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REDUCTION AND DUALITY IN GENERALIZED GEOMETRY

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Extending our reduction construction in (S. Hu, *Hamiltonian symmetries and reduction in generalized geometry*, Houston J. Math., to appear, math.DG/0509060, 2005.) to the Hamiltonian action of a Poisson Lie group, we show that generalized Kähler reduction exists even when only one generalized complex structure in the pair is preserved by the group action. We show that the constructions in string theory of the (geometrical) T-duality with H-fluxes for principle bundles naturally arise as reductions of factorizable Poisson Lie group actions. In particular, the groups involved may be non-abelian.

1. Introduction

In this article, we propose a candidate of the geometric realization of part of the ansatz of T-duality with H-flux in the physics literature, using reductions in generalized Kähler geometry. T-duality has long been intensively studied in physics and has made its marks in mathematics as well, e.g., via mirror symmetry [**34**]. The context of our reduction construction is the Hamiltonian Poisson action of Poisson Lie group. Classically, such reduction in symplectic category was first discussed in [**24**] and our construction here should be viewed as the generalization of it to generalized geometry.

Generalized geometry is introduced by Hitchin [**13**] in the context of generalized Calabi–Yau manifolds. The general theory of generalized complex and Kähler geometries is first developed by Gualtieri in his thesis [**12**]. Various reduction constructions in the context of generalized geometry are developed by [**7**, **15**, **23**, **33**, **37**]. The approach taken here follows the point of view of Hamiltonian symmetries [**15**].

It is now well known that a generalized complex structure induces a canonical Poisson structure, e.g., [**1**, **10**, **12**, **15**], also § 3.3. Let G be a Poisson Lie group with dual group $\hat{G}$, then the Hamiltonian Poisson action with moment map as defined in [**24**] (also see Definition B.12) can be adapted to the generalized complex geometry (Definition 3.8), as well as the generalized Kähler geometry (Definition 3.13). We then have the first results on reduction (cf. Theorem 3.12 and 3.16):

Theorem 1.1. *Suppose $(M, \mathcal{J})$ is an extended complex manifold with Hamiltonian G-action, whose moment map is $\mu : M \to \hat{G}$. Let $M_0 = \mu^{-1}(\hat{e})$, where $\hat{e} \in \hat{G}$ is the identity element. Suppose that $\hat{e}$ is a regular value and the geometrical action of G is proper and free on M_0. Then there is a natural extended complex structure on the reduced space $Q = M_0/G$.*

If furthermore, $(M, \mathcal{J}_1, \mathcal{J}_2)$ is an extended Kähler manifold and the G-action is $\mathcal{J}_1$-Hamiltonian. Then there is a natural extended Kähler structure on the manifold Q.

The notion *extended* (+ *structures*) is adopted to emphasize that we consider $\mathcal{T}M$ as an extension of TM by T^*M, instead of as a direct sum, with an exact Courant algebroid structure (cf. § 2.6). When a splitting is chosen, or equivalently, $\mathcal{T}M$ is identified with $\mathbb{T}M$ with an H-twisted Courant algebroid structure, we will use the notion *H-twisted generalized* (+ *structures*). Now, when the action of G preserves a splitting of $\mathcal{T}M$, then the reduced extended tangent bundle in the theorem naturally splits and the twisting form on Q can be explicitly written down (cf. Corollary A.5).

In investigating T-duality, we are guided by the detailed computation in [**15**] of the example of $\mathbb{C}^2 \setminus \{(0,0)\}$ with non-trivial twisting class, which we recall in Example 5.10. The following definition is crucial (cf. Definition 4.1).

Definition 1.2. Let $(\tilde{\mathfrak{g}}, \mathfrak{g}, \hat{\mathfrak{g}})$ be the Manin triple defined by a Poisson Lie group G (cf. Theorem B.6, also [**24**]), with dual group $\hat{G}$. The (infinitesimal) action of $\tilde{\mathfrak{g}}$ on M is *bi-Hamiltonian* if it is induced by a $\mathcal{J}_1$-Hamiltonian action of G together with a $\mathcal{J}_2$-Hamiltonian action of $\hat{G}$.

Suppose that the Manin triple $(\tilde{\mathfrak{g}}, \mathfrak{g}, \hat{\mathfrak{g}})$ is the Lie algebras of the (local) double Lie group $(\tilde{G}, G, \hat{G})$ (cf. Theorem B.6). We impose two sets of assumptions, on the group G (Assumption 4.2(0)) and on the action of $\tilde{G}$ (the rest of Assumption 4.2). Our first result in this direction is the *factorizable reduction* (cf. Theorem 4.5).

Theorem 1.3. *Under Assumption* 4.2 *and suppose that the action of $\tilde{G}$ on M_0 is proper and free, then the reduced space $\tilde{Q} = M_0/\tilde{G}$ of a bi-Hamiltonian action of factorizable Poisson Lie group admits a natural transitive Courant algebroid* (*Definition* 4.4)*.*

With further restrictions, i.e., the reduction exists with respect to either of the actions of G and $\hat{G}$ as given in Theorem 3.16, the factorizable reduction as in Theorem 4.5 can be factored in two ways,

$$M_0 \xrightarrow{/G} Q \xrightarrow{/\hat{G}} \tilde{Q} \quad \text{or} \quad M_0 \xrightarrow{/\hat{G}} \hat{Q} \xrightarrow{/G} \tilde{Q}.$$

We then propose (cf. Definition 5.1).

Definition 1.4. The extended Kähler structures on Q and $\hat{Q}$ are *Courant dual* to each other.

We note that any of the groups $\tilde{G}, G$ or $\hat{G}$ could be non-abelian. Thus we have a candidate for the *non-abelian* duality *with background twistings.* The more stringent but natural assumption that G and $\hat{G}$ commute in $\tilde{G}$ implies that $\tilde{G}$ is in fact a torus $\tilde{T}$. The choice of terminology in the above is supported by the following theorem (cf. Theorem 5.8) when the action of $\tilde{T}$ preserves a splitting of $\mathcal{T}M$.

Theorem 1.5. *After applying a natural B-transformation on M, which does not change the reduced Courant algebroid on $\tilde{Q}$, the twisting forms h and $\hat{h}$ of the structures on Q and $\hat{Q}$, respectively, satisfy:*

$$\hat{\pi}^*\hat{h} - \pi^*h = d(\hat{\Theta} \wedge \Theta),$$

where π and $\hat{\pi}$ are the quotient maps and Θ and $\hat{\Theta}$ are connection forms of principle torus bundles.

We point out that the equation above appears as a part of the *definition* of T-duality with H-flux of principle torus bundles in the literature (also see below). Here, it appears as a geometrical consequence. The notion of T-duality group in the literature can be recovered (§ 6) with our construction.

We describe the content of the article in the following. It is helpful to recall the basics of Lu's construction (see also § 8 Appendix B). A Poisson Lie group G is a Lie group with a multiplicative Poisson structure, i.e., $m : G \times G \to G$ is a Poisson map. Let (M, ω) be a symplectic manifold, the action of G on M is called Poisson if the map $G \times M \to M$ defining the action is Poisson, with the product Poisson structure on $G \times M$. In [**24**], Lu defined momentum mapping for such Poisson actions (see also Definition B.12 and Theorem B.13) and went on to show that symplectic reduction can be carried out for Poisson actions with momentum mapping, although in general, the symplectic structure ω is not invariant under Poisson actions.

Section 2 recalls the useful facts concerning the action of the group of generalized symmetries $\tilde{\mathscr{G}} = \mathrm{Diff}(M) \ltimes \Omega^2(M)$, the H-twisted Lie bracket on $\mathscr{X} = \Gamma(TM) \oplus \Omega^2_0(M)$, Courant algebroid and generalized complex structures and explain in more detail the notion of extended structures. These results are not new and details may be found in, for example, [**8**, **12**, **13**, **15**] and the references therein.

We show in § 3 that the momentum mapping as defined in [**24**] can be extended to the generalized geometry (Definition 3.8), and the reduction construction for symplectic manifold can be extended to generalized complex manifold, as well as generalized Kähler manifold (Theorem 1.1). Along the way, we obtain Lemma 3.3, which can be viewed as an extension of Moser's argument for symplectic geometry (Remark 3.5). We note that similar to the case of symplectic geometry in [**24**], the generalized complex structure may not be preserved by the group action. In fact, in our construction of generalized Kähler reduction, none of the two generalized complex structures need to be preserved by the group action, as long as certain sub-bundle of $\mathbb{T}M$ is preserved (Remark 3.17). We remark that reduction of Courant algebroid (§ 7, Appendix A) as well as reduction of generalized Kähler structure have been discussed in various other works [**12**, **23**, **33**].

One of the features of generalized Kähler geometry is that the two generalized complex structures are on the same footing, which is not at all obvious in the classical Kähler geometry. In fact, this is one of the reasons that generalized Kähler geometry could serve as the natural category of discussing duality. Generalized Kähler geometry is relevant also from the result in [**12**], that it is equivalent to the bi-Hermitian geometry, which has been shown to be the string background for $N = (2, 2)$ supersymmetry ([**11**, **6**] and references therein). The notion of T-duality with H-flux in abelian case is proposed in [**3**] and then has been worked to much more general situations which involve non-commutative [**26**] and non-associative geometries [**5**]. The motivation in physics is that the physical theories on T-dual spaces are isomorphic and thus provides insights to what the physics is about. Here we concentrate on the more geometrical duality and leave the non-classical cases to furture work.

We first describe the construction of T-duality with H-flux from the existing literature in the following. To simplify matters, we restrict to $T = S^1$, where many complications do not arise. Let $p : E \to M$ be an S^1-principle bundle with connection form $\Theta \in \Omega^1(E)$ and curvature form $\Omega \in \Omega^2(M)$. Let $H \in \Omega^3(E)^{S^1}$ be a closed S^1-invariant 3-form representing integral class $[H] \in H^3(E, \mathbb{Z})$. By construction, there is a form $h \in \Omega^3(M)$ so that $p^*h = H - \Theta \wedge \hat{\Omega}$. Let $\hat{\Omega} \in \Omega^2(M)$ be the integration of H along the fibre of E, then $[\hat{\Omega}] \in H^2(M, \mathbb{Z})$ and there is a principle S^1-bundle $\hat{p} : \hat{E} \to M$ whose first Chern class is $[\hat{\Omega}]$. In particular, we may choose a connection form $\hat{\Theta} \in \Omega^1(\hat{E})$ whose curvature form is $\hat{\Omega}$. Let $\hat{H} = \hat{p}^*h + \hat{\Theta} \wedge \Omega$, then $\hat{H} \in \Omega^3(\hat{E})^{S^1}$ is closed and the pair $(\hat{E}, \hat{H})$ is said to be T-dual to the pair (E, H). One may also consider the correspondence space $E \times_M \hat{E}$, whose projection to E and $\hat{E}$ is denoted π and $\hat{\pi}$, respectively. Then the forms satisfy $\hat{\pi}^*\hat{H} - \pi^*H = d(\hat{\Theta} \wedge \Theta)$. We may summarize this description with the following diagram.

$$\begin{tikzcd} & E \times_M \hat{E} \arrow[dl, "\pi"'] \arrow[dr, "\hat{\pi}"] & \\ (E, H; \Theta) \arrow[dr, "p"'] & & (\hat{E}, \hat{H}; \hat{\Theta}) \arrow[dl, "\hat{p}"] \\ & (M; h, \Omega, \hat{\Omega}) & \end{tikzcd}$$

For higher-dimensional torus, it is argued (see [**26**, **5**] and references therein) that various conditions are needed, on the action and twisting form H, in order for the dual space to be classical. Otherwise, it would be of one of the non-classical geometries.

The idea of applying generalized geometry in describing T-duality is introduced by Gualtieri [**12**] and Cavalcanti [**8**], where the first efforts were made. The guiding example for us is given in Example 5.10. By this example, we see that it's possible for the same function to serve as moment map for Hamiltonian group actions with respect to either generalized complex structure and thus provides a diagram similar to the one above. Another important input is from [**8**], where Cavalcanti showed that the Courant algebroids defined by invariant sections on T-dual S^1-principle bundles are isomorphic.

On the physics side, there is vast literature on T-duality, both with or without H-flux, abelian or non-abelian, for principle bundles or fibration with singular fibres. The approach of realizing dual theories by quotient construction appeared in [**30**, **17**], where it's argued that gauging different chiral currents produces dual σ-models. More recently, there is work of Hull [**16**], which discusses T-duality in the *doubled formalism*. The formalism is to look at the correspondence space as principle bundle of a doubled torus, consisting of the product of a dual pair of torus with the natural pairing on the Lie algebra. Then the group automorphisms preserving the pairing corresponds to the T-duality group. The idea of looking to Poisson Lie group in considering duality goes back to a series of papers by Klimčík and/or Ševera starting with [**18**, **20**, **21**], where Poisson Lie target space duality was proposed as the framework of non-abelian T-duality. The papers [**19**, **16**, **29**, **35**] and references therein contain more recent developement in this direction.

Starting from § 4, we discuss T-duality with H-fluxes in the context of generalized (Kähler) geometry, which includes both abelian and non-abelian groups. In § 4, we define the notion of bi-Hamiltonian action (Definition 4.1) and discuss reduction of bi-Hamiltonian action of factorizable Poisson Lie groups (Theorem 4.5). The main point is that the reduced structure is a transitive Courant algebroid on the reduced space (Definition 4.4). We note that the reduction considered in § 4 can be factorized in two ways and in

§ 5 we define the two intermediate stages as being *Courant dual* to each other (Definition 5.1). Our construction then provides an isomorpism of Courant algebroids defined by the invariant sections of Courant dual structures (Proposition 5.3), extending the result in [**8**], with a more geometrical method. The upshot is that in Theorem 5.8, we show that T-duality, as described above, can arise from a special case of Courant duality. The notion of T-duality group is essential in the full picture of T-duality with H-fluxes and we discuss it in § 6. We note that it is more desirable that T-duality is constructed starting from $(E, H; \Theta)$ instead of from the correspondence space as the approach here. The construction of the correspondence space from one of the reduced space will be discussed in the forthcoming paper by S. Hu and B. Uribe.

To make the paper more self-contained, in § 7 Appendix A, we present a construction of reduction of extended tangent bundles which is used in this article. In § 8 Appendix B, we collect various facts on Lie bialgebra, Poisson Lie group and Hamiltonian action.

2. Preliminaries

We recall the preliminaries of generalized geometry and symmetries. As mentioned in the introduction, the results are not new and for details, the readers are referred to the literatures, for example [**8**, **12**, **13**, **15**], and the references therein.

