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SCALING LIMITS FOR EQUIVARIANT SZEGÖ KERNELS

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Suppose that the compact and connected Lie group G acts holomorphically on the irreducible complex projective manifold M, and that the action linearizes to the Hermitian ample line bundle L on M. Assume that 0 is a regular value of the associated moment map. The spaces of global holomorphic sections of powers of L may be decomposed over the finite dimensional irreducible representations of G. We study how the holomorphic sections in each equivariant piece asymptotically concentrate along the zero locus of the moment map. In the special case where G acts freely on the zero locus of the moment map, this relates the scaling limits of the Szego kernel of the quotient to the scaling limits of the invariant part of the Szego kernel of (M, L).

1. Introduction

Let (M, J) be an n-dimensional complex projective manifold, and let (L, h) be an Hermitian ample line bundle on M. Suppose that the unique compatible connection on L has curvature $\Theta = -2i\,\omega$, where ω is a Hodge form on M. The pair (ω, J) puts a Hermitian structure $H = g - i\omega$ on the (holomorphic) tangent bundle TM, hence a Riemannian structure g on M.

Let G be a compact connected g-dimensional Lie group, and suppose given a Hamiltonian holomorphic action of G on (M, ω, J) unitarily linearizing to (L, h). For every $k = 1, 2, \ldots$, there is a natural Hermitian structure on each space of holomorphic global sections $H^0(M, L^{\otimes k})$, and a naturally induced unitary representation of G on $H^0(M, L^{\otimes k})$.

Let $\{V_\varpi\}_{\varpi\in\Theta}$ be the finite dimensional irreducible representations of G, and for every $\varpi \in \Theta$ let $H^0(M, L^{\otimes k})_\varpi \subseteq H^0\left(M, L^{\otimes k}\right)$ be the maximal subspace equivariantly isomorphic to a direct sum of copies of V_ϖ. There are unitary equivariant isomorphisms

$$H^0(M, L^{\otimes k}) = \bigoplus_{\varpi\in\Theta} H^0(M, L^{\otimes k})_\varpi. \qquad (1.1)$$

The action of G on L dualizes to an action on the dual line bundle L^* in a natural manner; on the other hand, the G-invariant Hermitian metric h on L naturally induces a Hermitian metric on L^*, still denoted by h, which is also G-invariant.

Let $X \subseteq L^*$ be the unit circle bundle, with projection $\pi : X \to M$. Then, the action of G on L^* leaves X invariant. Furthermore, X is a contact manifold, with contact form given by the connection 1-form α. Since G preserves both the Hermitian metric and the holomorphic structure, it preserves the unique compatible connection, and therefore it acts on X as a group of contactomorphisms; given this, X has a standard G-invariant Riemannian metric. By these underlying structures, in the following we shall tacitly identify functions, densities and half-densities on X. In the following, to avoid cumbersome notation, we shall use the same symbol μ_g for the symplectomorphism of M and the contactomorphism of X induced by $g \in G$.

As is well known, the spaces of smooth sections $\mathcal{C}^\infty(M, L^{\otimes k})$ may be unitarily and equivariantly identified with the spaces $\mathcal{C}^\infty(X)_k$ of smooth functions on X of the k-th isotype for the S^1-action, that is, obeying the covariance law $f(e^{i\vartheta} \cdot x) = e^{ik\vartheta} f(x)$ for $x \in X$ and $e^{i\vartheta} \in S^1$. Let $H(X)_k \subseteq \mathcal{C}^\infty(X)_k$ be the subspace of functions corresponding to $H^0(M, L^{\otimes k})$ under this isomorphism, so that $H(X) =: \bigoplus_{k=0}^{+\infty} H(X)_k$ is the Hardy space of X. Thus (1.1) translates into

$$H(X)_k = \bigoplus_{\varpi \in \Theta} H(X)_{\varpi,k}. \tag{1.2}$$

In this paper, we are concerned with certain $\mathcal{C}^\infty$ functions $\Pi_{\varpi,k}$ on $X \times X$ naturally associated to each pair $(\varpi, k) \in \Theta \times \mathbb{N}$. Namely, let us choose for any $(\varpi, k) \in \Theta \times \mathbb{N}$ an orthonormal basis $\{s_j^{(\varpi,k)}\}_{j=1}^{N_{\varpi k}}$ of $H(X)_{\varpi,k}$, and let us define

$$\Pi_{\varpi,k}(x, y) =: \sum_{j=1}^{N_{\varpi k}} s_j^{(\varpi,k)}(x) \cdot \overline{s_j^{(\varpi,k)}(y)} \quad (x, y \in X).$$

Then $\Pi_{\varpi,k}$ is well defined, that is, independent of the choice of the orthonormal basis, and in fact it can be intrinsically described as the distributional kernel of the orthogonal projection $P_{\varpi,k} : L^2(X) \to H(X)_{\varpi,k}$. We shall study here the asymptotic properties of the functions $\Pi_{\varpi,k}$, as ϖ is fixed and $k \to +\infty$.

Let $\mathfrak{g}$ be the Lie algebra of G, and denote by $\Phi : M \to \mathfrak{g}^*$ the moment map of the action of G on $(M, 2\omega)$. In [**P1**], it has been shown that for fixed ϖ one has $\Pi_{\varpi,k}(x, x) = O(k^{-\infty})$ as $k \to +\infty$, unless $\Phi\big(\pi(x)\big) = 0$.

On the other hand, if $\Phi\big(\pi(x)\big) = 0$, and G acts freely on $\Phi^{-1}(0) \subseteq M$, then by Corollary 1 of [**P2**] (working with a different normalization convention for the total volume) there is an asymptotic expansion

$$\Pi_{\varpi,k}(x,x)=\sum_j\left|s_j^{(\varpi,k)}(x)\right|^2\sim\frac{\dim(V_\varpi)^2}{V_{\mathrm{eff}}(x)}\,k^{n-\mathrm{g}/2}+\sum_{l\geq 1}a_{l,\varpi}(x)\,k^{n-\mathrm{g}/2-l},$$

where $V_{\mathrm{eff}}:(\Phi\circ\pi)^{-1}(0)\to\mathbb{R}$ is the effective potential of the action [**BG**]; its value on $x\in(\Phi\circ\pi)^{-1}(0)$ is the volume of the G-orbit in M through $\pi(x)$. Thus the effective potential of the action controls the asymptotics of the restriction of $\Pi_{\varpi,k}$ to the diagonal of $X\times X$.

In the particular case of the trivial representation $\varpi=0$, V_{eff} relates the asymptotics of $\Pi_{0,k}$ and of the Szegö kernel of the symplectic reduction (M_0,ω_0,L_0) of (M,L,ω), expressing an obstruction to the conformal unitarity of the Guillemin–Sternberg map $H^0\left(M,L^{\otimes k}\right)^G\to H^0(M_0,L_0^{\otimes k})$. Further developments on this problem are due to Charles [**Ch**], Hall and Kirwin [**HK**], Hui Li [**L**], Ma and Zhang [**MZ**].

Turning momentarily to the action free case, the fast decay of Szegö kernels away from the diagonal has stimulated interest in the asymptotics of their scaling limits near the diagonal. More precisely, suppose $x\in X$, and let $\rho(z,\theta)$ be a Heisenberg local chart for X centered at x, as in (3.2) that follows; in particular, if $m=:\pi(x)$ this unitarily identifies (T_mM,H_m) and $\mathbb{C}^{\mathrm{n}}$ with its standard Hermitian structure. As shown in Theorem 3.1 of [**SZ**], for any $w,v\in\mathbb{C}^{\mathrm{n}}$ the following asymptotic expansion holds as $k\to+\infty$ for the level-k Szegö kernel Π_k:

$$\Pi_k\left(\rho\left(\frac{v}{\sqrt{k}},\theta\right),\rho\left(\frac{w}{\sqrt{k}},\theta'\right)\right)\sim\left(\frac{k}{\pi}\right)^n e^{ik(\theta-\theta')+\psi_2(u,v)}$$
$$\times\left(1+\sum_{j\geq 1}a_j(x,w,v)\,k^{-j/2}\right),\tag{1.3}$$

where

$$\psi_2(w,v)=:w\cdot\overline{v}-\frac{1}{2}\left(\|w\|^2+\|v\|^2\right),$$

and the a_j are polynomials in w and v (see also [**BSZ**] for the leading term). Recall that here H_m denotes the Hermitian structure of TM induced by $\omega=:(i/2)\Theta$; this normalization convention accounts for the factor $1/\pi^n$ in (1.3), unlike the earlier work [**Z**]. We shall conform here to [**SZ**]; thus the total volume of M is $\mathrm{vol}(M)=(\pi^n/n!)\int_M c_1(L)^n$.

In this article, we shall study the scaling limits of the equivariant Szegö kernels $\Pi_{\varpi,k}$, and show that to leading order they are still simply related to the effective volume, certain data associated to the representation ϖ, and (in the special case where G acts freely on $\Phi^{-1}(0)\subseteq M$) the scaling limits of the Szegö kernel of the symplectic reduction. Furthermore, we shall see that equivariant scaling limits can also be expressed by the product of an exponentially decaying factor in v, w times an asymptotic expansion whose coefficients are polynomials in v and w. We remark that in the toric case equivariant asymptotics have been studied in [**STZ**].

To express our results, we need some basic facts about the local geometry of M along $M' =: \Phi^{-1}(0)$ [**GS**], [**GGK**]. Recall that if $0 \in \mathfrak{g}^*$ is a regular value of Φ, then M' is a g-codimensional connected coisotropic submanifold of M, whose null-fibration is given by the orbits of the G-action.

At any $m \in M$, let us denote by $\mathfrak{g}_M(m) \subseteq T_mM$ the tangent space to the orbit through m, and by $J_m : T_mM \to T_mM$ the complex structure.

If $m \in M'$, let us denote by $Q_m \subseteq T_mM$ the Riemannian orthocomplement of $\mathfrak{g}_M(m)$ in T_mM'. Thus, Q_m is a complex subspace of T_mM', of complex dimension $n - \mathrm{g}$.

The Riemannian orthocomplement of $T_mM' \subseteq T_mM$ is $J_m(\mathfrak{g}_M(m))$. Therefore, we have orthogonal direct sum decompositions

$$T_mM = T_mM' \oplus J_m\Big(\mathfrak{g}_M(m)\Big), \quad T_mM' = Q_m \oplus \mathfrak{g}_M(m). \tag{1.4}$$

Given (1.4), if $m \in M'$ and $w \in T_mM$, we shall decompose w as $w = w_v + w_h + w_t$, where $w_v \in \mathfrak{g}_M(m)$, $w_h \in Q_m$, $w_t \in J_m(\mathfrak{g}_M(m))$. The labels stand for vertical, horizontal, and transverse. This hints to the fact that in the special case where G acts freely on M', the latter is a principal G-bundle on the symplectic reduction $M_0 = M'/G$; thus $\mathfrak{g}_M(m)$ is the vertical tangent fiber, whereas Q is a connection projecting unitarily to the tangent bundle of M_0.

