\mathbb{R}^3,\xi)$ which is equal to f outside B, and equal to g on B. Then $\widetilde{f}$ is a coarse equivalence between L_1 and L_2, as was f. However, the $\mathbb{Z}_2$-invariant associated by Corollary 4.15.1 with a contactomorphism is additive for contactomorphisms with disjoint support, and hence it is trivial for $\widetilde{f}$. Thus, $\widetilde{f}$ is isotopic to the identity inside the group $\mathrm{Diff}_0(\mathbb{R}^3,\xi)$, and in particular, L_1 and L_2 are Legendrian isotopic. □

Remark 4.16. 1) Using Corollary 4.15.1 and some algebraic topology, one can show (see [**16**]) that for any overtwisted contact manifold (M,ξ) the group $\mathrm{Diff}_0(M,\xi)$ is disconnected.

2) Yu Chekanov informed us that he proved that for an overtwisted contact structure ξ_n on S^3 with the Hopf invariant n, one has

$$\pi_0(\mathrm{Diff}_0(S^3,\xi_n)) = \begin{cases} \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{if } n=-1, \\ \mathbb{Z}_2 & \text{otherwise.} \end{cases}$$

[23] Note that a complement of an overtwisted disk in its arbitrarily small neighborhood is overtwisted, hence a complement of an overtwisted disk in any manifold is overtwisted at infinity.

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