2.1. For a smooth manifold M, let $\mathbb{T}M = TM \oplus T^*M$ and $\tilde{\mathscr{G}} = \mathrm{Diff}(M) \ltimes \Omega^2(M)$. Let $\lambda, \mu \in \mathrm{Diff}(M)$ and $\alpha, \beta \in \Omega^2(M)$, then the product on $\tilde{\mathscr{G}}$ is given by

$$(\lambda, \alpha) \cdot (\mu, \beta) = (\lambda\mu, \mu^*\alpha + \beta).$$

Let $\mathfrak{X} = X + \xi$ with $X \in TM$ and $\xi \in T^*M$, then the (left) action of $\tilde{\mathscr{G}}$ on $\mathbb{T}M$ is given by

$$(\lambda, \alpha) \circ (X + \xi) = \lambda_* X + (\lambda^{-1})^*(\xi + \iota_X \alpha).$$

The Lie algebra of $\tilde{\mathscr{G}}$ is $\tilde{\mathscr{X}} = \Gamma(TM) \oplus \Omega^2(M)$ with the following Lie bracket:

$$[(X, A), (Y, B)] = ([X, Y], \mathcal{L}_X B - \mathcal{L}_Y A).$$

The 1-parameter subgroup generated by (X, A) is given by

$$e^{t(X,A)} = (\lambda_t, \alpha_t) = \left(e^{tX}, \int_0^t \lambda_s^* A \, ds\right).$$

Following the above notation, for $B \in \Omega^2(M)$, we use e^B to denote the so-called B-transformation

$$e^B \circ (X + \xi) = X + \xi + \iota_X B.$$

2.2. Let $H \in \Omega^3_0(M)$, i.e., $dH = 0$. The H-twisted Loday bracket on $\mathbb{T}M$ is defined by

$$(X+\xi) *_H (Y+\eta) = [X,Y] + \mathcal{L}_X\eta - \iota_Y(d\xi - \iota_X H).$$

Let $\langle X+\xi, Y+\eta\rangle = \frac{1}{2}(\iota_X\eta + \iota_Y\xi)$, then $(\mathbb{T}M, *_H, \langle,\rangle, a)$ defines a structure of Courant algebroid, with $a : \mathbb{T}M \to TM$ the natural projection (cf. Definition 2.2 below). The Loday bracket is not skew-symmetric, indeed we have

$$(X+\xi) *_H (Y+\eta) + (Y+\eta) *_H (X+\xi) = d\langle X+\xi, Y+\eta\rangle.$$

The subgroup $\mathscr{G} = \mathrm{Diff}(M) \ltimes \Omega^2_0(M)$ is the group of symmetries of the Courant algebroid structure with $H = 0$. The Lie algebra of $\mathscr{G}$ is $\mathscr{X} = \Gamma(TM) \oplus \Omega^2_0(M)$ with the induced bracket. Let $\tilde{\mathscr{G}}_H \subset \tilde{\mathscr{G}}$ be the symmetry group of the Courant algebroid structure for general H and $\tilde{\mathscr{X}}_H$ be its Lie algebra. Consider the linear isomorphism:

$$\psi_H : \tilde{\mathscr{X}} \longrightarrow \tilde{\mathscr{X}} : (X, A) \longmapsto (X, A + \iota_X H),$$

and the H-twisted Lie bracket

$$[,]_H : \mathscr{X} \times \mathscr{X} \to \mathscr{X} : [(X,A),(Y,B)]_H = ([X,Y], \mathcal{L}_X B - \mathcal{L}_Y A + d\iota_Y\iota_X H),$$

then we have

Proposition 2.1 ([15]). *For $H, H' \in \Omega^3_0(M)$,*

$$[\psi_H(X,A), \psi_H(Y,B)]_{H+H'} = \psi_H[(X,A),(Y,B)]_{H'}$$

and $\psi_H : (\tilde{\mathscr{X}}_H, [,]) \to (\mathscr{X}, [,]_H)$ is a Lie algebra isomorphism.

Let $\mathfrak{X} = X + \xi \in \Gamma(\mathbb{T}M)$, then $(X, d\xi) \in \mathscr{X}$ and generates a 1-parameter subgroup in $\tilde{\mathscr{G}}_H$:

$$e^{\psi_H^{-1}(X,d\xi)} = (\lambda_t, \alpha_t) = \left(e^{tX}, \int_0^t \lambda_s^*(d\xi - \iota_X H)ds\right).$$

The infinitesimal action of $\mathfrak{X}$ on $\mathfrak{Y} \in \Gamma(\mathbb{T}M)$ that generates the above subgroup is:

$$\mathfrak{X} \circ_H \mathfrak{Y} = -\mathfrak{X} *_H \mathfrak{Y}.$$

2.3. Let $\mathbb{J} : \mathbb{T}M \to \mathbb{T}M$ be a generalized almost complex structure on M, that is, $\mathbb{J}^2 = -\mathbb{1}$ and $\mathbb{J}$ is orthogonal with respect to the pairing $\langle,\rangle$. Let $L \subset \mathbb{T}_{\mathbb{C}}M$ be the i-eigensubundle of $\mathbb{J}$, then L is isotropic and $\mathbb{J}$ defines an H-twisted generalized complex structure if L is involutive with respect to the H-twisted Loday bracket $*_H$. Examples of generalized complex structures include the symplectic and complex structures. Let ω (respectively, J) be a symplectic (respectively, complex) structure on M, then the corresponding generalized complex structure is defined by the respective isotropic subbundles:

$$L_\omega = \{X - i\iota_X\omega | X \in TM\}$$

and

$$L_J = \{X + \xi + i(J(X) - J^*(\xi)) | X \in TM, \xi \in T^*M\}.$$

2.4. The space of complex valued differential forms $\Omega^\bullet(M;\mathbb{C})$ is the spinor space of generalized geometry. Let $d_H = d - H\wedge$ be the H-twisted differential on $\Omega^\bullet(M;\mathbb{C})$. Each maximally isotropic sub-bundle $L \subset \mathbb{T}_\mathbb{C}M$ corresponds to a pure line sub-bundle U of $\wedge^\bullet T^*_\mathbb{C}M$:

$$U = \mathrm{Ann}_C(L) := \{\rho \in \wedge^\bullet T^*_\mathbb{C}M | \mathfrak{X}\cdot\rho = \iota_X\rho + \xi\wedge\rho = 0 \text{ for all } \mathfrak{X} = X + \xi \in L\},$$

where $\cdot$ stands for the Clifford multiplication. A (nowhere vanishing) local section ρ of U is called a *pure spinor* associated to the sub-bundle L. The integrability of L with respect to the H-twisted Courant bracket is equivalent to the condition $d_H(\Gamma(U)) \subset \Gamma(U_1)$, where $U_1 = \Gamma(\mathbb{T}_\mathbb{C}M)\cdot U$ via Clifford multiplication. More explicitly, there is a unique local section $\mathfrak{Y} = Y + \eta$ of $\overline{L}$, so that

$$d_H\rho = d\rho - H\wedge\rho = \mathfrak{Y}\cdot\rho = \iota_Y\rho + \eta\wedge\rho, \tag{2.1}$$

where we use the convention of d_H as in [**27**]. For a generalized complex structure $\mathbb{J}$, the complex line bundle U is called the *canonical bundle* of $\mathbb{J}$.

2.5. In this paper we use the following equivalent definition of a Courant algebroid from [**22**] (Definition 2.1 there):

Definition 2.2. Let $E \to M$ be a vector bundle. A *Loday bracket* $*$ on $\Gamma(E)$ is a $\mathbb{R}$-bilinear map satisfying the Jacobi identity, i.e., for all $\mathfrak{X},\mathfrak{Y},\mathfrak{Z} \in \Gamma(E)$,

$$\mathfrak{X} * (\mathfrak{Y} * \mathfrak{Z}) = (\mathfrak{X} * \mathfrak{Y}) * \mathfrak{Z} + \mathfrak{Y} * (\mathfrak{X} * \mathfrak{Z}). \tag{2.2}$$

E is a *Courant algebroid* if it has a *Loday bracket* $*$ and a non-degenerate symmetric pairing $\langle,\rangle$ on the sections, with an *anchor map* $a : E \to TM$ which is a vector bundle homomorphism so that

$$a(\mathfrak{X})\langle\mathfrak{Y},\mathfrak{Z}\rangle = \langle\mathfrak{X}, \mathfrak{Y} * \mathfrak{Z} + \mathfrak{Z} * \mathfrak{Y}\rangle \tag{2.3}$$

$$a(\mathfrak{X})\langle\mathfrak{Y},\mathfrak{Z}\rangle = \langle\mathfrak{X} * \mathfrak{Y}, \mathfrak{Z}\rangle + \langle\mathfrak{Y}, \mathfrak{X} * \mathfrak{Z}\rangle. \tag{2.4}$$

The notion can be complexified, where the structures are required to be $\mathbb{C}$-linear.

The skew-symmetrization $[,]$ of $*$ in the definition is also called the Courant bracket. The equivalence of the above definition to the more common variant, e.g., as in [**7**, **15**] etc, follows from Theorem 2.1 of [**22**] together with the thesis of Roytenberg [**31**] (see also [**36**]). In particular, the datum $(\mathbb{T}M, *_H, \langle,\rangle, a)$ as given in the previous subsections, for $H \in \Omega^3_0(M)$, are examples of Courant algebroids, where the corresponding Courant bracket is usually denoted $[,]_H$.

2.6. Let $\mathcal{T}M$ be a Courant algebroid which fits into the following extension:

$$0 \longrightarrow T^*M \longrightarrow \mathcal{T}M \xrightarrow{a} TM \longrightarrow 0,$$

so that a is the anchor map. Such Courant algebroid is called *exact* [**32**]. The set of isotropic splitting $s : TM \to \mathcal{T}M$ is non-empty and is a torsor over $\Omega^2(M)$. The choice of such s determines a form $H \in \Omega^3_0(M)$ and $\mathcal{T}M$ can then be identified with the datum $(\mathbb{T}M, *_H, \langle,\rangle, a)$ as discussed above. The action of $B \in \Omega^2(M)$ on the set of splittings translates into $H \mapsto H + dB$ on the corresponding forms. It follows that $[H] \in H^3(M; \mathbb{R})$ is well defined and is the *Ševera class* of $\mathcal{T}M$. We use the notion *extended* (+ *structures*) to emphasize the absence of a splitting while reserve *twisted generalized* for the situation where a splitting is (or can be explicitly) chosen. For example, an extended complex structure $\mathcal{J}$ will represent a twisted generalized complex structure $\mathbb{J}$ on $\mathbb{T}M$ (once a splitting is chosen), which is integrable with respect to a twisted Loday bracket $*_H$, where $[H]$ gives the Ševera class of the extended tangent bundle $\mathcal{T}M$ defined by the Courant algebroid structure $(\mathbb{T}M, *_H, \langle,\rangle, a)$. Given a different choice of splitting of $\mathcal{T}M$, $\mathcal{J}$ will represent $\mathbb{J}_B$, which is $\mathbb{J}$ transformed by some $B \in \Omega^2(M)$ and is integrable with respect to $*_{H+dB}$ on $\mathbb{T}M$. We note that the Courant algebroids are *identical* (not only isomorphic) in either cases, since the difference is only the choice of a splitting that gives the identification to $\mathbb{T}M$.

3. Poisson Lie actions and reductions

This and the next section contain the main results on reductions. In this section, we discuss the reduction under the Hamiltonian action of a Poisson Lie group in the context of generalized complex and Kähler geometries. This extends the reduction construction of [**15**] for Hamiltonian action of Lie groups and that of [**24**] for Poisson Lie action on symplectic manifolds, which we describe in the Appendix B (§ 8). The construction in this section form the basis of our duality constructions. Again, when we use $\mathbb{T}M$, $\mathbb{J}$ and etc, we assume a choice of the splitting of the extended tangent bundle $\mathcal{T}M$ and identify the corresponding structures as H-twisted generalized structures.

3.1. We first discuss the invariance of $\mathbb{J}$ under generalized actions. The result we obtain here (Lemma 3.3) can be seen as an extension of the Moser's argument in symplectic geometry (Remark 3.5, see also [**28**]). On the other hand, it also shows that for Hamiltonian Poisson Lie group actions, the generalized complex structure will in general not be preserved (Remark 3.10).

Direct computation shows

Lemma 3.1. *Let $(\lambda, \alpha) \in \tilde{\mathscr{G}}$ and ρ be the pure spinor defining $\mathbb{J}$, then $(\lambda, \alpha) \circ \rho := (\lambda^{-1})^*(e^{-\alpha}\rho)$ is the pure spinor defining $(\lambda, \alpha) \circ \mathbb{J}$. If $\mathbb{J}$ is H-twisted integrable, then $(\lambda, \alpha) \circ \mathbb{J}$ is $(\lambda, \alpha) \circ H$-twisted integrable, where $(\lambda, \alpha) \circ H = (\lambda^{-1})^*(H - d\alpha)$. We have $d_{(\lambda,\alpha)\circ H}(\lambda, \alpha) \circ \rho = (\lambda, \alpha) \circ d_H\rho$.*

Remark 3.2. We note that when considering generalized symmetries, we do not have to restrict to real 2-forms to stay with *real* twisting form, e.g., the group $\mathrm{Diff}(M) \ltimes (\Omega^2(M) \oplus i\Omega^2_0(M))$ acts on $\mathbb{T}_{\mathbb{C}}M$. The infinitesimal action of $(X, A) \in \tilde{\mathscr{X}} \oplus i\Omega^2_0(M)$ on the spinors is then given by

$$(X, A) \circ \rho = -\mathcal{L}_X\rho - A \wedge \rho.$$

For $\mathfrak{X} = X + \xi \in \Gamma(\mathbb{T}_{\mathbb{C}}M)$ so that $X \in \Gamma(TM)$, let $(X, A) = (X, d\xi - \iota_X H)$ and we compute the infinitesimal action on a section ρ of the canonical bundle of $\mathbb{J}$:

$$\mathfrak{X} \circ_H \rho = (-d_H + \mathfrak{Y}\cdot)\mathfrak{X} \cdot \rho - \langle \mathfrak{X}, \mathfrak{Y}\rangle\rho. \tag{3.1}$$

We caution that when a generalized complex structure is concerned, such *complex* actions in general might not preserve the real index.

Lemma 3.3. *Suppose that L defines an extended complex structure $\mathcal{J}$ and $\mathfrak{X}_t \in \Gamma(L \cap (\mathcal{T}M \oplus iT^*M))$ is a family of sections parametrized by $\mathbb{R}$. Let $\tilde{\lambda}_t$ be the family of generalized symmetries generated by $\mathfrak{X}_t$. Suppose that for each $p \in M$ there is an open neighbourhood U_p and a compact set V_p so that $\{\lambda_t \circ U_p\} \subset V_p$ for all t, where λ_t is the geometrical part of $\tilde{\lambda}_t$. Then $\tilde{\lambda}_t$ preserves $\mathcal{J}$ for all t.*

Proof. Choose a splitting and identify the structures with H-twisted structures. Write $\mathfrak{X}_t = X_t + \xi_t$ under the splitting. Starting from any $p \in M$ and $t_0 \in \mathbb{R}$. Suppose that $\rho_{t_0} = \rho$ is a local section of the canonical bundle U of $\mathbb{J}$ and $\rho_t = (\lambda_t, \alpha_t)^*\rho := (\lambda_t, \alpha_t)^{-1} \circ \rho = e^{\alpha_t}\lambda_t^*\rho$. Then ρ_t is a local section of the canonical bundle U_t of $\mathbb{J}_t = (\lambda_t, \alpha_t)^{-1} \circ \mathbb{J}$. Direct computation shows that

$$\frac{d}{dt}\rho_t = \left.\frac{d}{ds}\right|_{s=0}\rho_{t+s} = (\lambda_t, \alpha_t)^*((d_H - \mathfrak{Y}\cdot)\mathfrak{X}_t \cdot \rho + \langle \mathfrak{X}_t, \mathfrak{Y}\rangle\rho).$$

Then by the assumption we have $\frac{d}{dt}\rho_t = f_t\rho_t$ for $f_t = \lambda_t^*\langle \mathfrak{X}_t, \mathfrak{Y}\rangle$. The initial condition of $\rho_{t_0} = \rho$ then gives

$$\rho_t = e^{\int_{t_0}^t f_s ds}\rho.$$

It follows that $U_t = U$ wherever both ρ and ρ_t are defined, e.g. for a neighbourhood of p. Since t_0 is arbitrary and by the compactness assumption, we have $U_t = U$ for all t.

The argument above shows that L is preserved by the family of symmetries generated by $\mathfrak{X}_t$, which is independent of the splitting chosen. The proposition then follows. $\square$

Remark 3.4. Of course, when M is compact, the condition in the lemma automatically holds. From the proof, we also see that when $d_H\rho = 0$, not only the canonical line bundle is preserved, the spinor ρ is preserved as well.