Before stating our theorem, another definition is in order. To this end, recall that if $0 \in \mathfrak{g}^*$ is a regular value of the moment map then the action of G on $\Phi^{-1}(0) \subseteq M$ is locally free. Therefore, any $m \in \Phi^{-1}(0)$ has finite stabilizer subgroup $G_m \subseteq G$.

Suppose $x \in X$, $\Phi\big(\pi(x)\big) = 0$. If $G_{\pi(x)} \subseteq G$ is the (finite) stabilizer subgroup of $\pi(x)$, for every $g \in G_{\pi(x)}$ there exists a unique $h_g \in S^1$ such that $\mu_g(x) = h_g \cdot x$, where $\mu_g : X \to X$ is the contactomorphism induced by g. We shall then let

$$A_{\varpi,k}(x) =: 2^{\mathrm{g}/2} \frac{\dim(V_\varpi)}{V_{\mathrm{eff}}(x)} \cdot \frac{1}{\big|G_{\pi(x)}\big|} \sum_{g \in G_{\pi(x)}} \chi_\varpi(g) h_g^k, \tag{1.5}$$

where $\chi_\varpi : G \to \mathbb{C}$ is the character of the irreducible representation ϖ.

As before, $\omega =: (i/2)\Theta$, where Θ is the curvature, and h is the Hermitian metric on TM associated to ω.

Furthermore, as in (1.3) we shall express the asymptotic expansion for $\Pi_{\varpi,k}$ in a Heisenberg local chart ρ centered at x. However, given that the dependence of $\Pi_{\varpi,k}$ on θ and θ' is given by the factor $e^{ik(\theta-\theta')}$ and carries no geometric information, in the following we shall generally take $\theta = \theta' = 0$; with the identification $T_mM \cong \mathbb{C}^n$ induced by ρ understood, we shall set $x + w/\sqrt{k} =: \rho\big(w/\sqrt{k}, 0\big)$.

We then have the following theorem.

Theorem 1.1. *Suppose that $0 \in \mathfrak{g}^*$ is a regular value of Φ, and $x \in X$, $\Phi\big(\pi(x)\big) = 0$. Let us choose a system of Heisenberg local coordinates centered at x. For every $\varpi \in \Theta$ and $w, v \in T_{\pi(x)}M$, the following asymptotic expansion holds as $k \to +\infty$:*

$$\Pi_{\varpi,k}\left(x + \frac{w}{\sqrt{k}}, x + \frac{v}{\sqrt{k}}\right) \sim \left(\frac{k}{\pi}\right)^{n-\mathrm{g}/2} A_{\varpi,k}(x)\, e^{Q(w_v+w_t,v_v+v_t)}\, e^{\psi_2(w_h,v_h)} \cdot \left(1 + \sum_{j\geq 1} a_{\varpi j}(x,w,v)\, k^{-j/2}\right),$$

where

$$Q(w_v + w_t, v_v + v_t) = -\|v_t\|^2 - \|w_t\|^2 + i\left[\omega_m(w_v, w_t) - \omega_m(v_v, v_t)\right],$$

and the $a_{\varpi j}$s are polynomials in v, w with coefficients depending on x and ϖ.

We integrate the statement by the following remarks.

- The remainder term can be given a "large ball estimate" (that is, for $\|u\|$, $\|v\| \lesssim k^{1/6}$), similar to the ones in [**SZ**]. More precisely, let $R_N(x, v, w)$ be the remainder term following the first N summands in (1.6). Given the description of $\Pi_{\varpi,k}$ as an oscillatory integral (cfr (4.32 below), we may adapt the arguments in Section 5 of [**SZ**] to obtain that for $\|u\|, \|v\| \lesssim k^{1/6}$ we have
$$|R_N(x, v, w)| \leq C_N\, k^{n-(\mathrm{g}+N+1)/2} e^{-\frac{1-\epsilon}{2}\left(\|u_h - v_h\|^2 + 2\|v_t\|^2 + 2\|w_t\|^2\right)}.$$
The bound also holds in $\mathcal{C}^j$-norm.
- In the special case where G acts freely on $\Phi^{-1}(0)$, denote by $X_0 \subseteq L_0^*$ the circle bundle of the reduced pair (M_0, L_0), and by $\Pi_k^{(0)}$ the level k Szegö kernel of X_0. If $\Phi\big(\pi(x)\big) = 0$, let us denote by $\overline{x}$ its image in X_0, and if $w_h \in Q_{\pi(x)}$ let $\overline{w}_h$ be its isometric image in the tangent space to M_0. By (1.3) and Theorem 1.1, we obtain

$$\begin{aligned}
&\Pi_{\varpi,k}\left(x + \frac{w}{\sqrt{k}}, x + \frac{v}{\sqrt{k}}\right) \\
&\sim 2^{\mathrm{g}/2} \left(\frac{k}{\pi}\right)^{n-\mathrm{g}/2} \frac{\dim(V_\varpi)^2}{V_{\mathrm{eff}}(x)}\, e^{Q(w_v+w_t,v_v+v_t)}\, e^{\psi_2(w_h,v_h)} \\
&\quad + \sum_{j\geq 1} a_{\varpi j}(x,w,v)\, k^{n-(\mathrm{g}+j)/2} \\
&= \left(\frac{2k}{\pi}\right)^{\mathrm{g}/2} \cdot \left(\frac{\dim(V_\varpi)^2}{V_{\mathrm{eff}}(x)}\, e^{Q(w_v+w_t,v_v+v_t)}\right) \cdot \Pi_k^{(0)}\left(\overline{x} + \frac{\overline{w}_h}{\sqrt{k}}, \overline{x} + \frac{\overline{v}_h}{\sqrt{k}}\right) \\
&\quad + \mathrm{L.O.T.}
\end{aligned}$$

- Arguing as in Section 2.3 of [**DP**], one can see that $\Pi_{\varpi,k} = O\left(k^{-\infty}\right)$ uniformly on compact subsets of the complement in $X \times X$ of the locus
$$I(\Phi) =: \left\{(x,y) : x \in (G \times S^1) \cdot y, \Phi\big(\pi(y)\big) = 0\right\}.$$
Thus, it is natural to consider scaling limits at any $(x, y) \in I(\Phi)$. Given $g_0 \in G$, $h_0 \in S^1$, $x \in (\Phi \circ \pi)^{-1}(0)$ and $v, w \in T_{\pi(x)}(M)$, a minor modification of the arguments in the proof of Theorem 1.1 leads to an asymptotic expansion

$$\begin{aligned}(1.6)\qquad &\Pi_{\varpi,k}\left(\mu_{g_0} \circ r_{h_0}\left(x + \frac{w}{\sqrt{k}}\right), x + \frac{v}{\sqrt{k}}\right)\\ &\sim \left(\frac{k}{\pi}\right)^{n-\mathrm{g}/2} A_{\varpi,k}(x, g_0, h_0)\, e^{Q(w_v+w_t,v_v+v_t)}\, e^{\psi_2(w_h,v_h)}\\ &\cdot \left(1 + \sum_{j\ge 1} a_{\varpi j}(x, w, v)\, k^{-j/2}\right),\end{aligned}$$

where now
$$A_{\varpi,k}(x, g_0, h_0) =: 2^{\mathrm{g}/2} \frac{\dim(V_\varpi)}{V_{\mathrm{eff}}(x)} \cdot \frac{1}{\left|G_{\pi(x)}\right|} \sum_{g \in G_{\pi(x)}} \chi_\varpi\left(g\, g_0^{-1}\right) \cdot (h_0\, h_g)^k.$$

- We are primarily interested in the case of complex projective manifolds. In view of the microlocal description of almost complex Szegö kernels appearing in [**SZ**], the results of this paper can however be extended to the context of almost complex symplectic manifolds.

After this paper was completed, I learned of the rich paper [**MZ**] alluded to that discussed before. Using analytic localization techniques of Bismut and Lebeau for spin^c Dirac operators, Ma and Zhang obtained among other things an asymptotic expansion for the trivial representation.

2. Examples

In the non-equivariant case, a key feature of scaling asymptotics of Szegö kernels expressed by (1.3) is the universal nature of the leading term, essentially the level-one Szegö kernel of the reduced Heisenberg group $\mathbf{H}^n_{\mathrm{red}}$. To express this more precisely, recall that the latter may be viewed as the unit circle bundle of the trivial line bundle $L = \mathbb{C}^n \times \mathbb{C}$ over $\mathbb{C}^n$, endowed with the Hermitian metric
$$h\big((\mathbf{z}, w), (\mathbf{z}, w')\big) = w\,\overline{w'} e^{-\|\mathbf{z}\|^2} \quad \Big(\mathbf{z}, \mathbf{z}' \in \mathbb{C}^n,\ w, w' \in \mathbb{C}\Big).$$
The unit circle bundle is thus given by
$$X = \mathbf{H}^n_{\mathrm{red}} = \left\{(\mathbf{z}, w) \in \mathbb{C}^n \times \mathbb{C} : |w| = e^{-\|\mathbf{z}\|^2/2}\right\}.$$

A Heisenberg chart for X centered at $(\mathbf{0}, 1)$ is

$$\varphi_{\mathbf{0}} : \mathbb{C}^n \times (-\pi, \pi) \to X, \quad (\mathbf{z}, \theta) \mapsto \left(\mathbf{z}, e^{-\|\mathbf{z}\|^2/2 + i\theta}\right).$$

As shown in [**BSZ**], for every $k = 1, 2, \ldots$ the level-k Szegö kernel is

$$\Pi_k^{\mathbf{H}}\big(\varphi_{\mathbf{0}}(\mathbf{w}, \theta), \varphi_{\mathbf{0}}(\mathbf{v}, \theta')\big) = \left(\frac{k}{\pi}\right)^n e^{k\left[i(\theta-\theta') + \psi_2(\mathbf{w}, \mathbf{v})\right]}. \tag{2.1}$$

In the linear case, we shall derive from (2.1) an asymptotic expansion in the spirit of Theorem 1.1, at any $x = \left(\mathbf{z}_1, e^{-\|\mathbf{z}_1\|^2/2}\right)$ for which the map $\gamma_{\mathbf{z_1}} : g \in G \mapsto \mu_g(\mathbf{z_1}) \in \mathbb{C}^{\mathrm{d}}$ is an embedding (that is, $\mathbf{z}_1$ has trivial stabilizer in G); with minor changes, the arguments that follow apply when $\gamma_{\mathbf{z_1}}$ is an immersion (that is, $\mathbf{z}_1$ has finite stabilizer in G).

