

Correction: The Verlinde formulas as fixed point formulas

A. ALEKSEEV, E. MEINRENKEN, AND C. WOODWARD

Due to a publisher error, the correct version of sections 5.4, 5.5, and 5.6 was not printed. The correct version is included here,

5. Verlinde formula.

5.4. The fixed point sets.

The fixed point sets for the action on the holonomy manifold are symplectic tori:

Proposition 5.1. *The fixed point set for the action of $(t_{\lambda_1}, \dots, t_{\lambda_r})$ on $M(\Sigma_h^r) = G^{2(h+r-1)}$ is empty unless $\lambda_1 = \dots = \lambda_r =: \lambda$, and*

$$M(\Sigma_h^r)^{(t_{\lambda, \dots, t_{\lambda}})} = F := T^{2(h+r-1)}.$$

Proof. Since $M(\Sigma_h^r)$ is obtained from a direct product of $h+r-1$ copies of $M(\Sigma_0^2)$ by passing to diagonal actions for some of the G -factors, it suffices to prove Proposition 5.1 for Σ_0^2 . By (28), an element $(a, b) \in M(\Sigma_0^2)$ is fixed by $(t_{\lambda_1}, t_{\lambda_2})$ if and only if

$$(31) \quad t_{\lambda_1} = \text{Ad}_a t_{\lambda_2}, \quad t_{\lambda_2} = \text{Ad}_b t_{\lambda_1}.$$

Both t_{λ_1} and t_{λ_2} belong to the exponential of the alcove $\exp(\mathfrak{A})$. Since each conjugacy class meet $\exp(\mathfrak{A})$ only once, (31) holds if and only if $\lambda_1 = \lambda_2$. \square

Notice that the fixed point set is independent of λ ; in fact, F is fixed by the full diagonal torus $T \subset G^r$.

5.5. Evaluation of the fixed point contributions.

Let $\Sigma = \Sigma_h^r$ and $\mu = (\mu_1, \dots, \mu_r)$ with $\mu_j \in \Lambda_k^*$. By Theorem 4.3, the Spin_c -index is given by the formula

$$(32) \quad \chi(\mathcal{M}(\Sigma)_\mu) = \frac{1}{(\#T_{k+c})^r} \sum_{\lambda \in \Lambda_k^*} \chi_{\mu_1}(t_\lambda) \cdots \chi_{\mu_r}(t_\lambda) |J(t_\lambda)|^{2r} \chi(\nu_F, t_\lambda)^*.$$

Here we abbreviated $(t_\lambda, \dots, t_\lambda)$ to t_λ , viewing T as diagonally embedded into G^r . For the calculation of the fixed point contribution

$$\chi(\nu_F, t_\lambda) = \zeta_F(t_\lambda)^{1/2} \int_F \frac{\hat{A}(F) e^{\frac{1}{2}c_1(\mathcal{L}_F)}}{\mathcal{D}_{\mathbb{R}}(\nu_F, t_\lambda)},$$

note first of all that

$$(33) \quad \hat{A}(F) = 1$$

since F is a torus. Furthermore, since the normal bundle ν_F is T -equivariantly isomorphic to $(\mathfrak{g}/\mathfrak{t})^{2(h+r-1)}$, we have

$$(34) \quad \mathcal{D}_{\mathbb{R}}(\nu_F, t_\lambda) = J(t_\lambda)^{2(h+r-1)} = (-1)^{(h+r-1)\#\mathfrak{A}} |J(t_\lambda)|^{2(h+r-1)}.$$

It remains to work out the integral $\int_F \exp(\frac{1}{2}c_1(\mathcal{L}_F))$ and to calculate the phase factor $\zeta_F(t_\lambda)^{1/2}$.