Remark 3.5. The Moser's argument in symplectic geometry can be seen as a special case of the above lemma. The Moser's argument goes as following (see [**28**]). Consider a smooth family of symplectic forms $\omega_t = \omega_0 + d\beta_t$ and $\eta_t = \frac{d}{dt}\beta_t$. Let Y_t be defined by $\iota_{Y_t}\omega_t + \eta_t = 0$ and ϕ_t be the family of diffeomorphisms generated by Y_t via $\frac{d}{dt}\phi_t = \phi_{t*}(Y_t)$, then $\phi_t^*\omega_t = \omega_0$.

In light of Lemma 3.3, we consider $\varphi_t = \phi_t^{-1}$, which is generated by the family of vector fields $X_t = -\varphi_{t*}(Y_t)$. Then we define $\xi_t = \iota_{X_t}\omega_0 = -\phi_t^*(\eta_t)$. It follows that $\mathfrak{X}_t = X_t - i\xi_t \in \Gamma(L_{\omega_0})$. The lemma then implies that the following family of symmetries preserves L_{ω_0}:

$$(\varphi_t, \alpha_t) = \left(\varphi_t, -id\int_0^t \varphi_s^*\xi_s ds\right) = \left(\varphi_t, -id\int_0^t \eta_s ds\right).$$

In this case, we have $\rho_0 = e^{i\omega_0}$ and $d\rho_0 = 0$. It follows from Remark 3.4 that ρ_0 is preserved:

$$e^{i\omega_0} = (\varphi_t, \alpha_t)^* e^{i\omega_0} = e^{-id\int_0^t \eta_s ds}\varphi_t^*(e^{i\omega_0}) = e^{-id\beta_t}(\phi_t^*)^{-1}(e^{i\omega_0}),$$

which is equivalent to $\phi_t^*\omega_t = \omega_0$ as in Moser's argument.

3.2. We will use the following conventions.

Convention 3.6. *Given a Lie group G, the Lie algebra $\mathfrak{g}$ of G is identified as the tangent space T_eG at identity, as well as the space of right invariant vector fields, i.e., $\tau \to X_\tau(g) = (R_g)_*\tau$. Then the dual $\hat{\mathfrak{g}} = \mathfrak{g}^*$ of the Lie algebra is identified with the space of right invariant 1-forms on G. Let $\theta^r_{\hat\tau} \in \Omega^1(G)$ denote the right invariant 1-form on G with $\theta^r_{\hat\tau}(e) = \hat\tau$ and $\theta^l_{\hat\tau}$ the left invariant 1-form on G with $\theta^l_{\hat\tau}(e) = \hat\tau$, for $\hat\tau \in \hat{\mathfrak{g}}$. Given a Poisson manifold P with Poisson tensor π_P, we consider π_P also as a map $\pi_P : TP^* \to TP$ defined by $\iota_{\pi_P(\xi)}\eta = \pi_P(\xi, \eta)$ for $\xi, \eta \in \Omega^1(P)$.*

We note that for $\tau \in \mathfrak{g}$, the right invariant vector field X_τ generates the left action on G by the 1-parameter subgroup $g_t = e^{t\tau}$. Thus the left action of G on M induces a homomorphism of Lie algebras $\xi \mapsto X_\tau^M$, where X_τ^M is the infinitesimal action generated by τ, while the right action of G induces an *anti*-homomorphism of Lie algebras. With this convention, the map π_P and the Lie algebra (anti)-homomorphism are opposite to the convention used in [**24**] and [**25**]. In the following, we will only consider *left* actions.

We collected the relevant definitions and results on Poisson Lie groups and actions in Appendix B (§ 8).

3.3. The basic setup is the following. Let $(M,\mathcal{J})$ be an extended complex manifold with extended tangent bundle $\mathcal{T}M$ and anchor $a:\mathcal{T}M\to TM$. Then there is a natural induced Poisson structure $\pi_{\mathcal{J}}$ on M defined by

$$\pi_{\mathcal{J}}:T^*M\longrightarrow \mathcal{T}M\xrightarrow{\mathcal{J}}\mathcal{T}M\xrightarrow{a}TM.$$

If a splitting is chosen, we use $\mathbb{T}M$, $\mathbb{J}$ and H-twisted when referring to the respective structures, while the Poisson structure $\pi_{\mathcal{J}}$ does not depend on the splitting. The following result is essential in establishing the definition of Hamiltonian property (Definition 3.8) for an extended complex structure.

Lemma 3.7. *Let (G,π_G) be a connected Poisson Lie group and $\sigma:G\times M\to M$ a (left) Poisson action with respect to the Poisson structure $\pi_{\mathcal{J}}$ on M, with an equivariant moment map $\mu:M\to\hat{G}$, as in Definition* B.12. *Let $\mathbb{J}(\mu^*\hat{\theta}_\tau)=\mathfrak{X}_\tau=X_\tau+\xi_\tau$ for $\tau\in\mathfrak{g}$, then*

$$\iota_{X_\tau}\mu^*(\hat{\theta}_\tau)=\iota_{X_\tau}\xi_\tau=0 \text{ and } [(X_\tau,d\xi_\tau),(X_\omega,d\xi_\omega)]_H=(X_{[\tau,\omega]},d\xi_{[\tau,\omega]}).$$

Proof. Let $\hat{\mathcal{X}}_\tau=\mu_*(X_\tau)$ be the dressing vector field on $\hat{G}$ generated by $\tau\in\mathfrak{g}$ (cf. Definition B.10). Then we have $\iota_{X_\tau}\mu^*\hat{\theta}_\tau=\mu^*(\iota_{\hat{\mathcal{X}}_\tau}\hat{\theta}_\tau)=\mu^*(\pi_{\hat{G}}(\hat{\theta}_\tau,\hat{\theta}_\tau))=0$. Because $\mathbb{J}$ preserves the pairing $\langle,\rangle$, it follows that

$$\iota_{X_\tau}\xi_\tau=\langle X_\tau+\xi_\tau,X_\tau+\xi_\tau\rangle=\langle\mathbb{J}(\mu^*\hat{\theta}_\tau),\mathbb{J}(\mu^*\hat{\theta}_\tau)\rangle=\langle\mu^*\hat{\theta}_\tau,\mu^*\hat{\theta}_\tau\rangle=0.$$

We then compute

$$\begin{aligned}&[\mathbb{J}(\mu^*\hat{\theta}_\tau)+i\mu^*\hat{\theta}_\tau,\mathbb{J}(\mu^*\hat{\theta}_\omega)+i\mu^*\hat{\theta}_\omega]_H\\&\quad=[X_\tau,X_\omega]+\mathcal{L}_{X_\tau}\xi_\omega-\iota_{X_\omega}d\xi_\tau+\iota_{X_\omega}\iota_{X_\tau}H+i(\mathcal{L}_{X_\tau}\mu^*\hat{\theta}_\omega-\iota_{X_\omega}d\mu^*\hat{\theta}_\tau),\end{aligned}$$

and the imaginary part is

$$\begin{aligned}\mathcal{L}_{X_\tau}\mu^*\hat{\theta}_\omega-\iota_{X_\omega}d\mu^*\hat{\theta}_\tau&=\mu^*(\mathcal{L}_{\mu_*X_\tau}\hat{\theta}_\omega-\iota_{\mu_*X_\omega}d\hat{\theta}_\tau)\\&=\mu^*(\mathcal{L}_{\hat{\mathcal{X}}_\tau}\hat{\theta}_\omega-\iota_{\hat{\mathcal{X}}_\omega}d\hat{\theta}_\tau)=\mu^*([\hat{\theta}_\tau,\hat{\theta}_\omega]^*)=\mu^*\hat{\theta}_{[\tau,\omega]}\end{aligned}$$

Thus $X_{[\tau,\omega]}=[X_\tau,X_\omega]$ and $\xi_{[\tau,\omega]}=\mathcal{L}_{X_\tau}\xi_\omega-\iota_{X_\omega}d\xi_\tau+\iota_{X_\omega}\iota_{X_\tau}H$. The lemma follows. □

We note that from the above lemma, the symmetry generated by $\mathfrak{X}_{[\tau,\omega]}$ coincides with that of $\mathfrak{X}_\tau *_H \mathfrak{X}_\omega$, which only depends on the Loday bracket.

Definition 3.8. The action of a Poisson Lie group G on an extended complex manifold $(M,\mathcal{J})$ is *Hamiltonian* with moment map $\mu:M\to\hat{G}$, if the action is Poisson with respect to $\pi_{\mathcal{J}}$, together with an equivariant moment map μ as in definition B.12, so that the G-action on $\mathcal{T}M$ is generated by $\mathcal{J}(\mu^*\hat{\theta})=\mathfrak{X}_\mu$, via the Loday bracket $*$.

Remark 3.9. In general, it is a non-trivial condition that $\mathfrak{X}_\mu$ generates a G-action on $\mathcal{T}M$. As an example, let $M = \mathbb{C}^2$, $H = 0$ and $\mathbb{J} = \omega \oplus J$, where ω and J are the standard symplectic and complex structure on $\mathbb{C}$. The Poisson structure here is $\pi_{\mathbb{J}} = \omega \oplus 0$. Let $G = S^1$ and $\mu(z_1, z_2) = \frac{1}{2}(|z_1|^2 + |z_2|^2)$, then the geometric action of G on M by rotating the first coordinate is Hamiltonian for the Poisson structure $\pi_{\mathbb{J}}$, and μ is an equivariant moment map. Let X be the infinitesimal action of $1 \in \mathbb{R}$, then it is easy to see that

$$\mathbb{J}(d\mu) = X + \xi \quad \text{where } \xi = -\frac{i}{2}(z_2 d\overline{z}_2 - \overline{z}_2 dz_2).$$

Since $d\xi = -\frac{i}{2} dz_2 \wedge d\overline{z}_2$, the $\mathbb{R}^1$-action on $\mathbb{T}M$ generated by $X + \xi$ can never be proper, i.e., $\mathbb{J}(d\mu)$ does not generate a G-action.

Remark 3.10. We recall that in the Poisson category, the Poisson action of a Poisson Lie group does not have to preserve the Poisson structure. Thus the action as defined above does not have to preserve the extended complex structure $\mathcal{J}$. Lemma 3.3 implies that the action on $\mathcal{T}M$ generated by $\mathcal{J}(\mu^*\hat{\theta}) + i\mu^*\hat{\theta}$ does preserve the structure $\mathcal{J}$. Thus the non-invariance under the action of $\mathcal{J}(\mu^*\hat{\theta})$ can be seen as due to the non-vanishing of $d\hat{\theta}$. When the Poisson structure on G is trivial, we have the definition for Hamiltonian actions of Lie groups [**15**]. By Theorem B.13, μ is a Poisson map. Let $M_0 = \mu^{-1}(\hat{e})$, then $\mu_*(\pi_{\mathcal{J}}|_{M_0}) = \pi_{\hat{G}}|_{\hat{e}} = 0$, and X_μ, i.e., the geometrical action of G, preserves M_0.

3.4. We may consider reduction by a Hamiltonian Poisson Lie group action. Assume that

(1) the identity $\hat{e} \in \hat{G}$ is a regular value of μ,
(2) (*the geometrical part of*) the G-action is free on M_0.

Lemma 3.11. *The sub-bundles* $(\mu^*\hat{\theta})$, $\mathcal{J}(\mu^*\hat{\theta})$ *and* $L \oplus (\mu^*\hat{\theta})$ *are* G-*equivariant.*

Proof. Choose a splitting. It is enough to show that the infinitesimal actions preserve the sub-bundles:

$$\begin{aligned}
(X_\omega + \xi_\omega) *_H (\mu^*\hat{\theta}_\tau) &= \mathcal{L}_{X_\omega}\mu^*\hat{\theta}_\tau = \mu^*(\mathcal{L}_{\hat{\mathcal{X}}_\omega}\hat{\theta}_\tau) = \mu^*(\hat{\theta}_{[\omega,\tau]} - \iota_{\hat{\mathcal{X}}_\tau} d\hat{\theta}_\omega),\\
(X_\omega + \xi_\omega) *_H (X_\tau + \xi_\tau) &= [X_\omega, X_\tau] + \mathcal{L}_{X_\omega}\xi_\tau - \iota_{X_\tau} d\xi_\omega + \iota_{X_\tau}\iota_{X_\omega} H\\
&= X_{[\omega,\tau]} + \xi_{[\omega,\tau]},\\
(X_\omega + \xi_\omega) *_H (Y + \eta) &= [X_\omega, Y] + \mathcal{L}_{X_\omega}\eta - \iota_Y d\xi_\omega + \iota_Y \iota_{X_\omega} H\\
&= (X_\omega + \xi_\omega + i\mu^*\hat{\theta}_\omega) *_H (Y + \eta) + i\iota_Y \mu^*(d\hat{\theta}_\omega),
\end{aligned}$$

for $Y + \eta \in \Gamma(L)$. We note that $\iota_Z \mu^*(d\hat{\theta}_\omega) = -\frac{1}{2}\iota_Z \mu^*([\hat{\theta}, \hat{\theta}]_\omega) \in (\mu^*\hat{\theta})$ for any $Z \in TM$. □

Theorem 3.12. *Suppose that G is compact and the assumptions in § 3.4 hold, then there is a natural extended complex structure on the quotient $Q = M_0/G$. When the action of G preserves a splitting of $\mathcal{T}M$, the reduced structure $\mathcal{T}_\mu Q$ admits a natural splitting up to a choice of connection form on $M_0 \to Q$.*

Proof. Because $\pi_{\hat{G}}|_{\hat{e}} = 0$, we compute on M_0:

$$\langle \mu^*\hat{\theta}_\tau, \mathcal{J}(\mu^*\hat{\theta}_\omega)\rangle = \iota_{X_\omega}\mu^*\hat{\theta}_\tau = \mu^*\iota_{\hat{\mathcal{X}}_\omega}\hat{\theta}_\tau = \mu^*\pi_{\hat{G}}(\hat{\theta}_\omega, \hat{\theta}_\tau) = 0.$$

Let $\mathcal{K} = \mu^*\hat{\theta}, \mathcal{K}' = \mathcal{J}(\mu^*\hat{\theta})$, it follows that $\mathcal{K} \oplus \mathcal{K}' \subset \mathrm{Ann}(\mathcal{K}, \mathcal{K}')$. $\mathcal{K}$ and $\mathcal{K}'$ satisfy the conditions (i)–(iii) of Lemma A.4 by definition. For example,

$$d(\mu^*\hat{\theta}_\omega) = -\tfrac{1}{2}\mu^*([\hat{\theta}, \hat{\theta}]_\omega) \subset \Gamma(\wedge^2\mathcal{K}).$$

Thus Lemma A.4 (1) applies and $\mathcal{T}_\mu M_0 = \mathrm{Ann}(\mathcal{K}, \mathcal{K}')/\mathcal{K} \oplus \mathcal{K}'$ descends to an extended tangent bundle $\mathcal{T}_\mu Q$ on Q.