Example 2.1. Let $A : G \to U(n)$, $g \mapsto A_g$, be a unitary representation, so that the underlying action on $(\mathbb{C}^n, \omega_0)$ is $\mu_g(\mathbf{z}) =: A_g\mathbf{z}$ ($\mathbf{z} \in \mathbb{C}^n$); here $\omega_0 =: (i/2)\sum_{j=1}^n dz_j \wedge d\overline{z}_j$ is the standard symplectic structure on $\mathbb{C}^n$. The standard Hermitian structure on $\mathbb{C}^n$ is then $H_0 = g_0 - i\omega_0$, where $g_0(\mathbf{w}, \mathbf{v}) = -\omega_0(J_0\mathbf{w}, \mathbf{v})$ (J_0 being multiplication by i).

A linearization to L is given by

$$\mu_g\big((\mathbf{z}, w)\big) =: \big(A_g\mathbf{z}, w\big).$$

For any $\mathbf{z}_1 \in \mathbb{C}^n$, a Heisenberg chart for X centered at $(\mathbf{z}_1, e^{-\|\mathbf{z}_1\|^2/2})$ is

$$\varphi_{\mathbf{z}_1} : (\mathbf{z}, \theta) \mapsto \varphi_{\mathbf{0}}\big(\mathbf{z}+\mathbf{z}_1, \omega_0(\mathbf{z}, \mathbf{z}_1)+\theta\big) = \left(\mathbf{z} + \mathbf{z}_1, e^{-\|\mathbf{z}+\mathbf{z}_1\|^2/2 + i\left(\omega_0(\mathbf{z}, \mathbf{z}_1)+\theta\right)}\right).$$

Thus, given $x = (\mathbf{z}_1, e^{-\|\mathbf{z}_1\|^2/2}) \in \mathbf{H}^n_{\mathrm{red}}$ and $\mathbf{v} \in \mathbb{C}^n$, in our notation

$$x + \mathbf{v} = \varphi_{\mathbf{z}_1}(\mathbf{v}, 0) = \varphi_{\mathbf{0}}\big(\mathbf{z}_1 + \mathbf{v}, \omega_0(\mathbf{v}, \mathbf{z}_1)\big).$$

Given an irreducible representation ϖ and $\mathbf{w}, \mathbf{v} \in \mathbb{C}^n$, by a straightforward computation using (1.7) we obtain

$$\begin{aligned}
\Pi^{\mathbf{H}}_{\varpi,k}\left(x + \frac{\mathbf{w}}{\sqrt{k}}, x + \frac{\mathbf{v}}{\sqrt{k}}\right) &= \dim(V_\varpi) \\
&\cdot \int_G \chi_\varpi(g)\, \Pi_k^{\mathbf{H}}\left(\varphi_{\mathbf{0}}\left(A_g\mathbf{z}_1 + \frac{A_g\mathbf{w}}{\sqrt{k}}, \frac{1}{\sqrt{k}}\,\omega_0\left(\mathbf{w}, \mathbf{z}_1\right)\right),\right. \\
&\left.\varphi_{\mathbf{0}}\left(\mathbf{z}_1 + \frac{\mathbf{v}}{\sqrt{k}}, \frac{1}{\sqrt{k}}\,\omega_0\left(\mathbf{v}, \mathbf{z}_1\right)\right)\right) dg \\
&= \dim(V_\varpi)\left(\frac{k}{\pi}\right)^n \int_G \chi_\varpi(g)\, e^{S_k(\mathbf{z}_1, \mathbf{w}, \mathbf{v})}\, dg,
\end{aligned} \tag{2.2}$$

where dg is the density on G associated to a Haar metric with $\int_G dg = 1$, and

$$S_k(\mathbf{z}_1, \mathbf{w}, \mathbf{v}) =: k\, H_0\big(A_g\mathbf{z}_1 - \mathbf{z}_1, \mathbf{z}_1\big) + \sqrt{k}\,\Big[H_0\left(\mathbf{v}, A_g^{-1}\mathbf{z}_1 - \mathbf{z}_1\right) + H_0\left(A_g\mathbf{z}_1 - \mathbf{z}_1, \mathbf{w}\right)\Big] + \psi_2(A_g\mathbf{v}, \mathbf{w}). \tag{2.3}$$

Given the simplifying assumption that $\mathbf{z}_1$ has trivial stabilizer in G, there exists $C > 0$ such that $\|A_g\mathbf{z}_1 - \mathbf{z}_1\| \geq C\,\mathrm{dist}_G(g, e)$, where $e \in G$ is the unit and dist_G is the Riemannian metric on G. Thus, it follows from (2.1) that the integrand of (2.2) is $O\left(k^{-\infty}\right)$ on the loci $A_k \subseteq G$ where, say, $\mathrm{dist}_G(g, e) \geq k^{-1/3}$. On the loci B_k where $\mathrm{dist}_G(g, e) \leq 2\,k^{-1/3}$, on the other hand, we can transfer the integration to the Lie algebra $\mathfrak{g}$ by the exponential map $\exp_G : \mathfrak{g} \to G$, $\eta \mapsto e^{\eta}$, and apply the rescaling $\eta = (1/\sqrt{k})\,\xi$. Let $\widehat{A} : \mathfrak{g} \to \mathfrak{u}(n)$, $\eta \mapsto \widehat{A}_\eta$, be the differential of the morphism of Lie groups $A : G \to U(n)$. Thus

$$A_{e^{\xi/\sqrt{k}}} = e^{\widehat{A}_{\xi/\sqrt{k}}} = \mathrm{id}_V + \frac{\widehat{A}(\xi)}{\sqrt{k}} + \frac{1}{2}\,\frac{\widehat{A}(\xi)^2}{k} + \cdots .$$

On the upshot, after some computations we obtain

$$\begin{aligned} \Pi_{\varpi,k}\left(x + \frac{w}{\sqrt{k}}, x + \frac{v}{\sqrt{k}}\right) &\sim \frac{\dim(V_\varpi)}{\pi^n}\,k^{n-\mathrm{g}/2}\int_{\mathfrak{g}} \chi_\varpi\left(e^{\xi/\sqrt{k}}\right)\,e^{\mathcal{S}_k(\xi,\mathbf{w},\mathbf{v},\mathbf{z}_1)}\,d\xi \\ &= \frac{\dim(V_\varpi)^2}{\pi^n}\,k^{n-\mathrm{g}/2} \\ &\quad\times \int_{\mathfrak{g}} e^{\mathcal{S}_k(\xi,\mathbf{w},\mathbf{v},\mathbf{z}_1)}\,d\xi \cdot \left(1 + O\left(k^{-1/2}\right)\right), \end{aligned} \tag{2.4}$$

where now

$$\begin{aligned} \mathcal{S}_k(\xi, \mathbf{w}, \mathbf{v}, \mathbf{z}_1) &=: i\sqrt{k}\,\omega_0\left(\mathbf{z}_1, \widehat{A}_\xi(\mathbf{z}_1)\right) + \psi_2(\mathbf{w}, \mathbf{v}) \\ &\qquad - \frac{1}{2}\left\|\widehat{A}_\xi(\mathbf{z}_1)\right\|^2 + H_0\left(\widehat{A}_\xi(\mathbf{z}_1), \mathbf{v}\right) - H_0\left(\mathbf{w}, \widehat{A}_\xi(\mathbf{z}_1)\right) \\ &= i\sqrt{k}\,\Phi^\xi\big(\mathbf{z}_1\big) + \psi_2(\mathbf{w}, \mathbf{v}) - \frac{1}{2}\left\|\widehat{A}_\xi(\mathbf{z}_1)\right\|^2 + H_0\left(\widehat{A}_\xi(\mathbf{z}_1), \mathbf{v}\right) \\ &\qquad - H_0\left(\mathbf{w}, \widehat{A}_\xi(\mathbf{z}_1)\right); \end{aligned} \tag{2.5}$$

here $\Phi : V \to \mathfrak{g}^*$ is the moment map, and $\Phi^\xi =: \langle \Phi, \xi \rangle$.

Suppose, to begin with, that $\Phi\big(\mathbf{z}_1\big) \neq \mathbf{0}$. Then the linear phase $\xi \mapsto \Phi^\xi\big(\mathbf{z}_1\big)$ has no stationary point in ξ, and since by (2.5) the integrand in (2.4) is absolutely convergent, the stationary-phase lemma applies to show that $\Pi_{\varpi,k}(x + w/\sqrt{k}, x + v/\sqrt{k}) = O(k^{-\infty})$.

If $\Phi\big(\mathbf{z}_1\big)=\mathbf{0}$, on the other hand, we have

$$\begin{aligned}
&\Pi_{\varpi,k}\left(x+\frac{w}{\sqrt{k}},x+\frac{v}{\sqrt{k}}\right)\\
&=\frac{\dim(V_\varpi)^2}{\pi^n}\,k^{n-\mathrm{g}/2}\,e^{\psi_2(\mathbf{w},\mathbf{v})}\\
&\quad\times\int_{\mathfrak{g}}e^{(-1/2)\,\left\|\widehat{A}_\xi(\mathbf{z}_1)\right\|^2+H_0\left(\widehat{A}_\xi(\mathbf{z}_1),\mathbf{v}\right)-H_0\left(\mathbf{w},\widehat{A}_\xi(\mathbf{z}_1)\right)}\,d\xi\cdot\left(1+k^{-1/2}\right)\\
&=\frac{\dim(V_\varpi)^2}{\pi^n\cdot V_{\mathrm{eff}}(\mathbf{z}_1)}\,k^{n-\mathrm{g}/2}\,e^{\psi_2(\mathbf{w},\mathbf{v})}\int_{\mathfrak{g}(\mathbf{z}_1)}e^{(-1/2)\,\|\mathbf{s}\|^2+H_0(\mathbf{s},\mathbf{v})-H_0(\mathbf{w},\mathbf{s})}\,d\mathbf{s}\\
&\quad\cdot\left(1+k^{-1/2}\right),
\end{aligned}$$

wherein the latter equality integration has been shifted from $\mathfrak{g}$ to the tangent space $\mathfrak{g}(\mathbf{z}_1)\subseteq\mathbb{C}^n$ at $\mathbf{z}_1$ to the G-orbit of $\mathbf{z}_1$ by the change of variables $\mathbf{s}=\widehat{A}_\xi(\mathbf{z}_1)$; hence $d\mathbf{s}=V_{\mathrm{eff}}(\mathbf{z}_1)\,d\xi$.