Proposition 5.2. *The integral of $\exp(\frac{1}{2}c_1(\mathcal{L}_F))$ over F equals $(\#T_{k+c})^{h+r-1}$.*

Proof. The line bundle $\mathcal{L} = L(\Sigma)^2 \otimes K^{-1}$ is \widehat{LG}^r -equivariant at levels $2(k+c), \dots, 2(k+c)$. Since $\mathcal{M}(\Sigma)$ carries up to isomorphism a unique line bundle at every level [29, 3.12], it follows that \mathcal{L} is the pre-quantum line bundle for the symplectic structure defined by $B_{2(k+c)}$. Hence \mathcal{L}_F is a pre-quantum line bundle for the corresponding symplectic structure on F , and $\int_F \exp(\frac{1}{2}c_1(\mathcal{L}_F))$ is the symplectic volume $\text{Vol}_{B_{k+c}}(F)$ for the 2-form defined using B_{k+c} . We claim that the symplectic volume coincides with the Riemannian volume, which will complete the proof since $\text{Vol}_{B_{k+c}}(T^2) = \#T_{k+c}$ (see e.g., Beauville [8, Remark 9.9], Bismut-Labourie [10, Prop. 1.2, 1.3]). By our description of $M(\Sigma_h^r)$ as a fusion product, the fixed point manifold $F = F(\Sigma_h^r)$ is obtained from the fixed point manifold $F(\Sigma_0^2)$ (viewed as a group valued Hamiltonian T^2 -space) by fusion: $F(\Sigma_1^1) = F(\Sigma_0^2)_{\text{fus}}$ and

$$F(\Sigma_h^r) = F(\Sigma_1^1) \otimes \cdots \otimes F(\Sigma_1^1) \otimes F(\Sigma_0^2) \otimes \cdots \otimes F(\Sigma_0^2),$$

with h factors $F(\Sigma_1^1)$ and $(r - 1)$ factors $F(\Sigma_0^2)$. Lemma C.2 from Appendix C says that the symplectic volume of group valued Hamiltonian torus spaces does not change under fusion. Hence $\text{Vol}_{B_{k+c}}(F(\Sigma_h^r)) = \text{Vol}_{B_{k+c}}(F(\Sigma_0^2))^{h+r-1}$. Finally, the expression (30) for the 2-form on $M(\Sigma_0^2)$ shows that $\text{Vol}_{B_{k+c}}(F(\Sigma_0^2))$ coincides with the Riemannian volume of T^2 with respect to B_{k+c} . \square

Proposition 5.3. *The phase factor is given by $\zeta_F(t_\lambda)^{1/2} = (-1)^{(h+r-1)\#\mathfrak{R}_+}$.*

Proof. The point $m = (e, \dots, e) \in F$ lies in identity level set of Φ , and its stabilizer in G^r is the image of the diagonal embedding of G . The 2-form ω restricts to a symplectic form on the tangent space $E = T_m M(\Sigma)$. The factor $\zeta_F(t_\lambda)^{1/2} \in U(1)$ may be computed using (21).

We first apply this recipe for the 2-holed sphere Σ_0^2 , so that $E = T_m M(\Sigma_0^2) = \mathfrak{g} \oplus \mathfrak{g}$. Formula (30) shows that ω_m is the standard 2-form on $\mathfrak{g} \oplus \mathfrak{g}$, given by the inner product B . A compatible complex structure is given by the endomorphism $(\xi, \eta) \mapsto (-\eta, \xi)$. Thus, as a complex G -representation E is just the complexification $E = \mathfrak{g}^{\mathbb{C}}$. It follows that the eigenvalues of A (other than 1) come in complex conjugate pairs

$$e^{i\phi_j}, e^{-i\phi_j}, \quad 0 < \phi_j \leq \pi/2,$$

and the corresponding eigenvalues of $A^{1/2}$ are $e^{i\phi_j/2}$ and $e^{i\pi-i\phi_j/2} = -e^{-i\phi_j/2}$. Hence

$$\zeta_F(t_\lambda)^{1/2} = (-1)^{\#\mathfrak{R}_+}.$$

Now consider the case $r \geq 1, h$ arbitrary. The tangent space is $T_m M(\Sigma_h^r) = (\mathfrak{g} \oplus \mathfrak{g})^{h+r-1}$, but because of the fusion terms the symplectic form is not the standard symplectic form defined by the inner product on \mathfrak{g} . However, by Appendix C, Lemma C.3 it is equivariantly and symplectically *isotopic* to the standard symplectic form. Since the phase factor $\zeta_F(t_\lambda)^{1/2}$ is a root of unity, it is invariant under equivariant symplectic isotopies, and we conclude as before that $\zeta_F(t_\lambda)^{1/2} = (-1)^{(h+r-1)\#\mathfrak{R}_+}$. \square