Lemma 3.11 implies that $L \oplus (\mu^*\hat{\theta})$ is involutive with respect to the bracket $*$. It follows that $(L \oplus (\mu^*\hat{\theta})) \cap \mathrm{Ann}(\mu^*\hat{\theta}, \mathcal{J}(\mu^*\hat{\theta}))$ induces a sub-bundle L_0 in $\mathcal{T}_{\mu,\mathbb{C}}M_0$ which coincides with the image of L under the subquotient. By Lemma 3.11, the bundle L_0 is G-equivariant and descends to a sub-bundle L_μ of $\mathcal{T}_{\mu,\mathbb{C}}Q$. That L_μ is maximally isotropic with real index 0 and integrable follows from the same properties of L, i.e., L_μ defines an extended complex structure $\mathcal{J}_\mu$. Corollary A.5 gives the last sentence. □

3.5. Let $(M, \mathcal{J}_1)$ be an extended complex manifold. A second extended complex structure $\mathcal{J}_2$ makes $(M, \mathcal{J}_1, \mathcal{J}_2)$ into an *extended Kähler* manifold if they are both defined on the same extended tangent bundle $\mathcal{T}M$ and $\mathcal{G} = -\mathcal{J}_1\mathcal{J}_2 = -\mathcal{J}_2\mathcal{J}_1$ defines a generalized metric (see [**14**]) on $\mathcal{T}M$, i.e., $\langle \mathcal{G}\cdot, \cdot\rangle$ defines a metric on $\mathcal{T}M$. We show that just as symplectic reduction admits induced Kähler structure when the original manifold is Kähler with G preserving the complex structure, generalized complex reduction with respect to $\mathcal{J}_1$ would admit extended Kähler structure if $\mathcal{J}_2$ is preserved.

Definition 3.13. A Poisson action of Poisson Lie group G on an extended Kähler manifold $(M, \mathcal{J}_1, \mathcal{J}_2)$ is $\mathcal{J}_1$-*Hamiltonian* if it is Hamiltonian with respect to $\mathcal{J}_1$ and preserves $\mathcal{J}_2$.

3.6. We note that $\mathrm{Ann}(\mu^*\hat{\theta}, \mathcal{J}_1(\mu^*\hat{\theta}))$ is not preserved by $\mathcal{J}_2$:

$$\begin{aligned}&\mathrm{Ann}(\mu^*\hat{\theta}, \mathcal{J}_1(\mu^*\hat{\theta})) \cap \mathcal{J}_2(\mathrm{Ann}(\mu^*\hat{\theta}, \mathcal{J}_1(\mu^*\hat{\theta})))\\ &\quad = \mathrm{Ann}(\mu^*\hat{\theta}, \mathcal{J}_1(\mu^*\hat{\theta}), \mathcal{J}_2(\mu^*\hat{\theta}), \mathcal{G}(\mu^*\hat{\theta})).\end{aligned}$$

The right-hand side of the above equation is again a G-equivariant sub-bundle when restricting to M_0, as the two terms on the left are both so.

The next two lemmata concern the linear algebra of the generalized Kähler reduction. Alternatively, they can be seen as the linear case for the reduction construction of Theorem 3.16.

Lemma 3.14. *Let $\mathbb{V} = V \oplus V^*$ and $(\mathbb{J}_1, \mathbb{J}_2; \mathbb{G})$ be a linear generalized Kähler structure. Given subspace $K \subset V^*$, let $\mathbb{U}_K^j = \operatorname{Ann}(K, \mathbb{J}_j(K))$, $\mathbb{W}_K = \operatorname{Ann}(K, \mathbb{J}_1(K), \mathbb{J}_2(K), \mathbb{G}(K))$ and $(\mathbb{U}_K^j)_{\mathbb{C}}$, $(\mathbb{W}_K)_{\mathbb{C}}$ be the respective complexified versions, then $L_j \cap (\mathbb{W}_K)_{\mathbb{C}} = L_j \cap (\mathbb{U}_K^l)_{\mathbb{C}}$ for $j \neq l$. If*

(1) *$K + \mathbb{J}_1(K) \subset \mathbb{U}_K^1$, then the following decomposition holds*

$$\mathbb{U}_K^1 = \mathbb{W}_K \oplus (K + \mathbb{J}_1(K)).$$

The $+$ above becomes $\oplus$ if we suppose further that

(2) *$\mathbb{J}_1(K) \cap V^* = \{0\}$.*

Proof. Obviously $L_j \cap (\mathbb{W}_K)_{\mathbb{C}} \subset L_j \cap (\mathbb{U}_K^l)_{\mathbb{C}}$. Let $\mathfrak{X} \in L_j \cap (\mathbb{U}_K^l)_{\mathbb{C}}$, then $\mathbb{J}_j(\mathfrak{X}) = i\mathfrak{X}$ and $\langle \mathfrak{X}, K \rangle = \langle \mathfrak{X}, \mathbb{J}_l(K) \rangle = 0$. Then by orthogonality of $\mathbb{J}_j$, we find that $\langle \mathfrak{X}, \mathbb{J}_j(K) \rangle = \langle \mathfrak{X}, \mathbb{G}(K) \rangle = 0$, i.e., $\mathfrak{X} \in L_j \cap (\mathbb{W}_K)_{\mathbb{C}}$.

For any subspace $W \subset \mathbb{V}$ we have $\mathbb{V} = W \oplus \operatorname{Ann}(\mathbb{G}W)$. In particular

$$\begin{aligned} \mathbb{V} &= \mathbb{W}_K \oplus (K + \mathbb{J}_1(K) + \mathbb{J}_2(K) + \mathbb{G}(K)) \\ &= \tilde{\mathbb{W}}_K \oplus (\mathbb{J}_1(K) + \mathbb{J}_2(K) + \mathbb{G}(K)) \\ &= \mathbb{U}_K^j \oplus (\mathbb{J}_l(K) + \mathbb{G}(K)) \quad \text{for } j \neq l, \end{aligned} \tag{3.2}$$

where $\tilde{\mathbb{W}}_K = \operatorname{Ann}(K, \mathbb{J}_1(K), \mathbb{J}_2(K))$. With condition (1), by the last expression in (3.2) for $j = 1$ and $l = 2$, we see that the decomposition in the statement holds. With condition (2), we have $K + \mathbb{J}_1(K) = K \oplus \mathbb{J}_1(K)$ and it follows that all $+$s in (3.2) are $\oplus$s. ☐

Lemma 3.15. *Continue from Lemma 3.14 and let $N = a \circ \mathbb{J}_1(K)$ where $a : \mathbb{V} \to V$ is the projection, then the restriction $\langle , \rangle_K$ of $\langle , \rangle$ on $\mathbb{W}_K$ is a non-degenerate pairing. There is a self-dual exact sequence with respect to $\langle , \rangle_K$*

$$0 \longrightarrow W_K^* \xrightarrow{a_K^*} \mathbb{W}_K \xrightarrow{a_K} W_K \to 0$$

where

$$W_K = \frac{\operatorname{Ann}_V(K)}{N} \quad \text{and} \quad W_K^* = \frac{\operatorname{Ann}_{V^*}(N)}{K}.$$

Furthermore, the restriction $(\mathbb{J}_{1,K}, \mathbb{J}_{2,K}; \mathbb{G}_K)$ of $(\mathbb{J}_1, \mathbb{J}_2; \mathbb{G})$ to $\mathbb{W}_K$ is a generalized Kähler structure with respect to the pairing $\langle , \rangle_K$. Let $\mathbb{V}_K = \operatorname{Ann}(K, K')/K \oplus K'$ as in Lemma A.1, where $K' = \mathbb{J}_1(K)$, then the inclusion $\mathbb{W}_K \hookrightarrow \operatorname{Ann}(K, \mathbb{J}_1(K))$ induces a natural isomorphism $\mathbb{W}_K \simeq \mathbb{V}_K$, and the extension sequences correspond.

Proof. Note that $\mathbb{W}_K$ is preserved by $\mathbb{G}$ we see that for any $\mathfrak{X} \in \mathbb{W}_K$ such that $\langle \mathfrak{X}, \mathbb{W}_K \rangle = 0$, it must satisfy $\langle \mathfrak{X}, \mathbb{G}(\mathfrak{X}) \rangle = 0$, i.e., $\mathfrak{X} = 0$. It implies that the restriction $\langle , \rangle_K$ is non-degenerate. It is obvious that the generalized Kähler structure restricts.

Let $a_K : \mathbb{W}_K \to W_K$ be the map induced from the projection a. The kernel of $\mathbb{U}^1_K \to W_K$ is $\mathrm{Ann}_{V^*}(N) \oplus \mathbb{J}_1(K)$. It follows that the kernel of a_K is

$$\ker a_K = (\mathrm{Ann}_{V^*}(N) \oplus \mathbb{J}_1(K)) \cap \mathbb{W}_K.$$

Note that $\mathbb{U}^1_K = \mathbb{W}_K \oplus (K \oplus \mathbb{J}_1(K))$ and $K \oplus \mathbb{J}_1(K) \subset \mathrm{Ann}_{V^*}(N) \oplus \mathbb{J}_1(K)$, we find that $\mathrm{Ann}_{V^*}(N) \oplus \mathbb{J}_1(K) = \ker a_K \oplus (K \oplus \mathbb{J}_1(K))$, thus $\ker a_K \simeq \mathrm{Ann}_{V^*}(N)/K$. Now $\ker a_K$ is maximally isotropic with respect to $\langle , \rangle_K$ and the self-duality follows. The last sentence follows from direct checking. $\square$

Similar to the classical Kähler case, we have now the generalized Kähler reduction.

Theorem 3.16. *Let $(M, \mathcal{J}_1, \mathcal{J}_2; \mathcal{G})$ be an extended Kähler manifold. Suppose that the action of G on M is $\mathcal{J}_1$-Hamiltonian with moment map $\mu : M \to \hat{G}$. When the assumptions in § 3.4 hold, then there is a natural extended Kähler structure on the quotient $Q = M_0/G$. If furthermore, the G-action preserves a splitting of $\mathcal{T}M$ in to H-twisted generalized tangent bundle, the reduced structure splits, up to a choice of connection form on $M_0 \to Q$.*

Proof. All the bundles in the proof will be on the various spaces at $\mu = \hat{e}$, either the level set or the reduced space. Let

$$\mathcal{T}'_\mu M_0 = \mathrm{Ann}(\mu^*\hat{\theta}, \mathcal{J}_1(\mu^*\hat{\theta}), \mathcal{J}_2(\mu^*\hat{\theta}), \mathcal{G}(\mu^*\hat{\theta}))$$

be the sub-bundle of $\mathcal{T}M|_{M_0}$, then it is a G-equivariant sub-bundle (§ 3.6). Consider $\mathfrak{X}, \mathfrak{Y} \in \Gamma(\mathrm{Ann}(\mu^*\hat{\theta}, \mathcal{J}_1(\mu^*\hat{\theta})))^G$ and $\mathfrak{X}_\tau = \mathcal{J}_1(\mu^*\hat{\theta}_\tau)$, $\tau \in \mathfrak{g}$, then

$$\mathfrak{X}_\tau * (\mathfrak{X} * \mathfrak{Y}) = (\mathfrak{X}_\tau * \mathfrak{X}) * \mathfrak{Y} + \mathfrak{X} * (\mathfrak{X}_\tau * \mathfrak{Y}) = 0 \Rightarrow \mathfrak{X} * \mathfrak{Y} \text{ is invariant.}$$

That $\mathfrak{X} * \mathfrak{Y} \in \Gamma(\mathrm{Ann}(\mu^*\hat{\theta}, \mathcal{J}_1(\mu^*\hat{\theta})))$ follows from the proof of Lemma A.4 case (1) where we set $\mathcal{K} = (\mu^*\hat{\theta})$ and $\mathcal{K}' = \mathcal{J}_1(\mu^*\hat{\theta})$. From Lemma 3.14 we get the decomposition

$$\mathrm{Ann}(\mu^*\hat{\theta}, \mathcal{J}_1(\mu^*\hat{\theta})) = \mathcal{T}'_\mu M_0 \oplus ((\mu^*\hat{\theta}) \oplus \mathcal{J}_1(\mu^*\hat{\theta})), \tag{3.3}$$

which by Lemma 3.11 is G-equivariant. Let $\pi_1 : \mathrm{Ann}(\mu^*\hat{\theta}, \mathcal{J}_1(\mu^*\hat{\theta})) \to \mathcal{T}'_\mu M_0$ be the projection to the first factor in the above decomposition. Let $\langle , \rangle_\mu$ be the restriction of $\langle , \rangle$ to $\mathcal{T}'_\mu M_0$ and define the bracket $*_1$ on $\Gamma(\mathcal{T}'_\mu M_0)^G$ by:

$$\mathfrak{X} *_1 \mathfrak{Y} = \pi_1(\mathfrak{X} * \mathfrak{Y}), \quad \text{for } \mathfrak{X}, \mathfrak{Y} \in \Gamma(\mathcal{T}'_\mu M_0)^G \subset \Gamma(\mathrm{Ann}(\mu^*\hat{\theta}, \mathcal{J}_1(\mu^*\hat{\theta})))^G.$$

The structure $(\mathcal{T}'_\mu M_0, \langle , \rangle_\mu, *_1)$ descends to an extended tangent bundle $\mathcal{T}'_\mu Q$ on Q. Due to the decomposition (3.3), the extended tangent bundles $\mathcal{T}'_\mu Q$

and $\mathcal{T}_\mu Q$ (as given by Theorem 3.12, using $\mathcal{J} = \mathcal{J}_1$) are naturally isomorphic via the inclusion $\mathcal{T}'_\mu M_0 \hookrightarrow \mathrm{Ann}(\mu^*\hat{\theta}, \mathcal{J}_1(\mu^*\hat{\theta}))$.

Then Lemma 3.15 implies that the restriction $(\mathcal{J}'_1, \mathcal{J}'_2)$ of $(\mathcal{J}_1, \mathcal{J}_2)$ to $\mathcal{T}'_\mu M_0$ defines a generalized almost Kähler structure $(\mathcal{J}_{1,\mu}, \mathcal{J}_{2,\mu})$ on $\mathcal{T}'_\mu Q$, i.e., $\mathcal{J}_{j,\mu}$ are generalized almost complex structures and $\mathcal{G}_\mu = -\mathcal{J}_{1,\mu}\mathcal{J}_{2,\mu}$ defines a generalized metric on $\mathcal{T}'_\mu Q$.

We check the integrability. Let L_j be the i-eigensub-bundle of $\mathcal{J}_j$ in $\mathcal{T}_\mathbb{C} M$. Then L_2 is G-equivariant because $\mathcal{J}_2$ is preserved. Lemma 3.14 implies that

$$L_2 \cap \mathrm{Ann}(\mu^*\hat{\theta}, \mathcal{J}_1(\mu^*\hat{\theta})) = L_2 \cap \mathcal{T}'_{\mu,\mathbb{C}} M_0,$$

which defines $\mathcal{J}'_2$. It follows that $\mathcal{J}'_2$ is involutive with respect to $*_1$ and $\mathcal{J}_{2,\mu}$ is integrable in $\mathcal{T}'_\mu Q$. As to $\mathcal{J}_1$, it is easy to check that

$$(L_1 \oplus (\mu^*\hat{\theta})) \cap \mathcal{T}'_{\mu,\mathbb{C}} M_0$$

is involutive and isotropic. In fact, it gives the sub-bundle L_0 of $\mathcal{T}_{\mu,\mathbb{C}} M_0$ in the proof of Theorem 3.12 under the natural identification. Thus $\mathcal{J}_{1,\mu}$ is integrable. $\square$

Remark 3.17. We notice from the proof that, in order to have extended Kähler reduction, even the extended complex structure $\mathcal{J}_2$ does not have to be preserved by the G-action either. The only thing that needs to be preserved is the intersection $L_2 \cap \mathcal{T}'_{\mu,\mathbb{C}} M_0$. Here, unlike the case in Theorem 3.12, where $L_1 \oplus (\mu^*\hat{\theta})$ being equivariant provides descending of $\mathcal{J}_1$, Lemma 3.14 implies that such flexibility does not apply to $\mathcal{J}_2$.

Remark 3.18. Generalized Kähler reduction have been constructed by several other works, e.g., [**7**], [**23**] and [**33**], with various generalities. The construction we describe here, which fits our needs for discussing duality, has not appeared in the stated form.