Given the equalities $H_0\big(\mathbf{w},\mathbf{s}\big)=H_0\big(\mathbf{w}_t+\mathbf{w}_v,\mathbf{s}\big)$, $\psi_2\big(\mathbf{w},\mathbf{v}\big)=\psi_2\big(\mathbf{w}_h,\mathbf{v}_h\big)+\psi_2\big(\mathbf{w}_t+\mathbf{w}_v,\mathbf{v}_t+\mathbf{v}_v\big)$, we obtain with a few calculations

$$\begin{aligned}
&\Pi_{\varpi,k}\left(x+\frac{w}{\sqrt{k}},x+\frac{v}{\sqrt{k}}\right)\\
&=\frac{\dim(V_\varpi)^2}{\pi^n\cdot V_{\mathrm{eff}}(\mathbf{z}_1)}\,k^{n-\mathrm{g}/2}\,e^{\psi_2(\mathbf{w},\mathbf{v})+(1/2)\|\mathbf{w}_v-\mathbf{v}_v\|^2}\\
&\quad\times\int_{\mathfrak{g}(\mathbf{z}_1)}e^{-i\omega_0(\mathbf{s},\mathbf{v}_t+\mathbf{w}_t)-(1/2)\,\|\mathbf{s}-(\mathbf{w}_v-\mathbf{v}_v)\|^2}\,d\mathbf{s}\cdot\left(1+O\left(k^{-1/2}\right)\right).
\end{aligned}\tag{2.6}$$

The Gaussian integral in (2.6) is $(2\pi)^{\mathrm{g}/2}\,e^{i\omega_0(\mathbf{v}_t+\mathbf{w}_t,\mathbf{w}_v-\mathbf{v}_v)-(1/2)\|\mathbf{v}_t+\mathbf{w}_t\|^2}$, and from this one computes

$$\begin{aligned}
&\Pi_{\varpi,k}\left(x+\frac{w}{\sqrt{k}},x+\frac{v}{\sqrt{k}}\right)\\
&=2^{\mathrm{g}/2}\,\frac{\dim(V_\varpi)^2}{V_{\mathrm{eff}}(\mathbf{z}_1)}\,\left(\frac{k}{\pi}\right)^{n-\mathrm{g}/2}\,e^{\psi_2(\mathbf{w}_h,\mathbf{v}_h)-\|\mathbf{w}_t\|^2-\|\mathbf{v}_t\|^2+i\left(\omega_0(\mathbf{w}_v,\mathbf{w}_t)-\omega_0(\mathbf{v}_v,\mathbf{v}_t)\right)}\\
&\quad\cdot\left(1+O\left(k^{-1/2}\right)\right).
\end{aligned}$$

Before considering the next example, let us recall from [**BSZ**] that for $k=1,2,\ldots$ an orthonormal basis of $H^0\left(\mathbb{P}^{\mathrm{d}},\mathcal{O}_{\mathbb{P}^{\mathrm{d}}}(k)\right)$ is $\left\{s^k_{\mathbf{J}}\right\}_{|\mathbf{J}|=k}$, where

$$s^k_{\mathbf{J}}=:\sqrt{\frac{(k+\mathrm{d})!}{\pi^d\,\mathbf{J}!}}\,z^{\mathbf{J}};\tag{2.7}$$

here $\mathbf{J}!=:\prod_{l=0}^d j_l!$, $z^{\mathbf{J}}=:\prod_{l=0}^d z_l^{j_l}$.

Example 2.2. The unitary representation of S^1 on $\mathbb{C}^2$ given by $t\cdot(z_0, z_1) =: \big(t^{-1}z_0, tz_1\big)$ descends to a symplectic action on $\mathbb{P}^1$, with a built-in linearization to the hyperplane line bundle. The associated moment map is

$$\Phi : \mathbb{P}^1 \to \mathbb{R}, \quad [z_0 : z_1] \mapsto \frac{-|z_0|^2 + |z_1|^2}{|z_0|^2 + |z_1|^2}.$$

Clearly, any $[z_0 : z_1] \in \Phi^{-1}(0)$ has stabilizer subgroup $\{\pm 1\}$. Since any S^1-orbit in S^3 has length 2π and doubly covers its image in $\mathbb{P}^1$, the effective volume is identically equal to $\pi = 2\pi/2$ on $\Phi^{-1}(0)$. Therefore,

$$A_{\varpi,k}\big([z_0 : z_1]\big) = \frac{\sqrt{2}}{\pi} \cdot \frac{1}{2}\left[1 + (-1)^{\varpi}(-1)^k\right] = \begin{cases} \sqrt{2}/\pi & \text{if } k \equiv \varpi \,(\text{mod. } 2)\\ 0 & \text{if } k \not\equiv \varpi \,(\text{mod. } 2). \end{cases}$$

Given that

$$\mu_t\left(z_0^l\, z_1^{k-l}\right) = \left(z_0^l\, z_1^{k-l}\right) \circ \mu_{t^{-1}} = (tz_0)^l\, \left(t^{-1}z_1\right)^{k-l} = t^{2l-k}\, z_0^l\, z_1^{k-l},$$

we have for $\varpi \in \mathbb{Z}$ and $k \in \mathbb{N}$:

$$H^0\left(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(k)\right)_{\varpi} = \begin{cases} \operatorname{span}\left\{z_0^{\frac{\varpi+k}{2}}\, z_1^{\frac{k-\varpi}{2}}\right\} & \text{if} \quad k \equiv \varpi \ (\text{mod. } 2),\\ 0 & \text{if} \quad k \not\equiv \varpi \ (\text{mod. } 2). \end{cases} \tag{2.8}$$

By the Stirling formula, if b is fixed and $a \to +\infty$ we have

$$(a+b)! \sim \sqrt{2\pi\, a}\, \left(\frac{a^{a+b}}{e^a}\right). \tag{2.9}$$

Suppose then $k = \varpi + 2s$, $s \in \mathbb{N}$, and choose $(z_0, z_1) \in S^3$ lying over $[z_0 : z_1]$; in view of (2.7), (2.8) and (2.9),

$$\begin{aligned} \Pi_{\varpi,k}\big([z_0 : z_1], [z_0 : z_1]\big) &= \frac{(\varpi + 2s + 1)!}{\pi\,(\varpi + s)!\, s!}\, |z_0|^{2(\varpi+s)}\, |z_1|^{2s} \\ &\sim \frac{1}{\pi}\sqrt{\frac{s}{\pi}}\, 2^{\varpi+2s+1}\, |z_0|^{2(\varpi+s)}\, |z_1|^{2s}, \end{aligned} \tag{2.10}$$

as $s \to +\infty$.

If $[z_0 : z_1] \in \Phi^{-1}(0)$, so that $|z_0|^2 = |z_1|^2 = 1/2$, we obtain

$$\Pi_{\varpi,k}\big([z_0 : z_1], [z_0 : z_1]\big) \sim \frac{2}{\pi}\sqrt{\frac{s}{\pi}} \sim \frac{\sqrt{2}}{\pi}\sqrt{\frac{k}{\pi}} = A_{\varpi,k}\big([z_0 : z_1]\big)\sqrt{\frac{k}{\pi}},$$

which fits with the asymptotic expansion of Theorem 1.1.

If $|z_0| \neq |z_1|$, (2.10) is rapidly decreasing as $s \to +\infty$.

3. Preliminaries

In this section we shall collect some preliminaries and set some notation.

If (M, J) is a complex manifold, any Kähler form ω on it determines an Hermitian metric h on the tangent bundle of M, and $\omega = -\Im(h)$. The Riemannian metric $g =: \Re(h)$ is $g_m(w, v) = \omega_m(w, J_m v)$ ($m \in M$, $w, v \in T_m M$).

Since Heisenberg local coordinates centered at a given $x \in X$ will be a key tool in the following, we shall briefly recall their definition [**SZ**].

Thus we now assume that $L \to M$ is an Hermitian ample line bundle, and $\omega = (i/2)\,\Theta$, where Θ is the curvature of the unique compatible covariant derivative. Let us choose an *adapted* holomorphic coordinate system $(z_1, \ldots, z_n)$ for M centered at $\pi(x)$. This means that, when expressed in the z_is, ω evaluated at $\pi(x)$ is the standard symplectic structure on $\mathbb{C}^n$, that is, $\omega\big(\pi(x)\big) = (i/2)\sum_{j=1}^n dz_j \wedge d\overline{z}_j$. Thus the choice of the z_is determines a unitary isomorphism $T_{\pi(x)}M \cong \mathbb{C}^n$.

Let us next choose a *preferred* local frame e_L for L at $\pi(x)$, in the sense of [**SZ**]. Thus e_L is a holomorphic local section for L in the neighborhood of $\pi(x)$, satisfying

(3.1)
$$\big\|e_L\big(\pi(x)\big)\big\| = 1, \quad \nabla e_L\big(\pi(x)\big) = 0, \;\; \nabla^2 e_L\big(\pi(x)\big) = -\overline{h}_{\pi(x)} \otimes e_L\big(\pi(x)\big),$$

where ∇ is the covariant derivative of the connection, and $\overline{h} = g + i\omega$. The local holomorphic frame for L uniquely determines a holomorphic dual local frame e_L^* for L^*, determined by the condition $(e_L^*, e_L) = 1$,

For $\delta > 0$, let $B_{2n}(0;\delta) \subseteq \mathbb{C}^n \cong \mathbb{R}^{2n}$ be the ball of radius δ centered at the origin. For an appropriate $\delta > 0$, a system of Heisenberg local coordinates for X centered at x is then given by the map

$$\rho : B_{2n}(0;\delta) \times (-\pi, \pi) \to X, \quad (z, \theta) \mapsto e^{i\theta}\, a(z)^{-1/2}\, e_L^*(z), \tag{3.2}$$

where $a(z) =: \|e_L^*\|^2 = \|e_L\|^{-2}$. If $w \in T_{\pi(x)}M \cong \mathbb{C}^n$, we shall denote by $x + w$ the point in X with Heisenberg local coordinates $(w, 0)$.

It will simplify our exposition to make a little equivariant adjustment to the previous construction. Suppose that $m \in M$ has finite stabilizer subgroup $G_m \subseteq G$ (this will always be the case when $\Phi(m) = 0$ if $0 \in \mathfrak{g}^*$ is a regular value of the moment map). Let $U \subseteq M$ be a G_m-invariant open neighborhood of the identity, and suppose that a local holomorphic frame $\sigma = e_L^*$ satisfying (3.1) has been chosen on U. Clearly, for every $g \in G_m$ we have $g^*(\sigma)(m) = h_g \cdot \sigma(m)$ (recall that $g^*(\sigma) = \mu_g \circ \sigma \circ \mu_{g^{-1}}$). We may then consider the new frame

$$\overline{\sigma} = \frac{1}{|G_m|} \sum_{g \in G_m} h_g^{-1}\, g^*(\sigma).$$

Then $\overline{\sigma}(m) = e_L(m)$, and since the metric and the connection are G-invariant $\overline{\sigma}$ also satisfies (3.1). Moreover, we now have

$$g^*(\overline{\sigma}) = h_g \cdot \overline{\sigma}, \quad \forall\, g \in G_m. \tag{3.3}$$

In the following, the underlying preferred local holomorphic frame in the definition of Heisenberg local coordinates will be assumed to satisfy (3.3).

For $\xi \in \mathfrak{g}$, we shall denote by ξ_M and ξ_X the vector fields on M and X, respectively, associated to ξ by the infinitesimal actions of $\mathfrak{g}$. The moment map $\Phi : M \to \mathfrak{g}^*$ for the action on $(M, 2\omega)$ is related to the G-invariant connection form α on X by the relation $\Phi^\xi = -\iota(\xi_X)\,\alpha$, where $\Phi^\xi = \langle \Phi, \xi\rangle$.

4. Proof of Theorem 1.1

To begin with, let us fix an invariant Haar metric on G, and let dg denote the associated measure; by Haar metric we mean that $\int_G dg = 1$. Now if $\rho : G \to GL(W)$ is a linear representation on a complex vector space, for any $\varpi \in \Theta$ the projection P_ϖ of W onto the the ϖ-isotypical component W_ϖ is given by

$$P_\varpi = \dim(V_\varpi) \int_G \chi_\varpi\left(g^{-1}\right)\, \rho(g)\, dg \tag{4.1}$$

[**Di**]. On the other hand, the unitary representation of G on $H_k(X) \subseteq L^2(X)$ induced by the action on X is given by $(g \cdot f)(y) =: f\left(\mu_{g^{-1}}(y)\right)$ ($f \in L^2(X)$, $y \in X$). Therefore, the equivariant Szegö kernel $\Pi_{\varpi,k}$ is given by

$$\Pi_{\varpi,k}(y, y') = \dim(V_\varpi) \int_G \chi_\varpi\left(g^{-1}\right)\, \Pi_k\left(\mu_{g^{-1}}(y), y'\right)\, dg, \tag{4.2}$$

where $\mu_g : X \to X$ is the contactomorphism associated to $g \in G$.

Suppose $x \in X$, $\Phi(x) = 0$, and set $m =: \pi(x)$. We assume given a system of Heisenberg local coordinates for X centered at x. This choice gives a meaning to the expression $x + w$, for any $w \in T_mM \cong \mathbb{C}^n$.

Then for every $\varpi \in \Theta$ and $k \in \mathbb{N}$ we have

$$\begin{aligned} &\Pi_{\varpi,k}\left(x + \frac{w}{\sqrt{k}}, x + \frac{v}{\sqrt{k}}\right) \\ &\quad = \dim(V_\varpi) \int_G \chi_\varpi\left(g^{-1}\right)\, \Pi_k\left(\mu_{g^{-1}}\left(x + \frac{w}{\sqrt{k}}\right), x + \frac{v}{\sqrt{k}}\right)\, dg, \end{aligned} \tag{4.3}$$

where $\chi_\varpi : G \to \mathbb{C}$ is the character of the irreducible representation V_ϖ.

We shall now split the integration in $d\mu$ as the sum of two terms, one which is rapidly decaying as $k \to +\infty$, and another where integration is over a suitably shrinking neighborhood of the (finite) stabilizer subgroup $G_m \subseteq G$.

To this end, let us define for every $k \in \mathbb{N}$ an open cover $\{A_k,\, B_k\}$ of G by setting

$$A_k =: \left\{ g \in G : \mathrm{dist}_G\big(g, G_m\big) > k^{-1/3} \right\},$$
$$B_k =: \left\{ g \in G : \mathrm{dist}_G\big(g, G_m\big) < 2\, k^{-1/3} \right\}$$

(towards application of the stationary-phase lemma later in the proof, the exponent $-1/3$ used in the definition of A_k and B_k, could be replaced by $-a$, for any $a \in (0, 1/2)$). Here $\mathrm{dist}_G : G \times G \to \mathbb{R}$ is the Riemannian distance function. Let $a_k + b_k = 1$ be a G_m-invariant partition of unity on G subordinate to the open cover $\{A_k,\, B_k\}$ (thus, $\mathrm{supp}(a_k) \subseteq A_k$, $\mathrm{supp}(b_k) \subseteq B_k$).

We may then split (4.3) as

$$(4.4)\quad \Pi_{\varpi,k}\left(x + \frac{w}{\sqrt{k}}, x + \frac{v}{\sqrt{k}}\right) = \Pi_{\varpi,k}\left(x + \frac{w}{\sqrt{k}}, x + \frac{v}{\sqrt{k}}\right)_a + \Pi_{\varpi,k}\left(x + \frac{w}{\sqrt{k}}, x + \frac{v}{\sqrt{k}}\right)_b;$$

the first (respectively, second) summand in (4.4) is (4.3) with the integrand multiplied by a_k (respectively, b_k).

Proposition 4.1. *$\Pi_{\varpi,k}(x + w/\sqrt{k}, x + v/\sqrt{k})_a = O(k^{-\infty})$ as $k \to +\infty$.*

Proof of Proposition 4.1. Let $\mathrm{dist}_M : M \times M \to M$ be the Riemannian distance function. We have the following.

Lemma 4.1. *There exists a positive constant C (dependent on w and v, but independent of k) such that for all $k \gg 0$ and $g \in A_k$ we have*

$$(4.5)\quad \mathrm{dist}_M\left(\mu_{g^{-1}}\left(m + \frac{w}{\sqrt{k}}\right), m + \frac{v}{\sqrt{k}}\right) \geq C\, \mathrm{dist}_G(g, G_m) \geq C\, k^{-1/3}.$$

Proof of Lemma 4.1. If not, we can find $\mathbb{N} \ni k_j \uparrow +\infty$ and $g_j \in A_{k_j}$ such that $\forall j = 1, 2, \ldots$ we have

$$(4.6)\quad \mathrm{dist}_M\left(\mu_{g_j}\left(m + \frac{w}{\sqrt{k_j}}\right), m + \frac{v}{\sqrt{k_j}}\right) \leq \frac{1}{j}\, \mathrm{dist}_G(g_j, G_m).$$

Since $\mathrm{dist}_G(g_j, G_m)$ is bounded above by the diameter of the compact Lie group G, we have in particular $\mathrm{dist}_M(\mu_{g_j}(m + w/\sqrt{k_j}), m + v/\sqrt{k_j}) \to 0$, hence also $\mathrm{dist}_M(\mu_{g_j}(m), m) \to 0$. Therefore, $g_j \to G_m$; after passing to a subsequence, therefore, we may assume that $g_j \to g_0$ for some $g_0 \in G_m$. Let us write $g_j = g_0\, h_j$, where $h_j \to e$, and $\mathrm{dist}_G(h_j, e) = \mathrm{dist}_G(g_j, G_m) \geq k_j^{-1/3}$. Using the exponential map $\exp_G : \mathfrak{g} \to G$, for all $j \gg 0$ we can write $h_j = e^{\nu_j}$, for unique $\nu_j \in \mathfrak{g}$ such that $\|\nu_j\| = \mathrm{dist}_G(h_j, e)$. Since G acts locally freely on

$\Phi^{-1}(0)$, there exists $c > 0$ such that $\|\nu_M(m)\| \geq c\,\|\nu\|$, $\forall\, m \in \Phi^{-1}(0)$, $\nu \in \mathfrak{g}$ (the former norm is in T_mM, the latter in $\mathfrak{g}$). Hence,

$$\left\| \left(\nu_j\right)_M(m) \right\| \geq c\, k_j^{-1/3} \quad (j \gg 0). \tag{4.7}$$

Working in preferred local coordinates centered at m, we have

$$\mu_{e^{\nu_j}}\left(m + \frac{w}{\sqrt{k_j}}\right) = m + \left(\nu_j\right)_M(m) + \frac{w}{\sqrt{k_j}} + O\left(k_j^{-2/3}\right),$$

$$\mu_{g_0^{-1}}\left(m + \frac{v}{\sqrt{k_j}}\right) = m + O\left(k_j^{-1/2}\right). \tag{4.8}$$

By definition of the preferred local coordinates, it follows from (4.7) and (4.8) that for $j \gg 0$ we have

$$\mathrm{dist}_M\left(\mu_{e^{\nu_j}}\left(m + \frac{w}{\sqrt{k_j}}\right), \mu_{g_0^{-1}}\left(m + \frac{v}{\sqrt{k_j}}\right)\right) \geq \frac{c}{2}\,\|\nu_j\|.$$

On the other hand, we can rewrite (4.6) as

$$\mathrm{dist}_M\left(\mu_{e^{\nu_j}}\left(m + \frac{w}{\sqrt{k_j}}\right), \mu_{g_0^{-1}}\left(m + \frac{v}{\sqrt{k_j}}\right)\right) \leq \frac{1}{j}\,\|\nu_j\|, \tag{4.9}$$

a contradiction. □

Returning to the proof of Proposition 4.1, by Lemma 4.1 and the off-diagonal estimate on the Szegö kernel in (6.1) of [**C**], we conclude

$$\left|\Pi_k\left(\mu_{g^{-1}}\left(m + \frac{w}{\sqrt{k}}\right), m + \frac{v}{\sqrt{k}}\right)\right| \leq C\, e^{-C_2\, k^{1/6}} \tag{4.10}$$

whenever $k \gg 0$ and $g \in A_k$. The statement follows easily from (4.10). □

Since our focus is on asymptotic expansions, we shall henceforth disregard the a term. Let us set $\beta_\varpi(g) = (\dim(V_\varpi)/2\pi)\,\chi_\varpi\left(g^{-1}\right)$ $(g \in G)$. Then we have

$$\begin{aligned}\Pi_{\varpi,k}&\left(x + \frac{w}{\sqrt{k}}, x + \frac{v}{\sqrt{k}}\right)_b \\ &= \int_{-\pi}^{\pi}\int_{B_k} \beta_\varpi(g)\; b_k(g)\, e^{-ik\vartheta}\,\Pi\left(\mu_{g^{-1}} \circ r_{e^{i\vartheta}}\left(x + \frac{w}{\sqrt{k}}\right),\right. \\ &\qquad \left. x + \frac{v}{\sqrt{k}}\right) d\vartheta\, dg.\end{aligned} \tag{4.11}$$

Suppose $G_m = \{g_1 = e, g_2, \ldots, g_{N_x}\}$. Let us define

$$E_k =: \left\{g \in G : \mathrm{dist}_G(g, e) < 2\, k^{-1/3}\right\}.$$

Then $B_k = \bigcup_{j=1}^{N_x} B_{jk}$, where $B_{jk} = g_j \cdot E_k$ $(1 \le j \le N_x,\ k \in \mathbb{N})$. Thus

$$\begin{aligned}
(4.12)\qquad & \Pi_{\varpi,k}\left(x + \frac{w}{\sqrt{k}}, x + \frac{v}{\sqrt{k}}\right)_b \\
&= \sum_j \int_{-\pi}^{\pi} \int_{B_{jk}} \beta_\varpi(g)\; b_k(g)\, e^{-ik\vartheta} \\
&\quad \Pi\left(\mu_{g^{-1}} \circ r_{e^{i\vartheta}}\left(x + \frac{w}{\sqrt{k}}\right), x + \frac{v}{\sqrt{k}}\right) d\vartheta\, dg \\
&= \sum_j \Pi_{\varpi,k}\left(x + \frac{w}{\sqrt{k}}, x + \frac{v}{\sqrt{k}}\right)_j,
\end{aligned}$$

where the j-th summand in (4.12) is

$$\begin{aligned}
(4.13)\qquad & \Pi_{\varpi,k}\left(x + \frac{w}{\sqrt{k}}, x + \frac{v}{\sqrt{k}}\right)_j \\
&= e^{-ik\vartheta_j} \int_{-\pi}^{\pi} \int_{E_k} \beta_\varpi(g_j\, g)\; b_k(g_j g)\, e^{-ik\vartheta} \\
&\quad \Pi\left(\mu_{g^{-1}g_j^{-1}} \circ r_{e^{i(\vartheta+\vartheta_j)}}\left(x + \frac{w}{\sqrt{k}}\right), x + \frac{v}{\sqrt{k}}\right) d\vartheta\, dg;
\end{aligned}$$

here $e^{i\vartheta_j} = h_{g_j}$ for every j. Notice that $b_k(g_j g) = b_k(g)$ for every $k \in \mathbb{N}$ and j, since b_k is G_m-invariant.