4. Bi-Hamiltonian action and factorizable reduction

In this section, we describe one more reduction construction that is central to our geometric approach to T-duality. Let G be a Poisson Lie group with dual group $\hat{G}$. Let $(\tilde{\mathfrak{g}}, \mathfrak{g}, \hat{\mathfrak{g}})$ be the associated Manin triple (cf. Theorem B.6, also [**24**]). Suppose that G and $\hat{G}$ both act on the extended Kähler manifold $(M, \mathcal{J}_1, \mathcal{J}_2)$, so that the actions are $\mathcal{J}_1$- and $\mathcal{J}_2$-Hamiltonian, respectively. One observation we gain from the computation of the example of $\mathbb{C}^2 \backslash \{(0,0)\}$ in [**15**] (also see Example 5.10) is that the two moment map could coincide. This observation leads to the following definition critical in our construction:

Definition 4.1. The (infinitesimal) (left) action of $\tilde{\mathfrak{g}}$ on M is *bi-Hamiltonian* if it is induced by a (left) $\mathcal{J}_1$-Hamiltonian action of G together with a (left) $\mathcal{J}_2$-Hamiltonian action of $\hat{G}$.

We will use μ and $\hat{\mu}$ to denote the moment maps of the G and $\hat{G}$ actions, respectively. Suppose that G is a factorizable Poisson Lie group (Definition B.8). Let $S : \hat{G} \dashrightarrow G$ be the local diffeomorphism defined by $\underline{s}$ and the exponential maps at the identity elements $\hat{e} \in \hat{G}$ and $e \in G$, then $dS(\hat{e}) = \underline{s}$. We consider the reduction by the bi-Hamiltonian action of $\tilde{G}$.

Assumption 4.2. We will need the following conditions.

(0) In the following, G is always a factorizable Poisson Lie group.
(1) The identity elements $e \in G$ and $\hat{e} \in \hat{G}$ are regular values of $\hat{\mu}$ and μ, respectively.
(2) $\hat{\mu}^{-1}(e) = \mu^{-1}(\hat{e})$ and is denoted M_0.
(3) Restricted over the identity elements, $d\hat{\mu} = dS \circ d\mu (= \underline{s} \circ d\mu)$.

Remark 4.3. It follows from condition (2) that M_0 is preserved by the $\tilde{G}$ action. By condition (3), we see that $\hat{\mu}^* = \mu^* \circ \underline{s}^* = \mu^* \circ \underline{s}$ when restricted to M_0, since s is symmetric. Thus on M_0 we have

$$\hat{\mu}^*\theta_{\hat{\tau}} = \mu^* \circ \underline{s}(\theta_{\hat{\tau}}) = \mu^*\hat{\theta}_{\underline{s}(\hat{\tau})} \quad \text{for } \hat{\tau} \in \hat{\mathfrak{g}}.$$

The reduced structure will be a Courant algebroid of a more general type instead of an extended tangent bundle, (also see [**36**] and the references therein for the following definition):

Definition 4.4. A Courant algebroid E on M is a *transitive Courant algebroid* if it fits in the following diagram.

$$\begin{tikzcd}
0 \arrow[r] & T^*M \arrow[r] & E \arrow[r, "p"] \arrow[dr, dotted, "a"'] & E_0 \arrow[r] \arrow[d] & 0 \\
& & & TM \arrow[d] \arrow[dr] & \\
& & & 0 & 0
\end{tikzcd}$$

where a is the anchor map and the sequences are all exact.

The usual constructions of B-transformation for $B \in \Omega^2(M)$ and twisting of the Courant bracket by $H \in \Omega_0^3(M)$ are valid for a transitive Courant algebroid E.

Theorem 4.5. *Given Assumption* 4.2, *and let* $(\tilde{G}, G, \hat{G})$ *be a* (*local*) *double Lie group whose Lie algebras form the Manin triple* $(\tilde{\mathfrak{g}}, \mathfrak{g}, \hat{\mathfrak{g}})$, *where* $\tilde{G}$ *is connected but not necessarily simply connected* (*compare to Theorem* B.6). *Suppose that the action of* $\tilde{\mathfrak{g}}$ *induces an action of* $\tilde{G}$, *which is proper and free on* M_0, *then there is a transitive Courant algebroid* $\mathcal{T}_e\tilde{Q}$ *on* $\tilde{Q} = M_0/\tilde{G}$.

Proof. Let $\Gamma(\cdot)^{\tilde{G}}$ denote the set of $\tilde{G}$-invariant sections. By Lemma 3.11 we see that

$$(\mu^*\hat{\theta}, \mathcal{J}_1(\mu^*\hat{\theta}), \mathcal{J}_2(\mu^*\hat{\theta}), \mathcal{G}(\mu^*\hat{\theta})) = (\mu^*\hat{\theta}, \mathcal{J}_1(\mu^*\hat{\theta})) \oplus \mathcal{J}_2(\mu^*\hat{\theta}, \mathcal{J}_1(\mu^*\hat{\theta}))$$

is preserved by the G-action. Similarly, it is also preserved by $\hat{G}$ and it follows that it is preserved by the action of $\tilde{G}$. Analogously, the bundles $(\mu^*\hat{\theta}, \mathcal{J}_1(\mu^*\hat{\theta}), \mathcal{J}_2(\mu^*\hat{\theta}))$ and $(\mu^*\hat{\theta})$ are preserved by the $\tilde{G}$-action. Let $\mathcal{K} = (\mu^*\hat{\theta})$ and $\mathcal{K}' = \mathcal{J}_1(\mu^*\hat{\theta}) \oplus \mathcal{J}_2(\mu^*\hat{\theta})$, then the conditions for Lemma A.4(2) are satisfied. Thus $\mathcal{T}''_\mu M_0 = \mathrm{Ann}(\mu^*\hat{\theta}, \mathcal{J}_1(\mu^*\hat{\theta}), \mathcal{J}_2(\mu^*\hat{\theta}))/(\mu^*\hat{\theta})$ descends to an Courant algebroid $\mathcal{T}_e\tilde{Q}$ on $\tilde{Q}$.

Another way to see the Courant algebroid structure is to follow Theorem 3.16. Using the decomposition in (3.2), where K there corresponds to $(\mu^*\hat{\theta})$, and $\tilde{\mathbb{W}}_K$ corresponds to the domain of the map below, we define the following projection

$$\pi : \mathrm{Ann}(\mu^*\hat{\theta}, \mathcal{J}_1(\mu^*\hat{\theta}), \mathcal{J}_2(\mu^*\hat{\theta})) \longrightarrow \mathcal{T}'_\mu M_0$$

and the bracket $*_\mu$:

$$\mathfrak{X} *_\mu \mathfrak{Y} = \pi(\mathfrak{X} *_H \mathfrak{Y}) \text{ for } \mathfrak{X}, \mathfrak{Y} \in \Gamma(\mathcal{T}'_\mu M_0)^{\tilde{G}} \subset \Gamma(\mathrm{Ann}(\mu^*\hat{\theta}, \mathcal{J}_1(\mu^*\hat{\theta}), \mathcal{J}_2(\mu^*\hat{\theta}))^{\tilde{G}}.$$

By definition, $\Gamma(\mathcal{T}'_\mu M_0)^{\tilde{G}}$ is closed under $*_\mu$. Then the inclusion:

$$\mathcal{T}'_\mu M_0 \hookrightarrow \mathrm{Ann}(\mu^*\hat{\theta}, \mathcal{J}_1(\mu^*\hat{\theta}), \mathcal{J}_2(\mu^*\hat{\theta}))$$

induces a natural isomorphism to $\mathcal{T}''_\mu M_0$, and the brackets coincide. □

Corollary 4.6. *With the same assumptions as in Theorem* 4.5*, let* $(M, \mathcal{J}'_1, \mathcal{J}'_2; \mathcal{G}')$ *be the* B_1*-transformed generalized Kähler structure for* $B_1 \in \Omega^2(M)^{\tilde{G}}$*. Let all other choices be the same. Then the transitive Courant algebroid* $\mathcal{T}'_e\tilde{Q}$ *induced from* $(\mathcal{J}'_1, \mathcal{J}'_2; \mathcal{G}')$ *is a* b*-transformation of* $\mathcal{T}_e\tilde{Q}$*, for some* $b \in \Omega^2(\tilde{Q})$*.*

Proof. Choose a connection form $\tilde{\theta}$ of the $\tilde{G}$-principle bundle $M_0 \to \tilde{Q}$ and with respect to a choice of basis of $\tilde{\mathfrak{g}}$ we have $\tilde{\theta}_j$ and $\tilde{X}_j$. Consider the form $\tilde{b} = \prod_j (1 - \tilde{\theta}_j \wedge \iota_{\tilde{X}_j}) B_1|_{M_0}$, where the terms in brackets are considered operators on $\Omega^2(M_0)$. Then $\tilde{b}$ is horizontal with respect to $\tilde{G}$-action and $\mathrm{Ann}(\mu^*\hat{\theta}, \mathcal{J}_1(\mu^*\hat{\theta}), \mathcal{J}_2(\mu^*\hat{\theta}))$ is preserved by the transformation $e^{\tilde{b}}$, from which the result follows. □

5. Courant and T-duality

Let $(\tilde{G}, G, \hat{G})$ be a double Lie group and suppose that there is a bi-Hamiltonian $\tilde{G}$-action on the generalized Kähler manifold $(M, \mathcal{J}_1, \mathcal{J}_2)$, as considered in Theorem 4.5. Since the actions of G and $\hat{G}$ are both Hamiltonian, although with respect to different generalized complex structures, they could be both reduced as described in Theorem 3.16. We introduce the following.

Definition 5.1. Consider a bi-Hamiltonian action as in Definition 4.1. Suppose that the reduction of G- (resp. $\hat{G}$-) action at $\hat{e} \in \hat{G}$ (resp. at $e \in G$) as given in Theorem 3.16 exists and denote it $(Q, \mathcal{J}_1, \mathcal{J}_2)$ (resp. $(\hat{Q}, \hat{\mathcal{J}}_1, \hat{\mathcal{J}}_2)$). The structures $(Q, \mathcal{J}_1, \mathcal{J}_2)$ and $(\hat{Q}, \hat{\mathcal{J}}_1, \hat{\mathcal{J}}_2)$ are *Hamiltonian dual* to each other. When the assumptions of Theorem 4.5 holds, the structures are said to be *Courant dual* to each other.

Geometrically, the Hamiltonian duality as defined above has a significant drawback: *a priori*, the level sets $M_{\hat{e}} = \mu^{-1}(\hat{e})$ and $M_e = \hat{\mu}^{-1}(e)$ might have nothing to do with each other and the relation between the geometry and topology of the quotients Q and $\hat{Q}$ may not be clear. For Courant duality, the relation of the topology and geometry can be understood much better.

Proposition 5.2. *Assume the conditions in Theorem* 4.5 *and that* $(\tilde{G}, G, \hat{G})$ *is a connected double Lie group, and all the groups are compact, we have the following diagram, where the maps are principle bundles of compact Lie groups.*

$$\begin{tikzcd}
 & M_0 = Q \times_{\tilde{Q}} \hat{Q} \arrow[dl, "\pi"', "/G"] \arrow[dr, "\hat{\pi}", "/\hat{G}"'] & \\
Q \arrow[dr, "/\hat{G}", "p"'] & & \hat{Q} \arrow[dl, "/G"', "\hat{p}"] \\
 & \tilde{Q} &
\end{tikzcd}$$

Proof. Recall that $G \times \hat{G} \to \tilde{G} : (g, \hat{g}) \mapsto g\hat{g}^{-1}$ as well $\hat{G} \times G \to \tilde{G} : (\hat{g}, g) \mapsto \hat{g}g^{-1}$ are diffeomorphisms for the double Lie group $(\tilde{G}, G, \hat{G})$. The left action of $\hat{G}$ on Q is induced from:

$$\hat{g} \circ g^{-1}x = \hat{g}g^{-1}x \quad \text{for } x \in M_0,$$

while the left action of G on $\hat{Q}$ is induced from

$$g \circ \hat{g}^{-1}x = g\hat{g}^{-1}x \quad \text{for } x \in M_0.$$

These two actions are both free with the same quotient space $\tilde{Q} = M_0/\tilde{G}$. $\square$

The choice of terminology is justified by the following:

Proposition 5.3. *With the same conditions as in Proposition* 5.2*, the Courant algebroids on* $\tilde{Q}$ *formed by the invariant sections of* $\mathcal{T}Q$ *and* $\mathcal{T}\hat{Q}$ *are isomorphic to the one defined by Theorem* 4.5.

Proof. Note that the invariant sections of $\mathcal{T}Q$ lifts to M_0 as the $\tilde{G}$-invariant sections of $\mathrm{Ann}(\mu^*\hat{\theta}, \mathcal{J}_1(\mu^*\hat{\theta}))/(\mu^*\hat{\theta}) \oplus \mathcal{J}_1(\mu^*\hat{\theta})$, which is isomorphic to $\mathcal{T}''_\mu M_0$ by Lemma 3.14. The proposition then follows. □

In case the subgroups G and $\hat{G}$ commute in $\tilde{G}$ we show that the double Lie group is abelian. Infinitesimally, we have the following.

Proposition 5.4. *Let $(\tilde{\mathfrak{g}}, \mathfrak{g}, \hat{\mathfrak{g}})$ be the Manin triple defined by a factorizable Lie bialgebra $\mathfrak{g}$. If $[\mathfrak{g}, \hat{\mathfrak{g}}] = 0$, then $[\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}] = 0$, i.e., $\tilde{\mathfrak{g}}$ is abelian, and we write $(\tilde{\mathfrak{g}}, \mathfrak{g}, \hat{\mathfrak{g}}) = (\tilde{\mathfrak{t}}, \mathfrak{t}, \hat{\mathfrak{t}})$.*

Proof. By (B.4), we have for $\tau \in \mathfrak{g}$ and $\hat{\omega} \in \hat{\mathfrak{g}}$:

$$[(\tau, \tau), (\underline{r_+}(\hat{\omega}), \underline{r_-}(\hat{\omega}))] = ([\tau, \underline{r_+}(\hat{\omega})], [\tau, \underline{r_-}(\hat{\omega})]) = 0,$$

which implies that $[\tau, \underline{s}(\hat{\omega})] = 0$. Since $\underline{s}$ is invertible, we see that $\mathfrak{g}$ is abelian. Then $(\tilde{\mathfrak{g}}, \mathfrak{g}, \hat{\mathfrak{g}})$ form a Manin triple implies that $\tilde{\mathfrak{g}}$ and $\hat{\mathfrak{g}}$ are abelian as well. □

Because of this, in the following we work under Assumption 4.2 and that $\tilde{G}$ is abelian. The notations $\tilde{T}$, T and $\hat{T}$ will mean that the respective groups are compact, i.e., torus. In this case, a new pairing can be defined for $\mathfrak{t}$ and $\hat{\mathfrak{t}}$:

Lemma 5.5. *Both $\mathcal{J}_1$ and $\mathcal{J}_2$ are preserved by the $\tilde{T}$-action. For any $\tau \in \mathfrak{t}$ and $\hat{\omega} \in \hat{\mathfrak{t}}$, we have $d\langle \mathcal{J}_1(\mu^*\hat{\theta}_\tau), \mathcal{J}_2(\hat{\mu}^*\theta_{\hat{\omega}})\rangle = 0$. We define the pairing*

$$P : \mathfrak{t} \otimes \hat{\mathfrak{t}} \to \mathbb{R} : \tau \otimes \hat{\omega} \mapsto 2\langle \mathcal{J}_1(\mu^*\hat{\theta}_\tau), \mathcal{J}_2(\hat{\mu}^*\theta_{\hat{\omega}})\rangle \tag{5.1}$$

then P is non-degenerate, i.e., $\tau = 0 \in \mathfrak{t} \iff P(\tau, \hat{\omega}) = 0$ for all $\hat{\omega} \in \hat{\mathfrak{t}}$ and vice versa for $\hat{\omega}$.

Proof. By definition, the $\mathfrak{t}$-action preserves $\mathcal{J}_2$ and $\hat{\mathfrak{t}}$-action preserves $\mathcal{J}_1$. Then by the proof of Lemma 3.11 and $d\hat{\theta} = 0$ (since $\mathfrak{t}$ is abelian), it follows that $\mathcal{J}_1$ as well as $\mu^*(\hat{\theta}_\tau)$ are preserved by $\mathfrak{t}$. Thus the $\tilde{T}$-action preserves $\mathcal{J}_1$. Similarly, $\mathcal{J}_2$ and $\hat{\mu}^*(\theta_{\hat{\omega}})$ are preserved by the $\tilde{T}$-action. We have

$$\mathcal{J}_1(\mu^*\hat{\theta}_\tau) *_H \mathcal{J}_2(\hat{\mu}^*\theta_{\hat{\omega}}) = 0 \quad \text{and} \quad \mathcal{J}_2(\hat{\mu}^*\theta_{\hat{\omega}}) *_H \mathcal{J}_1(\mu^*\hat{\theta}_\tau) = 0.$$

Add the above two equations, we see that

$$d\langle \mathcal{J}_1(\mu^*\hat{\theta}_\tau), \mathcal{J}_2(\hat{\mu}^*\theta_{\hat{\omega}})\rangle = 0.$$

The non-degeneracy of P follows from that of the generalized metric $\mathcal{G}$. □

Definition 5.1 for the case of the double Lie group $(\tilde{T}, T, \hat{T})$ is of special interest and we give a seperate notion in this case.

Definition 5.6. The structures $\mathcal{T}Q$ and $\mathcal{T}\hat{Q}$ are said to be (*Courant*) *T-dual* to each other.

Assumption 5.7. In the rest of this section, we assume that the $\tilde{T}$-action preserves a splitting of $\mathcal{T}M$ into H-twisted generalized tangent bundle $\mathbb{T}M$.

Consider the reduced structures on $Q = M_0/T$ and $\hat{Q} = M_0/\hat{T}$. By Corollary A.5, the structures are both twisted generalized Kähler structures, whose twisting form can be described with a choice of connection forms. Let $\tilde{\Theta}$ be a connection form on M_0 as principle $\tilde{T}$-bundle. Choose basis $\{\tau_j\}$ and $\{\hat{\tau}_j\}$ of $\mathfrak{t}$ and $\hat{\mathfrak{t}}$ respectively, and denote $\theta_j, X_j + \xi_j, \Theta_j$ and $\hat{\theta}_j, \hat{X}_j + \hat{\xi}_j, \hat{\Theta}_j$ the corresponding components. We define:

(5.2)

$$\begin{aligned}\tilde{B} &= B + \hat{B}\\ &= \left(\Theta \wedge \xi - \frac{1}{2}\sum_{j,k} \Theta_j \wedge \Theta_k \cdot \iota_{X_k}\xi_j\right) + \left(\hat{\Theta} \wedge \hat{\xi} - \frac{1}{2}\sum_{j,k} \hat{\Theta}_j \wedge \hat{\Theta}_k \cdot \iota_{\hat{X}_k}\hat{\xi}_j\right).\end{aligned}$$

Then $\tilde{B}$ is $\tilde{T}$-invariant on M_0. When the actions generated by $\mathfrak{t}$ and $\tilde{\mathfrak{t}}$ are proper, the forms Θ and $\hat{\Theta}$ become connection forms on M_0 as, respectively T and $\hat{T}$ principle bundles.

Theorem 5.8. *Suppose Assumption* 4.2 *holds and let* $(M, \mathbb{J}_1^{\tilde{B}}, \mathbb{J}_2^{\tilde{B}}; \mathbb{G}^{\tilde{B}})$ *be the* $-\tilde{B}$*-transformed structure on* M*, with* $\tilde{B}$ *defined above. Then the induced Courant algebroid on* $\tilde{Q} = M_0/\tilde{T}$ *remains unchanged. Let* h *(resp.* $\hat{h}$*) be the twisting form of the corresponding reduced structure on* Q *(resp.* $\hat{Q}$*), then*

$$\hat{\pi}^*\hat{h} - \pi^*h = d(\hat{\Theta} \wedge \Theta), \tag{5.3}$$

where on the right-hand side we use also the pairing (5.1)*.*

Proof. Direct computation shows that the $\tilde{T}$-horizontal part of $\tilde{B}$ is 0. Thus by Corollary 4.6, the Courant algebroid structure on $\tilde{Q}$ remains unchanged under the $-\tilde{B}$-transformation.

The $-\tilde{B}$-transformed structures on M has twisting form $\tilde{H} = H + d\tilde{B}$. Let $\mathbb{J}_1^{\tilde{B}} = e^{-\tilde{B}}\mathbb{J}_1 e^{\tilde{B}}$ and so on. We compute

$$\iota_{X_l}\tilde{B} = \xi_l - \hat{\Theta} \cdot \iota_{X_l}\hat{\xi} = \xi_l - \sum_j \hat{\Theta}_j \cdot \iota_{X_l}\hat{\xi}_j$$

and it follows that $\mathbb{J}_1^{\tilde{B}}(\mu^*\hat{\theta}_l) = X_l + \xi_l'$, where $\xi_l' = \sum_j \hat{\Theta}_j \cdot \iota_{X_l}\hat{\xi}_j$. We note that $\iota_{X_j}\xi_l' = 0$, and the twisting form h satisfies

$$\pi^*h = \tilde{H} + d(\Theta \wedge \xi').$$

Similarly, we have $\hat{\pi}^*\hat{h} = \tilde{H} + d(\hat{\Theta} \wedge \hat{\xi}')$, where $\hat{\xi}'_l = \sum_j \Theta_j \cdot \iota_{\hat{X}_l}\xi_j$. More explicitly, we compute

$$\begin{aligned}\hat{\pi}^*\hat{h} - \pi^*h &= d\left(\sum_{j,k}\hat{\Theta}_j \wedge \Theta_k \cdot \iota_{\hat{X}_j}\xi_k - \sum_{j,k}\Theta_k \wedge \hat{\Theta}_j \cdot \iota_{X_k}\hat{\xi}_j\right)\\ &= d\sum_{j,k}\hat{\Theta}_j \wedge \Theta_k \cdot (\iota_{\hat{X}_j}\xi_k + \iota_{X_k}\hat{\xi}_j)\\ &= d\sum_{j,k}\hat{\Theta}_j \wedge \Theta_k \cdot 2\langle \mathbb{J}_2(\hat{\mu}^*\theta_j), \mathbb{J}_1(\mu^*\hat{\theta}_k)\rangle\\ &= d(\hat{\Theta} \wedge \Theta).\end{aligned}$$

where the last step we use the pairing P as given in (5.1). □

Remark 5.9. We note that the T- or $\hat{T}$-horizontal part of $\tilde{B}$, in general, do not vanish. Thus the structures on Q and $\hat{Q}$ are B-transformed from their respective original structures. With Proposition 5.3, the theorem above states that the Courant algebroid on $\tilde{Q}$ formed by the set of invariant sections of $\mathcal{T}Q$ or $\mathcal{T}\hat{Q}$ are still isomorphic to the original one. We note also that the equation (5.3) coincides with the equation in the physics literature, where M_0 is to be the correspondence space of the T-dual bundles Q and $\hat{Q}$. It is shown (e.g., [**3**]) that the twisted cohomology of T-dual principle bundles are isomorphic. Since the twisted cohomology only depends on the cohomology class of the twisting, the same is true for the structures on Q and $\hat{Q}$ before applying B-transformation. In [**8**], Proposition 5.3 is shown when Q and $\hat{Q}$ are T-dual S^1-principle bundles, with twisted generalized complex structures, by directly defining the isomorphism.

Example 5.10. The following example is considered in [**15**] and we recall the setup and point out its relevance to the current discussion. Let $M = \mathbb{C}^2 \setminus \{(0,0)\}$ and consider the coordinates $z = (z_1, z_2) = (x_1 + iy_1, x_2 + iy_2) = (x_1, y_1, x_2, y_2)$. Let $r^2 = |z_1|^2 + |z_2|^2$ and $J = \left(\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right)$. Consider the following structures:

$$\mathbb{J}_1 = \begin{pmatrix} 0 & 0 & r^2J & 0\\ 0 & -J & 0 & 0\\ r^{-2}J & 0 & 0 & 0\\ 0 & 0 & 0 & -J\end{pmatrix},\ \mathbb{J}_2 = \begin{pmatrix} J & 0 & 0 & 0\\ 0 & 0 & 0 & -r^2J\\ 0 & 0 & J & 0\\ 0 & -r^{-2}J & 0 & 0\end{pmatrix},$$

where the labelling on rows are $(Tz_1, Tz_2, T^*z_1, T^*z_2)^T$. Then $(M, \mathbb{J}_1, \mathbb{J}_2)$ is an H-twisted generalized Kähler structure where

$$H = -\sin(2\lambda)d\lambda \wedge d\phi_1 \wedge d\phi_2$$

in the polar coordinates $(z_1, z_2) = r(e^{i\phi_1}\sin\lambda, e^{i\phi_2}\cos\lambda)$. In particular, $[H] \neq 0 \in H^3(M)$ (cf. [**12**]).

Let $\tilde{T} = S^1 \times S^1$ and $(e^{i\theta_1}, e^{i\theta_2})$ be the coordinates. It acts on M via

$$(e^{i\theta_1}, e^{i\theta_2}) \circ (z_1, z_2) = (e^{i\theta_1} z_1, e^{-i\theta_2} z_2).$$

Let T and $\hat{T}$ be the first and second S^1, respectively, then $(\tilde{T}, T, \hat{T})$ is a double Lie group and the action of $\tilde{T}$ is bi-Hamiltonian and satisfies Assumption 4.2 with the common moment map $f = \ln r$. The T and $\hat{T}$ actions are generated, respectively, by

$$\mathbb{J}_1(df) = \frac{\partial}{\partial\phi_1} - \cos^2\lambda d\phi_2 \quad \text{and} \quad \mathbb{J}_2(df) = -\frac{\partial}{\partial\phi_2} + \sin^2\lambda d\phi_1.$$

It follows that $2\langle \mathbb{J}_1(df), \mathbb{J}_2(df)\rangle = 1$, i.e., T and $\hat{T}$ are dual tori in the standard sense. We note that the actions are not free. Consider the submanifold $M' = M \setminus (\{z_1 = 0\} \cup \{z_2 = 0\})$, on which the actions are free. The reduced structures of the T and $\hat{T}$ action on M' are, respectively, the opposite and standard Kähler structures on $D^2 \setminus \{0\}$. Our results then state that they are T-dual to each other.

6. *T*-duality group

The group $O(m, m; \mathbb{Z})$ is called the T*-duality group* in the physics literature ([**27**, **16**] and the references therein). In the physics literature, for each element of $O(m, m; \mathbb{Z})$ it is associated a pair of related T-dual principle bundles with H-fluxes. The physical theory on such related structures are expected to be the same. In our picture, the group $O(m, m; \mathbb{Z})$ also enters naturally, as we will see in Corollary 6.3.

We first consider the linear case. Let $\mathbb{V} = V \oplus V^*$, $K \subset V^*$ and $(\mathbb{J}_1, \mathbb{J}_2; \mathbb{G})$ a linear generalized Kähler structure. We note that the natural pairing on $\mathbb{V}$ induces a pairing P_K on $\mathbb{J}_1(K) \oplus \mathbb{J}_2(K)$, which can also be seen as induced from the pairing $\langle\cdot, \mathbb{G}(\cdot)\rangle$ defined on K by $\mathbb{G}$. By the positive definiteness of $\mathbb{G}$ we see that P_K has signature (m, m) where $m = \dim K$. Completely parallel to Lemma 3.15, we have

Lemma 6.1. *Let $\mathbb{V} = V \oplus V^*$, $a : \mathbb{V} \to V$ be the projection and $(\mathbb{J}_1, \mathbb{J}_2; \mathbb{G})$ a linear generalized Kähler structure. Let $K \subset V^*$ and $K' \subset \mathbb{J}_1(K) \oplus \mathbb{J}_2(K)$ a maximal isotropic subspace with respect to P_K and $N' = a(K')$. Assume that for $j = 1, 2$*

(1) $K + \mathbb{J}_j(K) \subset \mathrm{Ann}(K, \mathbb{J}_j(K))$
(2) $\mathbb{J}_j(K) \cap V^* = \{0\}$ *and*
(3) $N_1 \cap N_2 = \{0\}$, *where* $N_j = a \circ \mathbb{J}_j(K)$.

Then there is a self-dual exact sequence:

$$0 \longrightarrow W^*_{K'} \xrightarrow{a^*_{K'}} \mathbb{W}_K \xrightarrow{a_{K'}} W_{K'} \longrightarrow 0,$$

with

$$W_{K'} = \frac{\mathrm{Ann}_V(K)}{N'} \quad \textit{and} \quad W^*_{K'} = \frac{\mathrm{Ann}_{V^*}(N')}{K},$$

where $\mathbb{W}_K = \mathrm{Ann}(K, K', \mathbb{G}(K'), \mathbb{G}(K)) = \mathrm{Ann}(K, \mathbb{J}_1(K), \mathbb{J}_2(K), \mathbb{G}(K))$.

Proof. By the assumption (2), we see that $K' \cap V^* = \{0\}$, $K \oplus K' \subset \mathrm{Ann}(K, K')$. Let $a_{K'}$ be the map induced from a. Let $\mathbb{U}_{K'} = \mathrm{Ann}(K, K')$, then the kernel of the induced map $\mathbb{U}_{K'} \to W_{K'}$ is $\mathrm{Ann}_{V^*}(N') \oplus K'$ and thus the kernel of $a_{K'}$ is

$$\ker a_{K'} = (\mathrm{Ann}_{V^*}(N') \oplus K') \cap \mathbb{V}_K.$$

By the decomposition $\mathbb{U}_{K'} = \mathbb{W}_K \oplus (K \oplus K')$ and inclusion $K \oplus K' \subset \mathrm{Ann}_{V^*}(N') \oplus K'$, we see that $\ker a_{K'} \simeq \mathrm{Ann}_{V^*}(N')/K$. Since $\ker a_{K'}$ is maximally isotropic with respect to the induced pairing $\langle , \rangle_K$ on $\mathbb{W}_K$, we see that the exact sequence is self-dual. □

Using the notations in (the proof of) Theorem 3.16, we have

Proposition 6.2. *Under the condition of Theorem* 4.5 *and let* $T' \subset \tilde{T}$ *be a maximally isotropic subtorus of* $\tilde{T}$ *with respect to the pairing* P *as in* (5.1), *i.e., the Lie algebra* $\mathfrak{t}'$ *is a Lagrangian subspace of* $\tilde{\mathfrak{t}}$. *Then the reduced space* $Q' = M_0/T'$ *has a natural extended Kähler structure.*

Proof. By the proof of Theorem 4.5, the bundles $(\mu^*\hat{\theta})$, $(\mu^*\hat{\theta}, \mathcal{J}_1(\mu^*\hat{\theta}), \mathcal{J}_2(\mu^*\hat{\theta}))$ and $\mathcal{T}_\mu M_0$ are all preserved by the $\tilde{T}$-action, and thus are preserved by the T'-action. Let $\mathcal{K} = (\mu^*\hat{\theta})$ and $\mathcal{K}'$ be the sub-bundle generated by the infinitesimal fields $\{X_{\tau'} + \xi_{\tau'} | \tau' \in K'\}$, then it follows from the proofs of Lemmata 3.7 and 5.5 that $(\mu^*\hat{\theta}, \mathcal{K}')$ is preserved by the T'-action. Since T' is isotropic, we have $\mathcal{K} \oplus \mathcal{K}' \subset \mathrm{Ann}(\mathcal{K}, \mathcal{K}')$ and Lemma A.4(1) gives an extended tangent bundle $\mathcal{T}Q'$ on Q'. It follows from Proposition 5.5 that $\mathcal{J}_1$ and $\mathcal{J}_2$ are both invariant with respect to the T'-action and thus descend to $\mathcal{J}'_1$ and $\mathcal{J}'_2$ on $\mathcal{T}Q'$, which define an extended Kähler structure. □

The following provides an interpretation of the T-duality group $O(m, m; \mathbb{Z})$ in the context of our reduction construction:

Corollary 6.3. *Suppose that the action of* $\tilde{T}$ *preserves a splitting of* $\mathcal{T}M$. *Let* $g \in O(m, m; \mathbb{Z})$ *and consider the pair of Lagrangian subgroups* T_g *and* $\hat{T}_g$ *with Lie algebra* $g(\mathfrak{t})$ *and* $g(\hat{\mathfrak{t}})$. *Let* Q_g *and* $\hat{Q}_g$ *be the reduction of* M_0 *by the groups* T_g *and* $\hat{T}_g$, *respectively. Then the induced structures on* Q_g *and* $\hat{Q}_g$ *are twisted generalized Kähler structures and the equation* (5.3) *holds for this pair after applying certain* B*-transformation. The Courant algebroid on*

$\tilde{Q}$ defined by the $\hat{T}_g$-invariant sections of $\mathbb{T}Q_g$ is isomorphic to the one given by Theorem 4.5.