Let us examine the asymptotics of (4.13). To this end, fix $\epsilon > 0$ very small (but independent of k), and let $\gamma_0 + \gamma_1 = 1$ be a partition of unity on $(-\pi, \pi)$ with $\mathrm{supp}(\gamma_0) \subseteq (-\epsilon, \epsilon)$, $\mathrm{supp}(\gamma_1) \subseteq (-\pi, -\epsilon/2) \cup (\epsilon/2, \pi)$. Then

$$\Pi_{\varpi,k}\left(x + \frac{w}{\sqrt{k}}, x + \frac{v}{\sqrt{k}}\right)_j = \sum_{l=0}^{1} \Pi_{\varpi,k}\left(x + \frac{w}{\sqrt{k}}, x + \frac{v}{\sqrt{k}}\right)_{jl},$$

where $\Pi_{\varpi,k}(x + w/\sqrt{k}, x + v/\sqrt{k})_{jl}$ is given by (4.13) with the integrand multiplied by γ_l.

Lemma 4.2. $\Pi_{\varpi,k}(x + w/\sqrt{k}, x + v/\sqrt{k})_{j1} = O(k^{-\infty})$ *as* $k \to +\infty$.

Proof. If $k \gg 0$, $g \in B_{jk}$ and $|\vartheta| > \epsilon/2$, then

$$\mathrm{dist}_X\left(\mu_{g^{-1}g_j^{-1}} \circ r_{e^{i(\vartheta+\vartheta_j)}}\left(x + \frac{w}{\sqrt{k}}\right), x + \frac{v}{\sqrt{k}}\right) > \frac{\epsilon}{3}$$

(w and v are held fixed). Since the singular support of the Szegö kernel Π is the diagonal $\mathrm{diag}(X) \subseteq X \times X$, we conclude that

$$\Psi_{k,g}(h) =: \gamma_0(h)\, \beta_\varpi(g)\, b_k(g)\, \Pi\left(\mu_{g^{-1}g_j^{-1}} \circ r_{e^{i(\vartheta+\vartheta_j)}}\left(x + \frac{w}{\sqrt{k}}\right), x + \frac{v}{\sqrt{k}}\right)$$

is a bounded family of smooth functions on S^1 when $k \geq k_0$, $g \in B_{jk}$ and $|\vartheta| > \epsilon/2$; here γ_0 is interpreted as $\gamma_0(e^{i\vartheta})$, a cut-off function supported on a small open neighborhood of $1 \in S^1$.

In the same range, therefore, for every $l \in \mathbb{N}$ we can find a constant $C_l > 0$ such that $\left|\Psi^{(s)}_{k,g}\right| < C_l\, s^{-l}$ for every $s \in \mathbb{N}$, where $\Psi^{(s)}_{k,g}$ denotes the s-th Fourier coefficient of $\Psi_{k,g}$. In particular, this is true for $s = k$, hence $\left|\Psi^{(k)}_{k,g}\right| < C_l\, k^{-l}$. The same estimate then holds after integrating over B_k, and this implies the statement. □

We are reduced to studying the asymptotics of $\Pi_{\varpi,k}(x + w/\sqrt{k}, x + v/\sqrt{k})_{j1}$. To proceed, let us introduce the parametrix for the Szegö kernel contructed in [**BS**]. Thus, up to a smoothing term which does not contribute to the asymptotic expansion, we can represent Π as a Fourier integral operator of the form

$$\Pi(y, y') = \int_0^{+\infty} e^{it\psi(y,y')}\, s(y, y', t)\, dt \quad (y, y' \in X), \tag{4.14}$$

where the phase satisfies $\Im(\psi) \geq 0$, and the amplitude is a semiclassical symbol admitting an asymptotic expansion $s(y, y', t) \sim \sum_{t=0}^{+\infty} t^{n-j}\, s_j(y, y')$. In view of Lemma 4.2, inserting (4.14) into (4.13), and multiplying the integrand by γ_0, we obtain

$$\begin{aligned}
(4.15)\quad & \Pi_{\varpi,k}\left(x + \frac{w}{\sqrt{k}}, x + \frac{v}{\sqrt{k}}\right)_j \\
&\sim e^{-ik\vartheta_j} \int_0^{+\infty} \int_{-\epsilon}^{\epsilon} \int_{E_k} \gamma_0(\vartheta)\, \beta_\varpi(g_j\, g)\, b_k(g) \\
&\quad \times e^{i\left[t\psi\left(\mu_{(g_j g)^{-1}} \circ r_{e^{i(\vartheta+\vartheta_j)}}\left(x + w/\sqrt{k}\right), x + v/\sqrt{k}\right) - k\vartheta\right]} \\
&\quad \cdot s\left(\mu_{g^{-1} g_j^{-1}} \circ r_{e^{i(\vartheta+\vartheta_j)}}\left(x + \frac{w}{\sqrt{k}}\right), x + \frac{v}{\sqrt{k}}, t\right) dt\, d\vartheta\, dg \\
&= k\, e^{-ik\vartheta_j} \int_0^{+\infty} \int_{-\epsilon}^{\epsilon} \int_{E_k} e^{ik\Psi_{kj}(g,t,\vartheta)}\, A_{\varpi kj}(g, t, \vartheta)\, dt\, d\vartheta\, dg;
\end{aligned}$$

in the last equality we have performed the coordinate change $t \rightsquigarrow kt$, and set

$$\begin{aligned}
(4.16)\quad & \Psi_{kj}(g, t, \vartheta) =: t\psi\left(\mu_{(g_j g)^{-1}} \circ r_{e^{i(\vartheta+\vartheta_j)}}\left(x + w/\sqrt{k}\right), x + v/\sqrt{k}\right) - \vartheta, \\
(4.17)\quad & A_{\varpi kj}(g, t, \vartheta) =: \gamma_0(\vartheta)\, \beta_\varpi(g_j\, g)\, b_k(g) \\
&\qquad \cdot s\left(\mu_{g^{-1} g_j^{-1}} \circ r_{e^{i(\vartheta+\vartheta_j)}}\left(x + \frac{w}{\sqrt{k}}\right), x + \frac{v}{\sqrt{k}}, kt\right).
\end{aligned}$$

Let $\exp_G : \mathfrak{g} \to G$ be the exponential map, and let $E \subseteq \mathfrak{g}$ be a suitably small open neighborhood of the origin $0 \in \mathfrak{g}$, over which $\exp_G$ restricts to a diffeomorphism $E \to E' =: \exp_G(E)$. Since the shrinking open neighborhood $E_k \subseteq G$ of the identity $e \in G$ is definitely contained in E', we may express the integration in dg using the exponential chart. To this end, let us fix an orthonormal basis of $\mathfrak{g}$, so as to unitarily identify $\mathfrak{g}$ with $\mathbb{R}^{\mathrm{g}}$, and let us write ξ for the correspondig linear coordinates on $\mathfrak{g}$. We shall denote by $H_G(\xi)\, d\xi$ the local coordinate expression of the Haar measure dg under the exponential cart; the orthonormality of the chosen basis of $\mathfrak{g}$ implies that $H_G(0) = 1$.

With some abuse of language, we shall write b_k for the composition $b_k \circ \exp_G$, and assume that $b_k(\xi) = b(\sqrt[3]{k}\,\xi)$ for a fixed function $b = b_1$ on E. We shall also leave $\exp_G$ implicit in the expression for Ψ_{kj} and $A_{\varpi kj}$, which shall be viewed in the following as functions of $\xi \in E$. Thus, replacing g by ξ and dg by $H_G(\xi)\, d\xi$ in (4.15), and then performing the change of variable $\xi = \nu/\sqrt{k}$, we obtain

$$\begin{aligned}
\Pi_{\varpi,k}&\left(x + \frac{w}{\sqrt{k}}, x + \frac{v}{\sqrt{k}}\right)_j \\
&\sim k^{1-\mathrm{g}/2}\, e^{-ik\vartheta_j} \int_0^{+\infty} \int_{-\epsilon}^{\epsilon} \int_{\mathbb{R}^{\mathrm{g}}} e^{ik\Psi_{kj}\left(\frac{\nu}{\sqrt{k}},t,\vartheta\right)} \\
&\quad \cdot A_{\varpi kj}\left(\frac{\nu}{\sqrt{k}}, t, \vartheta\right)\, H_G\left(\frac{\nu}{\sqrt{k}}\right)\, dt\, d\vartheta\, d\nu.
\end{aligned} \tag{4.18}$$

Our next step will be to Taylor expand Ψ_{kj} in descending powers of $k^{1/2}$, by relying on (64) and (65) of [**SZ**]; to this end, we shall need the Heisenberg local coordinates of $\mu_{g^{-1}} \circ r_{e^{i(\vartheta+\vartheta_j)}}(\mu_{g_j^{-1}}(x + w/\sqrt{k}))$ when $g = \exp_G\left(\nu/\sqrt{k}\right)$.

Recalling that $m = \pi(x)$ and $G_m \subseteq G$ is the stabilizer subgroup, let us consider the isotropy representation $G_m \to \mathrm{GL}\big(T_mM\big)$, $g \mapsto d_m\mu_g$; for every $j = 1, \ldots, N_x$ and $w \in T_mM$, let us set $w_j =: d_m\mu_{g_j^{-1}}(w) \in T_mM$.

In view of our choice of $\omega = (i/2)\,\Theta$ as the reference Kähler form in our construction of Heisenberg local coordinates, we then have the following lemma.