Proof. Similar to (5.2), we choose basis $\{\tau_j^g\}$ and $\{\hat{\tau}_j^g\}$ of $\mathfrak{t}^g$ and $\hat{\mathfrak{t}}^g$ respectively and let $X_j^g+\xi_j^g$, Θ_j^g and $\hat{X}_j^g+\hat{\xi}_j^g$, $\hat{\Theta}_j^g$ be the corresponding components, and define

$$\begin{aligned}\tilde{B}^g &= B^g + \hat{B}^g\\ &= \left(\Theta^g \wedge \xi^g - \frac{1}{2}\sum_{j,k} \Theta_j^g \wedge \Theta_k^g \cdot \iota_{X_k^g}\xi_j^g\right)\\ &\quad + \left(\hat{\Theta}^g \wedge \hat{\xi}^g - \frac{1}{2}\sum_{j,k} \hat{\Theta}_j^g \wedge \hat{\Theta}_k^g \cdot \iota_{\hat{X}_k^g}\hat{\xi}_j^g\right).\end{aligned}$$

In particular, the basis $\{\tau_j^g\}$ and $\{\hat{\tau}_j^g\}$ can be taken as the transformation of $\{\tau_j\}$ and $\{\hat{\tau}_j\}$ by g. The proof of (5.3) is then completely parallel to that of Theorem 5.8. The isomorphism of courant algebroids is straight forward. □

Example 6.4. We consider in detail the special case when $g = e^b$, where $b : \mathfrak{t} \to \hat{\mathfrak{t}}$ is skew-symmetric with respect to the pairing P. Then $g(\mathfrak{t}) = \mathrm{graph}(b)$ and $g(\hat{\mathfrak{t}}) = \hat{\mathfrak{t}}$. Let $\{\tau_j\}$ and $\{\hat{\tau}_j\}$ be basis of $\mathfrak{t}$ and $\hat{\mathfrak{t}}$, respectively, and (b_{ij}) the matrix of b with respect to these basis. Then $\{\tau_j^b = \tau_j + \sum_k b_{jk}\hat{\tau}_k\}$ is a basis of $g(\mathfrak{t})$, where $b(\tau_j) = \sum_k b_{kj}\hat{\tau}_k \in \hat{\mathfrak{t}}$. Let ^{b} to denote the objects for the transformed structures, then

$$\begin{cases}\Theta_j^b = \Theta_j\\ \hat{\Theta}_j^b = \hat{\Theta}_j - \sum_k b_{jk}\Theta_k\end{cases} \quad\text{and}\quad \begin{cases}\hat{X}_j^b + \hat{\xi}_j^b = \hat{X}_j + \hat{\xi}_j\\ X_j^b + \xi_j^b = X_j + \xi_j + \sum_k b_{kj}(\hat{X}_k + \hat{\xi}_k).\end{cases}$$

Direct computation gives

$$\left.\begin{aligned}(\pi^b)^*h^b &= \pi^*h - d\sum_{j,k,l} b_{lj}\Theta_k \wedge \Theta_j \langle X_k + \xi_k, \hat{X}_l + \hat{\xi}_l\rangle\\ (\hat{\pi}^b)^*\hat{h}^b &= \hat{\pi}^*\hat{h} - d\sum_{j,k,l} b_{lj}\Theta_j \wedge \Theta_k \langle X_k + \xi_k, \hat{X}_l + \hat{\xi}_l\rangle\end{aligned}\right\}\text{ which implies}$$

$$(\hat{\pi}^b)^*\hat{h}^b - (\pi^b)^*h^b = d\sum_{j,k} \hat{\Theta}_j^b \wedge \Theta_k^b \cdot (\iota_{\hat{X}_j^b}\xi_k^b + \iota_{X_k^b}\hat{\xi}_j^b),$$

i.e., the equation (5.3) holds for the pair of reduced structures Q^b and $\hat{Q}^b$. Since $\hat{\mathfrak{t}}^b = \hat{\mathfrak{t}}$, we have $\hat{Q}^b = \hat{Q}$, while the twisting form $\hat{h}$ is changed by an exact term. As the situation for $\mathfrak{t}$ and $\hat{\mathfrak{t}}$ is symmetric, we may consider e^β for skew-symmetric $\beta : \hat{\mathfrak{t}} \to \mathfrak{t}$ and obtain similar result.

Example 6.5. The Example 5.10 discussed does not admit interesting T-duality group action, because $O(1,1;\mathbb{Z}) = \{\pm 1, \pm \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\}$. This can be compensated by considering a product of these, e.g. twisted structures on M^2, for example, and apply Example 6.4. Instead, here we consider another situation which is not quite covered by T-duality group. For Example 5.10, we consider the anti-diagonal action generated by

$$X_d + \xi_d = \left(\frac{\partial}{\partial \phi_1} + \frac{\partial}{\partial \phi_2}\right) - (\cos^2 \lambda d\phi_2 + \sin^2 \lambda d\phi_1),$$

then $\iota_{X_d}\xi_d = -1$. Let $\mathcal{K} = (d\mu)$ and $\mathcal{K}' = (X_d + \xi_d)$, then the condition for Lemma A.4 (1) does not hold. On the other hand, the condition for Lemma A.4(2) holds and there is an induced transitive Courant algebroid on the corresponding reduced space, i.e., S^2. A more general result holds.

Proposition 6.6. *Let $T^+ \subset \tilde{T}$ be a* non-degenerate *subtorus of $\tilde{T}$ with respect to P, i.e., the restriction of P on its Lie algebra $\mathfrak{t}^+$ is non-degenerate, then there is a natural transitive Courant algebroid on the reduced space $Q^+ = M_0/T^+$.*

7. Appendix A: Reduction of extended tangent bundle

Special case of the reduction of Courant algebroid has been discussed implicitly in our paper [**15**] showing that extended complex structure exists as the result of reduction of generalized complex manifold and, in general, it has been discussed explicitly in the works [**7**], [**33**] and [**37**]. For the sake of completeness, we prove the reduction of Courant algebroid in the relevant context of our construction in this article, i.e., for extended tangent bundles. We will use the notations in § 2.

Lemma A.1. *Let $\mathbb{V} = V \oplus V^*$ with the natural pairing $\langle , \rangle$, $K \subset V^*$ and $K' \subset \mathbb{V}$ so that $K' \cap V^* = \{0\}$. Define $N' = a \circ K'$, where $a : \mathbb{V} \to V$ is the projection. Assume that*

(1) *$K \oplus K' \subset \mathrm{Ann}(K, K')$ and*

Let $\mathbb{V}_K = \mathrm{Ann}(K, K')/K \oplus K'$, then $\langle , \rangle$ descends to non-degenerate pairing $\langle , \rangle_K$ on $\mathbb{V}_K$ and we have the self-dual exact sequence

$$0 \longrightarrow W_K^* \longrightarrow \mathbb{V}_K \xrightarrow{a_K} W_K \longrightarrow 0 \quad \textit{where } W_K = \frac{\mathrm{Ann}_V(K)}{N'}.$$

Proof. See [**15**, Lemma 4.3] □

Lemma A.2. *Using the same notations as in Lemma* A.1 *and replacing assumption* (1) *by one of the following statements that are equivalent to each other:*

(1′) *$K \subset \mathrm{Ann}(K, K')$ and $\langle , \rangle$ induces a non-degenerate pairing on $\mathbb{V}'_K = \mathrm{Ann}(K, K')/K$,* (1″) $K \subset \mathrm{Ann}(K, K')$ *and* $\langle , \rangle$ *restricts to a non-degenerate pairing on* K'. *then we have the exact sequence:*

$$0 \longrightarrow W_K^* \longrightarrow \mathbb{V}_K' \xrightarrow{a_K'} \mathrm{Ann}_V(K) \longrightarrow 0.$$

Proof. The surjectivity of $\mathrm{Ann}(K, K') \to \mathrm{Ann}_V(K)$ is easy and everything then follows. ☐

Definition A.3. Let S be a subspace of sections in $\mathcal{T}M$ which is closed with respect to $*$. A closed subspace $S' \subset S$ is a *two-sided null ideal* if $S * S' \subset S'$, $S' * S \subset S'$ and $\langle S', S\rangle = 0$.

It follows that when S' is a two-sided null idea of S, the structure $(S, *, \langle , \rangle)$ induces one such structure on the quotient space S/S', which also satisfies (2.3) and (2.4).

Lemma A.4. *Let* $(M, \mathcal{T}M)$ *be a manifold with an extended tangent bundle* $\mathcal{T}M$ *and* $M_0 \subset M$ *a submanifold. Let* $\mathcal{K} \subset T^*M|_{M_0}$ *and* $\mathcal{K}' \subset \mathcal{T}M|_{M_0}$ *be two subbundles of rank* m *and* m', *respectively, so that*

(i) $TM_0 = \mathrm{Ann}_{TM}(\mathcal{K})$ *and* $\mathcal{K}' \cap T^*M = \{0\}$,
(ii) $\mathcal{K}$ *is generated by sections* $\{\theta_j\}_{j=1}^m$ *so that* $d\theta_j \in \Gamma(\wedge^2\mathcal{K})$ *and*
(iii) $\mathcal{K}'$ *is generated by sections* $\{\mathfrak{X}_j\}_{j=1}^{m'}$.

Let $\tilde{\sigma}$ *be the infinitesimal action generated by* $\{\mathfrak{X}_j\}_{j=1}^{m'}$ *via the Loday bracket* $*$. *Suppose that* $\mathrm{Ann}(\mathcal{K}, \mathcal{K}')$ *is preserved by* $\tilde{\sigma}$.

If furthermore, we suppose that the action $\tilde{\sigma}$ *on* M_0 *is induced by a morphism* $G \to \tilde{\mathscr{G}}_H$, *where* G *is compact of dimension* m' *and the geometrical action* σ *is free. Let* $Q = M_0/G$ *then*

(1) *If* $\mathcal{K} \oplus \mathcal{K}' \subset \mathrm{Ann}(\mathcal{K}, \mathcal{K}')$, *then* $\mathrm{Ann}(\mathcal{K}, \mathcal{K}')/\mathcal{K} \oplus \mathcal{K}'$ *descends to an extended tangent bundle on* Q.
(2) *If* $\mathcal{K} \subset \mathrm{Ann}(\mathcal{K}, \mathcal{K}')$ *is preserved by* σ *and* $\langle , \rangle$ *induces a non-degenerate pairing on* $\mathcal{K}'$, *then* $\mathrm{Ann}(\mathcal{K}, \mathcal{K}')/\mathcal{K}$ *descends to a transitive Courant algebroid on* Q.

Proof. Let $a : \mathcal{T}M \to TM$ be the projection. Let $\mathfrak{X}, \mathfrak{X}' \in \Gamma(\mathrm{Ann}(\mathcal{K}, \mathcal{K}'))$, then

$$\langle \mathfrak{X}_j, \mathfrak{X}\rangle = 0 \quad \text{and} \quad \iota_X\theta_j = 0.$$

From the assumption (ii), $\mathrm{Ann}_{TM}(\mathcal{K})$ is an integrable distribution and M_0 is a leaf of this distribution. From the same assumption, we also obtain $\langle \mathfrak{X} *_H \mathfrak{X}', \theta_j\rangle = \iota_{[X,X']}\theta_j = 0$. It then follows that

$$\langle \mathfrak{X} *_H \theta_j, \mathfrak{X}'\rangle = X'\langle \mathfrak{X}, \theta_j\rangle - \langle \mathfrak{X} *_H \mathfrak{X}', \theta_j\rangle = 0,$$

i.e., $\mathfrak{X} *_H \theta_j \in \Gamma(\mathcal{K} \oplus \mathcal{K}')$. Similarly we have $\theta_j *_H \mathfrak{X} \in \Gamma(\mathcal{K} \oplus \mathcal{K}')$. Let $\mathcal{N}' = a(\mathcal{K}')$, then $\mathcal{N}' \subset \mathrm{Ann}_{TM}(\mathcal{K})$. We see that the (geometrical) action of G preserves M_0 and the quotient Q is well defined.
Case (1). Let

$$S_1 = \{\mathfrak{X} \in \Gamma(\mathrm{Ann}(\mathcal{K}, \mathcal{K}'))|\mathfrak{Y} *_H \mathfrak{X} \in \Gamma(\mathcal{K} \oplus \mathcal{K}') \text{ for all } \mathfrak{Y} \in \Gamma(\mathcal{K}')\}.$$

For any $\mathfrak{X}, \mathfrak{X}' \in S_1$, $\mathfrak{Y} \in \Gamma(\mathcal{K}')$ and $\mathfrak{Z} \in \Gamma(\mathrm{Ann}(\mathcal{K}, \mathcal{K}'))$ we compute

$$\begin{aligned}\langle \mathfrak{X} *_H \mathfrak{X}', \mathfrak{Y}\rangle &= a(\mathfrak{X})\langle \mathfrak{X}', \mathfrak{Y}\rangle - \langle \mathfrak{X}', \mathfrak{X} *_H \mathfrak{Y}\rangle \\ &= \langle \mathfrak{X}', \mathfrak{Y} *_H \mathfrak{X}\rangle - a(\mathfrak{X}')\langle \mathfrak{Y}, \mathfrak{X}'\rangle = 0, \\ \langle \mathfrak{X} *_H \mathfrak{Y}, \mathfrak{Z}\rangle &= a(\mathfrak{Z})\langle \mathfrak{X}, \mathfrak{Y}\rangle - \langle \mathfrak{Y} *_H \mathfrak{X}, \mathfrak{Z}\rangle = 0.\end{aligned}$$

Thus we have $\mathfrak{X} *_H \mathfrak{X}' \in \Gamma(\mathrm{Ann}(\mathcal{K}, \mathcal{K}'))$ as well as $\mathfrak{X} *_H \mathfrak{W}$ and $\mathfrak{W} *_H \mathfrak{X} \in \Gamma(\mathcal{K} \oplus \mathcal{K}')$ for all $\mathfrak{W} \in \Gamma(\mathcal{K} \oplus \mathcal{K}')$. It follows that

$$\mathfrak{Y} *_H (\mathfrak{X} *_H \mathfrak{X}') = (\mathfrak{Y} *_H \mathfrak{X}) *_H \mathfrak{X}' + \mathfrak{X} *_H (\mathfrak{Y} *_H \mathfrak{X}') \Rightarrow \mathfrak{X} *_H \mathfrak{X}' \in S_1,$$

i.e., S_1 is closed under $*_H$ and $\Gamma(\mathcal{K} \oplus \mathcal{K}')$ is a two-sided null ideal in S_1. Thus the structures $(*_H, \langle,\rangle)$ descend to $(*^G, \langle,\rangle^G)$ on

$$\frac{S_1}{\Gamma(\mathcal{K} \oplus \mathcal{K}')} = \Gamma\left(\frac{\mathrm{Ann}(\mathcal{K}, \mathcal{K}')}{\mathcal{K} \oplus \mathcal{K}'}\right)^G.$$

Lemma A.1 implies that $\mathrm{Ann}(\mathcal{K}, \mathcal{K}')/\mathcal{K} \oplus \mathcal{K}'$ descends to an extension E_Q of TQ by T^*Q. The equations (2.2), (2.3) and (2.4) holds by the comment above.
Case (2). Let

$$S_2 = \{\mathfrak{X} \in \Gamma(\mathrm{Ann}(\mathcal{K}, \mathcal{K}'))|\mathfrak{X}_j *_H \mathfrak{X} \in \Gamma(\mathcal{K})\}.$$

Completely parallel to case (1) above, we see that S_2 is closed under $*_H$. By definition $\Gamma(\mathcal{K})$ is a two-sided null ideal in S_2 with respect to $(*_H, \langle,\rangle)$. By Lemma A.2, $T^*Q \subset \ker a'_Q$. □

Let Θ be a connection form on $\pi : M_0 \to Q$, and Θ_j the component dual to X_j.