Lemma 4.3. *Suppose $x \in X$, $\Phi \circ \pi(x) = 0$, and fix a system of Heisenberg local coordinates centered at x. Then there exist $\mathcal{C}^\infty$ functions $Q_j, T_j : \mathbb{C}^n \times \mathbb{R}^g \to \mathbb{C}^n$, vanishing at the origin to third and second orders, respectively, such that the following holds. For every $w \in T_{\pi(x)}M$, $-\pi < \vartheta < \pi$, $\nu \in \mathfrak{g}$, as $k \to +\infty$ the Heisenberg local coordinates of*

$$X_{j,k}(x, w, \nu) =: \mu_{e^{-\nu/\sqrt{k}}}\left(r_{e^{i\vartheta_j}} \circ \mu_{g_j^{-1}}\left(x + \frac{w}{\sqrt{k}}\right)\right)$$

are given by

$$\left(\frac{1}{\sqrt{k}}\left[w_j - \nu_M(m)\right] + T_j\left(\frac{w}{\sqrt{k}}, \frac{\nu}{\sqrt{k}}\right), \frac{1}{k}\,\omega_m\Big(\nu_M(m), w_j\Big) + Q_j\left(\frac{w}{\sqrt{k}}, \frac{\nu}{\sqrt{k}}\right)\right).$$

Proof. Set $m = \pi(x)$. By definition of ν_M, $\mu_{e^{-\nu/\sqrt{k}}} \circ \mu_{g_j^{-1}}(m + w/\sqrt{k})$ has underlying preferred local coordinates $(1/\sqrt{k})\,\big(w_j - \nu_M(m)\big) + T((1/\sqrt{k})\,w_j, (1/\sqrt{k})\,\nu)$, where $T : \mathbb{C}^n \times \mathbb{R}^{\mathrm{g}} \to \mathbb{C}$ vanishes to second order at the origin (here, w, $\nu_M(m) \in T_m M$ are identified with their images in $\mathbb{C}^n$ under the unitary isomorphism $T_m M \to \mathbb{C}^n$ induced by the chosen preferred local coordinates centered at m).

Therefore, the Heisenberg local coordinates of $X_{j,k}(x, w)$ have the form $(\theta(k^{-1/2}), k^{-1/2}\,(w_j - \nu_M(m)) + T(k^{-1/2}\,w_j, k^{-1/2}\,\nu))$, for an appropriate smooth function $\theta : (-\delta, \delta) \to \mathbb{R}$.

In order to determine θ, set $\gamma_s(t) =: \mu_{e^{-t\nu}} \circ \mu_{g_j^{-1}}(x + sw)$, defined and smooth for all sufficiently small $s, t \in \mathbb{R}$. Let us write $w_j = p_{w_j} + iq_{w_j}$, $\nu_M(m) = p_\nu + iq_\nu$, where $p_{w_j}, q_{w_j}, p_\nu, q_\nu \in \mathbb{R}^n$. The preferred local coordinates of $\pi\big(\gamma_s(t)\big) = \mu_{e^{-t\nu}} \circ \mu_{g_j^{-1}}(m + sw)$ are $(sp_{w_j} - tp_\nu) + i(sq_w - tq_\nu) + T(sw, t\nu)$. Therefore, the Heisenberg local coordinates of $\gamma_s(t)$ have the form

$$\Big(\widetilde{\theta}(s,t), \big(sp_{w_j} - tp_\nu\big) + i\big(sq_{w_j} - tq_\nu\big) + T(sw, t\nu)\Big), \tag{4.19}$$

for an appropriate smooth function $\widetilde{\theta}(s,t)$; clearly, $\theta(u) = \widetilde{\theta}(u,u)$.

Claim 4.1. We have $\widetilde{\theta}(s,t) = -\vartheta_j + (st)\cdot d_0(w_j, \nu) + \widetilde{\theta}_1(sw_j, t\nu)$, for appropriate smooth functions $d_0, \widetilde{\theta}_1 : \mathbb{C}^n \times \mathbb{R}^{\mathrm{g}} \to \mathbb{R}$, with $\widetilde{\theta}_1$ vanishing to third order at the origin.

Proof of Claim 4.1. Recall that Heisenberg local coordinates depend on the choice of a preferred local holomorphic frame e_L of L, an open neighborhood $U \subseteq M$ of m; as discussed in Section 3, without loss of generality we may assume that U is G_m-invariant and $g^*(e_L^*) = h_g \cdot e_L^*$, $\forall\, g \in G_m$. Let us write $\sigma = e_L^*$. We have $x + sw = \sigma(m + sw)/\|\sigma(m + sw)\|$, where $m + sw \in U$ is the point with local preferred holomorphic coordinates $w \in \mathbb{C}^{\mathrm{n}}$. Therefore,

$$\begin{aligned}\mu_{g_j}^{-1}(x + sw) &= \mu_{g_j}^{-1}\Big(\sigma(m + sw)\Big)\Big/\|\sigma(m + sw)\| \\ &= h_{g_j}^{-1}\,\sigma\Big(\mu_{g_j}^{-1}(m + sw)\Big)\Big/\Big\|\sigma\Big(\mu_{g_j}^{-1}(m + sw)\Big)\Big\|\end{aligned}$$

has Heisenberg local coordinates $(-\vartheta_j, z_j(w, s))$, where $z_j(w, s)$ are the local preferred holomorphic coordinates of $\mu_{g_j}^{-1}(m + sw)$. Therefore, $\widetilde{\theta}(s, 0) = -\vartheta_j$ for all s.

We conclude that $\widetilde{\theta}(s,t) = -\vartheta_j + t\,R(s,t)$, for some smooth function R.

On the other hand, X' is G-invariant, and G acts horizontally on it (in other words, for every $x \in X'$ and $\xi \in \mathfrak{g}$ we have $\xi_X(x) = \xi_M^\sharp(\pi(x))$, where $\xi_M^\sharp$ denotes the horizontal lifting of ξ_M). Lemmata 2.4 and 3.3 of [**DP**] then imply that $\theta(0,t) = t^3\,S(t)$ for a smooth function $S(t)$. Thus, $R(s,t) = t^2 R_1(t) + s\,d(s,t)$ for smooth functions $R_1(t)$, $d(s,t)$. We conclude that $\widetilde{\theta}(s,t) = t^3\,R_1(t) + st\,d(s,t)$, and the statement follows by setting $d_0 = d(0,0)$.

Returning to the proof of Lemma 4.3, in order to determine d_0 we recall that the expression for α in Heisenberg local coordinate is $\alpha = d\theta + p\,dq - q\,dp + \beta(\|z\|^2)$. Inserting the local expression for $\gamma_s(t)$ that we obtain from (4.19) and Claim 4.1, we obtain

$$\begin{aligned}\gamma_s^*(\alpha) &= \Big\{ \Big[sd_0 + \big(sp_{w_j} - tp_\nu\big)\cdot(-q_\nu) - \big(sq_{w_j} - tq_\nu\big)\cdot(-p_\nu)dt \Big] \\ &\quad + G_1(s,t,\nu,w_j)\Big\}\ dt \\ &= \Big\{ s\,\Big[d_0 + \omega_m(\nu_M, w_j)\Big] + G_1(s,t,\nu,w_j)\Big\}\ dt, \end{aligned} \tag{4.20}$$

where G_1 vanishes to second order for $(s,t) = (0,0)$.

On the other hand, we have $d_t(\gamma_s)(1) = -\nu_X\big(\gamma_s(t)\big)$. Therefore

$$\gamma_s^*(\alpha)(t) = -\iota(\nu_X)\alpha\Big(\pi\big(\gamma_s(t)\big)\Big)\,dt = \Phi^\nu\Big(\pi\big(\gamma_s(t)\big)\Big)\,dt, \tag{4.21}$$

having used that $\Phi^\nu = -\iota(\nu_X)\alpha$.

Because of the G-equivariance of Φ, $\Phi\circ\pi(\gamma_0(t)) = \Phi\circ\pi(\mu_{g_j e^{t\nu}}(x)) = 0$ for every sufficiently small t; therefore, $\partial\Phi\circ\gamma/\partial t|_{(0,t)} = 0$ identically, where with abuse of language we have written Φ for $\Phi\circ\pi : X \to \mathfrak{g}$. This implies

$$\begin{aligned}\Phi^\nu\Big(\pi\big(\gamma_s(t)\big)\Big) &= s\,d_m\Phi^\nu(w_j) + G_3(s,t,\nu,w_j) \\ &= 2s\,\omega_m(\nu_M, w_j) + G_3(s,t,\nu,w_j),\end{aligned} \tag{4.22}$$

where $m = \pi(x)$, and G_3 vanishes to second order for $(s,t) = (0,0)$.

Comparing (4.21) and (4.22) with (4.20), we obtain $d_0 = \omega_m(\nu_M, w)$, $G_2 = G_3$. To complete the proof of Lemma 4.3, we need only take $s = t = 1/\sqrt{k}$, and remark that in Heisenberg local coordinates $r_{e^{i\vartheta_j}}$ is simply translation by ϑ_j.

Let us set $\psi_2(u,v) = u\cdot\overline{v} - 1/2\left(\|u\|^2+\|v\|^2\right)$ $(u,\,v\in\mathbb{C}^n)$. Invoking (63)–(65) of [**SZ**], in view of Lemma 4.3 we obtain that Ψ_{kj} in (4.16) has the form:

$$\begin{aligned}\Psi_{kj}(g,t,\vartheta) &= i\,t\,\left(1-e^{i\vartheta}\right)-\vartheta \\ &\quad+\frac{t}{k}\,e^{i\vartheta}\left[\omega_m\big(\nu_M(m),w_j\big)-i\,\psi_2\big(w_j-\nu_M(m),v\big)\right] \\ &\quad+\,t\,e^{i\vartheta}\,R_j\left(\frac{\nu_M(m)}{\sqrt{k}},\frac{w}{\sqrt{k}},\frac{v}{\sqrt{k}}\right),\end{aligned} \tag{4.23}$$

where $R_j:(\mathbb{C}^n)^3\to\mathbb{C}$ is a smooth function vanishing to third order at the origin.