Corollary A.5. *In the above lemma, suppose that the action of G on $\mathcal{T}M$ preserves a splitting into $\mathbb{T}M$ with H-twisted structures. Let $\mathfrak{X}_j = X_j + \xi_j$ under the splitting. Then $\mathcal{T}Q$ splits into $\mathbb{T}Q$ with h-twisted structures, where $\pi^*h = H + dB$ with*

$$B = \Theta \wedge \xi_\mu - \tfrac{1}{2}\sum_{j,k} \Theta_j \wedge \Theta_k \cdot \iota_{X_k}\xi_j.$$

Proof. The action preserving the splitting implies that $d\xi_j = \iota_{X_j}H$ and $\mathcal{L}_{X_j}H = 0$. It follows that $\mathcal{L}_{X_\tau}B = 0$ for all $\tau \in \mathfrak{g}$. Let $B' = \prod_j(1 - \Theta_j \wedge \iota_{X_j})B$ be the horizontal part of B, where $\Theta_j \wedge \iota_{X_j}$ is interpreted as an operator on $\Omega^2(M)$. Direct computation gives $\iota_{X_\tau} B = \xi_\tau$ and $B' = 0$. Apply B-transformation (or choose a different splitting), we have $\mathrm{Ann}(\mathcal{K}, \mathcal{K}') \mapsto \mathrm{Ann}(\mathcal{K}, \{X_j\})$ and it defines a splitting of $\mathcal{T}Q$. Under the B-transformation, the twisting form becomes $H' = H + dB$ and we compute

$$\iota_{X_j}(H + dB) = \iota_{X_j} H + \mathcal{L}_{X_j} B - d\iota_{X_j} B = \iota_{X_j} H - d\xi_j = 0.$$

It follows that there is $h \in \Omega^3_0(Q)$, so that $\pi^* h = H + dB$, which gives the twisting form of the induced splitting of $\mathcal{T}Q$. $\square$

Remark A.6. We note that $\mathcal{L}_{X_j} H = 0$ and $d\xi_j = \iota_{X_j} H$ for all j implies that $d_G(H + \sum_j \xi_j u_j) = 0$, where d_G is the equivariant differential in the equivariant Cartan complex. Then h in the above gives an explicit description of the image of $[H + \sum_j \xi_j u_j]$ under the isomorphism $H^*_G(M_0) \xrightarrow{\cong} H^*(Q)$. From here, it again follows that $[h]$ is independent of the choice of the connection form.

8. Appendix B: Poisson Lie group and actions

The material in this subsection is taken from [**9**, **24**, **25**] (the first three chapters). More details can be found there as well as the references therein. We follow Convention 3.6.

Definition B.1. A Lie group G is called a *Poisson Lie group* if it is also a Poisson manifold such that the multiplication map $m : G \times G \to G$ is a Poisson map, where $G \times G$ is equipped with the product Poisson structure.

Let π_G be a multiplicative Poisson tensor on G, then $\pi_G|_e = 0$, where $e \in G$ is the identity, and the linearization of π_G at e defines on $\hat{\mathfrak{g}} = \mathfrak{g}^*$ a structure of Lie algebra $[,\hat]$. From [**38**],

Theorem B.2. *The right* (*left*) *invariant* 1*-forms on a Poisson Lie group* (G, π_G) *form a Lie subalgebra of* $\Omega^1(G)$ *with respect to the bracket*

$$\begin{aligned} [\theta^{\cdot}_{\hat\tau}, \theta^{\cdot}_{\hat\omega}]^* &= -d\pi_G(\theta^{\cdot}_{\hat\tau}, \theta^{\cdot}_{\hat\omega}) + \mathcal{L}_{\pi_G(\theta^{\cdot}_{\hat\tau})}\theta^{\cdot}_{\hat\omega} - \mathcal{L}_{\pi_G(\theta^{\cdot}_{\hat\omega})}\theta^{\cdot}_{\hat\tau} \\ &= \mathcal{L}_{\pi_G(\theta^{\cdot}_{\hat\tau})}\theta^{\cdot}_{\hat\omega} - \iota_{\pi_G(\theta^{\cdot}_{\hat\omega})} d\theta^{\cdot}_{\hat\tau} \quad \textit{for } \hat\tau, \hat\omega \in \hat{\mathfrak{g}}. \end{aligned} \tag{B.1}$$

The corresponding Lie algebra structure on $\hat{\mathfrak{g}}$ *coincides with the one given by linearizing* π_G *at the identity* $e \in G$. *In particular,* $\theta^{\cdot}_{[\hat\tau, \hat\omega\hat]} = [\theta^{\cdot}_{\hat\tau}, \theta^{\cdot}_{\hat\omega}]^*$ *for* $\hat\tau, \hat\omega \in \hat{\mathfrak{g}}$ *and* $\cdot = l$ *or* r.

The tangent Lie algebra $\mathfrak{g}$ of a Poisson Lie group G is an example of Lie bialgebra, as defined below.

Definition B.3. A *Lie bialgebra* is a vector space $\mathfrak{g}$ with a Lie algebra structure and a Lie coalgebra structure, which are compatible in the following sense: the cocommutator mapping $\delta : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ must be a 1-cocycle ($\mathfrak{g}$ acts on $\mathfrak{g} \otimes \mathfrak{g}$ by means of the adjoint representation). A triple of Lie algebras $(\mathfrak{p}, \mathfrak{p}_1, \mathfrak{p}_2)$ is called *Manin triple* if $\mathfrak{p}$ has a nondegenerate invariant pairing $\langle , \rangle$ and isotropic Lie subalgebras $\mathfrak{p}_1$ and $\mathfrak{p}_2$ such that as vector space $\mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_2$.

The cocommutator δ induces a Lie bracket on the dual $\hat{\mathfrak{g}}$ of $\mathfrak{g}$ and $(\tilde{\mathfrak{g}} = \mathfrak{g} \oplus \hat{\mathfrak{g}}, \mathfrak{g}, \hat{\mathfrak{g}})$ with the natural pairing between $\mathfrak{g}$ and $\hat{\mathfrak{g}}$ form a Manin triple. Conversely, the $\mathfrak{p}_i$ are dual to each other via the non-degenerate pairing $\langle , \rangle$.

Definition B.4. Let $\hat{G}$ be a Lie group with Lie algebra $(\hat{\mathfrak{g}}, \hat{[,]})$ with a Poisson Lie structure $\pi_{\hat{G}}$ so that the linearization of $\pi_{\hat{G}}$ at the $\hat{e} \in \hat{G}$ gives $(\mathfrak{g}, [,])$, then $(\hat{G}, \pi_{\hat{G}})$ is *a dual* Poisson Lie group of G. When $\hat{G}$ is simply connected, the structure $\pi_{\hat{G}}$ always exists and $\hat{G}$ is called *the dual group*.

Definition B.5. Three Lie groups $(\tilde{G}, G_+, G_-)$ form a *double Lie group* if $G_\pm$ are both closed Lie subgroups of G such that the map $G_+ \times G_- \to \tilde{G} : (g_+, g_-) \mapsto g_+ g_-$ is a diffeomorphism. They form a *local double Lie group* if there exist Lie subgroups $G'_\pm$ of $\tilde{G}$ such that G'_i is locally isomorphic to G_i for $i = +, -$ and the map $G'_+ \times G'_- \to \tilde{G} : (g'_+, g'_-) \mapsto g'_+ g'_-$ is a local diffeomorphism near the identities.

Theorem B.6. *Let G be a Poisson Lie group with dual group $\hat{G}$, then $\mathfrak{g}$ is naturally a Lie bialgebra. Let $\tilde{G}$ be the connected and simply connected Lie group with Lie algebra $\tilde{\mathfrak{g}} = \mathfrak{g} \oplus \hat{\mathfrak{g}}$ as given above, then $(\tilde{G}, G, \hat{G})$ form a local double Lie group.*

The local double Lie group $(\tilde{G}, G, \hat{G})$ in the theorem will be called *the local double group* of G. In general, if the Lie algebras of a (local) double Lie group $(\tilde{G}, G, \hat{G})$ coincide with the Manin triple defined by the Lie bialgebra $\mathfrak{g}$, then we say that $(\tilde{G}, G, \hat{G})$ is *a* (*local*) *double group* of G.

8.2. Let $r = \sum_i a_i \otimes b_i \in \mathfrak{g} \otimes \mathfrak{g}$, then it defines a cocommutator δ via

$$\delta : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g} : X \mapsto \mathrm{ad}_X r, \tag{B.2}$$

which is a 1-cocycle because it is in fact a 1-coboundary. We write $r = s + a$, where s (respectively a) is the symmetric (respectively antisymmetric) part of r, then δ as given in (B.2) defines a Lie bialgebra iff

(1) s is ad-invariant and
(2) $[\![r, r]\!]$ is ad-invariant, where

$$[\![r, r]\!] = \sum_{i,j} \left([a_i, a_j] \otimes b_i \otimes b_j + a_i \otimes [b_i, a_j] \otimes b_j + a_i \otimes a_j \otimes [b_i, b_j] \right). \tag{B.3}$$

Definition B.8. The Lie bialgebra defined by $r \in \mathfrak{g} \otimes \mathfrak{g}$ as above is called a *coboundary Lie bialgebra*. It is *factorizable* if $[\![r, r]\!] = 0$ and s is invertible. In this case, r is also called a *factorizable* r-*matrix*. A (local) double Lie group $(\tilde{G}, G, \hat{G})$ is called *factorizable* if the corresponding Lie bialgebra is factorizable. In this case, we will also call G a *factorizable* Poisson Lie group.

For an element $r \in \mathfrak{g} \otimes \mathfrak{g}$, let $\underline{r} : \hat{\mathfrak{g}} \to \mathfrak{g}$ be the map defined by $\underline{r}(\tau^*)(\omega^*) = (\tau^* \otimes \omega^*)(r)$. Suppose that r is factorizable and let $(\tilde{\mathfrak{g}}, \mathfrak{g}, \hat{\mathfrak{g}})$ be the associated Manin triple, then $\tilde{\mathfrak{g}} \simeq \mathfrak{g} \oplus \mathfrak{g}$ as Lie algebra. The isomorphism is given by $\mathfrak{g} \hookrightarrow \mathfrak{g} \oplus \mathfrak{g} : \tau \mapsto (\tau, \tau)$ and

$$\hat{\mathfrak{g}} \hookrightarrow \mathfrak{g} \oplus \mathfrak{g} : \hat{\omega} \mapsto (\underline{r_+}(\hat{\omega}), \underline{r_-}(\hat{\omega})) \quad \text{with } r_\pm = a \pm s. \tag{B.4}$$

B.9. It follows from a general fact for Poisson manifolds that $\pi_G([\theta^{\cdot}_{\hat{\tau}}, \theta^{\cdot}_{\hat{\omega}}]^*) = [\pi_G(\theta^{\cdot}_{\hat{\tau}}), \pi_G(\theta^{\cdot}_{\hat{\omega}})]$, in Convention 3.6. Thus the map

$$\rho^{\cdot} : \hat{\mathfrak{g}} \to \Gamma(TG) : \hat{\tau} \longmapsto \mathcal{X}^{\cdot}_{\hat{\tau}} = \pi(\theta^{\cdot}_{\hat{\tau}}) \tag{B.5}$$

is a Lie algebra homomorphism, where $\cdot$ stands for l or r.

Definition B.10. For each $\hat{\tau} \in \hat{\mathfrak{g}}$, the *left* (respectively *right*) *dressing* vector field on G is

$$\mathcal{X}^l_{\hat{\tau}} = \pi_G(\theta^l_{\hat{\tau}}) \text{ (resp. } \mathcal{X}^r_{\hat{\tau}} = -\pi_G(\theta^r_{\hat{\tau}})),$$

and $\theta^{\cdot}_{\hat{\tau}}$ is the left or right invariant 1-form on G determined by $\hat{\tau}$. Integrating $\mathcal{X}^{\cdot}_{\hat{\tau}}$ gives rise to a local (global if the dressing vector fields are complete) *left* (or *right*) *dressing action* of *the* dual group $\hat{G}$ on G, and we say that this left (or right) dressing action consists of *left* (or *right*) *dressing transformations*. The Poisson Lie group (G, π) is complete if each left (or equivalently, right) dressing vector field is complete. Analogously, we may define the corresponding concepts on $\hat{G}$.

The dressing actions as defined above are the same as those in [**24**, **25**]. Following [**24**]:

Definition B.11. A left action $\sigma_l : G \times P \to P$ of Poisson Lie group (G, π_G) on a Poisson manifold (P, π_P) is *Poisson* if σ_l is a Poisson map, where $G \times P$ is endowed with the product Poisson structure. Similarly a right action $\sigma_r : P \times G \to P$ is *Poisson* when σ_r is Poisson.

Definition B.12. A C^∞ map $\mu : P \to \hat{G}$ is called a *momentum mapping* for the left (respectively right) Poisson action $\sigma : G \times P \to P$ (respectively $\sigma_r : P \times G \to P$) if for each $\tau \in \mathfrak{g} = \hat{\mathfrak{g}}^*$, the infinitesimal action X^l_τ (respectively X^r_τ) of τ is given by

$$X^l_\tau = \pi_P(\mu^* \hat{\theta}^l_\tau) \text{ (resp. } X^r_\tau = -\pi_P(\mu^* \hat{\theta}^r_\tau)),$$

where $\hat{\theta}^{\cdot}_\tau$ is the left (or right) invariant 1-form on $\hat{G}$ determined by τ. The moment map μ of the Poisson action $\sigma_{\cdot}$ is G-equivariant if it is equivariant with respect to the left (or right) dressing action of G on $\hat{G}$.

In particular, when the moment map is equivariant, we have $\mu_*(X^{\cdot}_\tau) = \hat{\mathcal{X}}^{\cdot}_\tau$, where $\hat{\mathcal{X}}^{\cdot}_\tau$ is the dressing vector field on $\hat{G}$ defined by $\tau \in \mathfrak{g}$. Theorem 4.8 of [**24**] then states

Theorem B.13. *Let G be a connected complete Poisson Lie group. A momentum mapping $\mu : P \to \hat{G}$ for a Poisson action σ is G-equivariant iff μ is a Poisson map.*

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