Let us now insert (4.23) in (4.18). We obtain

$$\begin{aligned}&\Pi_{\varpi,k}\left(x+\frac{w}{\sqrt{k}},x+\frac{v}{\sqrt{k}}\right)_j \sim k^{1-\mathrm{g}/2}e^{-ik\vartheta_j}\int_0^{+\infty} \\ &\qquad\int_{-\epsilon}^{\epsilon}\int_{\mathbb{R}^{\mathrm{g}}} e^{ik\Psi(t,\vartheta)}\widetilde{A}_{\varpi kj}\left(\nu,w,v,t,\vartheta\right)dtd\vartheta d\nu,\end{aligned} \tag{4.24}$$

where we have set

$$\Psi\left(t,\vartheta\right) =: i\,t\,\left(1-e^{i\vartheta}\right)-\vartheta, \tag{4.25}$$

$$\begin{aligned}\widetilde{A}_{\varpi kj}\left(\nu,w,v,t,\vartheta\right) &=: e^{t\,e^{i\vartheta}\left[\psi_2\left(w_j-\nu_M(m),v\right)+i\omega_m\left(\nu_M(m),w_j\right)\right]} \\ &\quad\cdot e^{i\,k\,t\,e^{i\vartheta}\,R_j(\nu_M(m)/\sqrt{k},\ w/\sqrt{k},\ v/\sqrt{k})} \\ &\quad\cdot A_{\varpi kj}(\nu/\sqrt{k},t,\vartheta)\,H_G(\nu/\sqrt{k}).\end{aligned} \tag{4.26}$$

A straightforward computation shows that

$$\psi_2\big(w_j-\nu_M(m),v\big)+i\omega_m\big(\nu_M(m),w_j\big) = T_h+T_t+T_v+T_{vt},$$

where

$$\begin{aligned}&T_h =: \psi_2\big(w_h,v_h\big),\quad T_t =: -\frac{1}{2}\big\|w_t-v_t\big\|^2,\quad T_v =: -\frac{1}{2}\big\|w_{jv}-\nu_M(m)-v_{jv}\big\|^2, \\ &T_{tv} =: -\,i\,\omega_m\big(w_{jv}-\nu_M(m)-v_{jv},w_{jt}+v_{jt}\big)+i\Big[\omega_m\big(w_v,w_t\big)-\omega_m\big(v_v,v_t\big)\Big]\end{aligned}$$

(notice that the map $w\mapsto w_j$ induced by the isotropy action of $g_j^{-1}\in G_m\subseteq G$ is an isometry of T_mM, since G preserves the metric of M).

We may insert in (4.17) the asymptotic expansion for the classical symbol $s(x,y,t)$ appearing in the parametrix for Π, and use Taylor expansion in

$g = \nu/\sqrt{k}$, $w/\sqrt{k}$ and $v/\sqrt{k}$ in descending powers of $k^{1/2}$, to deduce that

$$\widetilde{A}_{\varpi kj}\,(\nu, w, v, t, \vartheta) \sim \sum_{l\geq 0} a_{\varpi jl}\,(\nu, w, v, t, \vartheta)\; k^{n-l/2}, \tag{4.27}$$

where every coefficient has the form

$$a_{\varpi jl}\,(\nu, w, v, t, \vartheta) = e^{te^{i\vartheta}(T_h+T_t+T_v+T_{vt})}\, p_{\varpi jl}\,(\nu, w, v, t, \vartheta)$$

and each $p_{\varpi jl}\,(\nu, w, v, t, \vartheta)$ is a polynomial in ν, w and v with coefficients depending on x, t, ϑ and ϖ. In particular, the leading coefficient is

$$a_{\varpi j0}\,(\nu, w, v, t, \vartheta) = e^{t\,e^{i\vartheta}\left(T_h+T_t+T_v+T_{vt}\right)}\,\gamma_0(\vartheta)\,\beta_\varpi(g_j)\,s_0(x,x)\,t^n. \tag{4.28}$$

Thus,

$$\begin{aligned}\Pi_{\varpi,k}\left(x+\frac{w}{\sqrt{k}}, x+\frac{v}{\sqrt{k}}\right)_j &\sim k^{1-\mathrm{g}/2}\,e^{-ik\vartheta_j} \\ &\sum_{l\geq 0}\left(\int_0^{+\infty}\int_{-\epsilon}^{\epsilon}\int_{\mathbb{R}^{\mathrm{g}}} e^{ik\Psi(t,\vartheta)}\right. \\ &\left.\times a_{\varpi jl}\,(\nu, w, v, t, \vartheta)\;k^{n-l/2}\,dt\,d\vartheta\,d\nu\right).\end{aligned} \tag{4.29}$$

To determine the leading asymptotics of (4.29), let us first integrate (4.28) in $d\nu$. By our choice of Heisenberg local coordinates, we may unitarily identify (T_mM, ω_m) with $(\mathbb{C}^n, \omega_0)$, where ω_0 is the standard symplectic structure on $\mathbb{C}^n$; let g_0 be the standard scalar product on $\mathbb{C}^n$, so that $\omega_0\big(\mathbf{a}, \mathbf{b}\big) = -g_0\big(\mathbf{a}, J_0(\mathbf{b})\big)$, $\forall\, \mathbf{a},\, \mathbf{b} \in \mathbb{C}^n$, where J_0 is multiplication by i. We shall view $S_m : \nu \mapsto \nu_M(m)$ as a map $\mathfrak{g} \to \mathbb{C}^n$. Let us set $\lambda = t\,e^{i\vartheta}$. Up to a multiplicative factor, we are led to integrating

$$e^{\lambda\left[-1/2\left\|w_{jv}-\nu_M(m)-v_{jv}\right\|^2+i\,g_0\left(w_{jv}-\nu_M(m)-v_{jv},J_0(w_{jt}+v_{jt})\right)\right]} \tag{4.30}$$

in $d\nu$.

Recall that the ν coordinates are induced by the choice of an orthonormal basis of $\mathfrak{g}$; we can shift the integration to the tangent space of the G-orbit through m, $\mathfrak{g}_M(m) \subseteq T_mM$. Let us then choose an orthonormal basis of $\mathfrak{g}_M(m)$, and let β be the corresponding linear coordinates. We can use β as integration variable, by the relation $\beta = S_m(\nu)$. By Lemma 3.9 of [**DP**], after performing the change of variables $\beta \mapsto \beta - (w_{jv} - v_{jv})$ we are left with

$$\begin{aligned}&\frac{1}{V_{\text{eff}}(m)\,|G_m|}\int_{\mathbb{R}^{\mathrm{g}}} e^{t\,e^{i\vartheta}\left[-1/2\|\beta\|^2-i\,g_0\left(\beta,J_0(w_{jt}+v_{jt})\right)\right]}\,d\beta \\ &\quad= \frac{(2\pi)^{\mathrm{g}/2}}{V_{\text{eff}}(m)\,|G_m|}\cdot\frac{1}{\sqrt{t}\,e^{i\vartheta/2}}\,e^{-\frac{1}{2}\,t\,e^{i\vartheta}\,\|w_{tj}+v_{tj}\|^2};\end{aligned} \tag{4.31}$$

in fact, since $t > 0$ (4.31) is valid when $\vartheta = 0$ because $-(1/2)\|\beta\|^2$ equals its own Fourier transform, and consequently by analytic continuation it holds for all $\vartheta \in (-\pi/2, \pi/2)$.

Let us next consider the case a general $a_{\varpi jl}(\nu, w, v, t, \vartheta)$. Up to multiplicative factors polynomial in w and v, we are led to integrate the product of (4.30) times a monomial in ν. Again up to an appropriate scalar factor, this amounts to multiplying the integrand in (4.31) by a monomial in β, hence evaluating an appropriate higher derivative of $e^{-\|\beta\|^2/2}$ in $J_0(w_{j\mathrm{t}} + v_{j\mathrm{t}}) \in \mathfrak{g}_M(m)$. We are thus left with the product of the right hand side in (4.31) times a polynomial in w_t and v_t.

We can now insert (4.31) in (4.27) and (4.24) to obtain

$$\begin{aligned} \text{(4.32)} \quad & \Pi_{\varpi,k}\left(x + \frac{w}{\sqrt{k}}, x + \frac{v}{\sqrt{k}}\right)_j \\ & \sim \quad k^{1-\mathrm{g}/2}\, e^{-ik\vartheta_j} \int_0^{+\infty} \int_{-\epsilon}^{\epsilon} e^{ik\Psi(t,\vartheta)}\, S_{\varpi j}(w, v, t, \vartheta, k)\, dt\, d\vartheta \end{aligned}$$

where $S_{\varpi j}(w, v, t, \vartheta, k) \sim \sum_{l \ge 0} S_{\varpi jl}(w, v, t, \vartheta)\, k^{n-l/2}$ and the coefficients of the expansion are as follows.

First, the leading coefficient is

(4.33)
$$S_{\varpi j0}(w, v, t, \vartheta) = \frac{(2\pi)^{\mathrm{g}/2}}{V_{\mathrm{eff}}(m)\,|G_m|} \cdot \frac{1}{\sqrt{t\, e^{i\vartheta/2}}}\, \gamma_0(\vartheta)\, \beta_\varpi(g_j) s_0(x, x)\, t^n\, e^{t\, e^{i\vartheta}\, \Gamma(w,v)}$$

where $\Gamma(w, v) = \psi_2(w_h, v_h) - \|w_t\|^2 - \|v_t\|^2 + i[\omega_m(w_v, w_t) - \omega_m(v_v, v_t)]$.

Next, for every $l \ge 1$ we have $S_{\varpi jl}(w, v, t, \vartheta) = p_{\varpi jl}(w, v, t, \vartheta)\, e^{t\, e^{i\vartheta}\, \Gamma(w,v)}$, where $p_{\varpi jl}(w, v, t, \vartheta)$ is a poynomial in w and v.

Thus we are left with an oscillatory integral whose phase Ψ, given by (4.25), is the same phase appearing in the discussion of the scaling asymptotics of non-equivariant Szegö kernels in Section 3 of [**SZ**]. In particular, Ψ has non-negative imaginary part, and a unique stationary point for $t = 1$ and $\vartheta = 0$; furthermore, at this point the Hessian of Ψ is

$$\Psi''(1, 0) = \begin{pmatrix} 0 & 1 \\ 1 & i \end{pmatrix}$$

Hence $(1, 0)$ is a non-degenerate stationary point of Ψ. Arguing as in *loc. cit.*, the contribution coming from $|t| \ge 2$, say, is rapidly decreasing, and by the stationary phase method for complex oscillatory integrals (Theorem 7.7.5

of [**H**]) there is an asymptotic expansion:

$$\Pi_{\varpi,k}\left(x+\frac{w}{\sqrt{k}},x+\frac{v}{\sqrt{k}}\right)_j \sim k^{1-\mathrm{g}/2}\,e^{-ik\vartheta_j}\,\frac{1}{\sqrt{\det\big(k\Psi''(1,0)/2\pi i\big)}} \times\sum_{s=0}^{+\infty}k^{-s}\ \ L_s\big(S_{\varpi j}\,(w,v,t,\vartheta,k)\,\big)\big|_{t=1,\vartheta=0}\,, \tag{4.34}$$

where L_0 is the identity, and L_s is a suitable differential operator of degree $2s$ in (t,θ) for any $s=0,1,2\ldots$. The statement then follows from the previous description of the phase; in particular, each coefficient in the asymptotic expansion is the product of $e^{\Gamma(w,v)}$ and a polynomial in w, v.

The statement of the theorem follows by summing over j.

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