We conclude that the fixed point integral is given by

$$(35) \quad \chi(\nu_F, t_\lambda) = \left(\frac{\#T_{k+c}}{|J(t_\lambda)|^2} \right)^{h+r-1}$$

5.6. Verlinde formula.

From (35) and (32) we obtain:

Theorem 5.4 (Verlinde Formula). *Let G be a simply connected Lie group and k a given integral level. The Spin_c -index of the moduli space of flat connections on Σ_h^r at level k , with markings $\mu = (\mu_1, \dots, \mu_r) \in (\Lambda_k^*)^r$ is given by the formula*

$$(36) \quad \chi(\mathcal{M}(\Sigma_h^r)_\mu) = (\#T_{k+c})^{h-1} \sum_{\lambda \in \Lambda_k^*} |J(t_\lambda)|^{2-2h} \chi_{\mu_1}(t_\lambda) \cdots \chi_{\mu_r}(t_\lambda).$$

Remarks 5.5. (a) Theorem (5.4) also covers the case without boundary, since $M(\Sigma_h^0) = M(\Sigma_h^1, 0)$. One obtains

$$\chi(\mathcal{M}(\Sigma_h^0)) = (\#T_{k+c})^{h-1} \sum_{\lambda \in \Lambda_k^*} |J(t_\lambda)|^{2-2h}.$$

- (b) For the two-holed sphere Σ_0^2 , formula (36) simplifies by the orthogonality relations for level k characters, and gives $\chi(M(\Sigma_0^2)_{\mu_1, \mu_2}) = \delta_{\mu_1, * \mu_2}$.
- (c) In Bismut-Labourie [10] the Spin_c -indices $\chi(\mathcal{M}(\Sigma_h^r)_\mu)$ are computed by direct application of the Kawasaki-Riemann-Roch formula to the reduced spaces. Their approach involves a description of all orbifold strata of the reduced space. The equality with the above sum over level k weights is non-trivial; it is established in [10] for sufficiently high level k .
- (d) Theorem 5.4 gives a formula for a Spin_c -index rather than the dimension of a space of holomorphic sections. Vanishing results for higher cohomology groups have recently been proved by Teleman [34, Section 8].

A more involved computation leads to a formula for semi-simple, compact, connected groups which need not be simply connected, for the case of a single boundary component $r = 1$. An interesting feature of our approach is that we obtain a formula for each connected component of the moduli space of flat bundles. We will state the result without proof. Let $G' = G/\Gamma$, where G is simply connected and $\Gamma \subset Z(G)$ is a subgroup of the center $Z(G)$ of G . For $j = 1, \dots, s$, let $\Gamma_j \subset G_j$ be the image of Γ under projection to the j th simple factor. Let

$$M'(\Sigma_h^1) = M(\Sigma_h^1)/\Gamma^{2h} = G^{2h}/\Gamma^{2h} = (G')^{2h}.$$

The product of commutators, $\Phi : M(\Sigma_h^1) = G^{2h} \rightarrow G$ is invariant under the action of Γ^{2h} and descends to the G -valued moment map $\Phi' : M'(\Sigma_h^1) \rightarrow G$. For any $g' \in G'$, the moduli space of flat G' -connections with holonomy conjugate to g' is a disjoint union of symplectic quotients $M'(\Sigma)_g$ where g varies over all pre-images of g' in G . The reduced spaces at central elements $\gamma \in \Gamma \subset G$ may also be interpreted as moduli spaces of flat connections on the G' -bundle over Σ_h^0 , with topological type given by γ . Let $k = (k_1, \dots, k_s)$, where each k_j is a positive multiple of the greatest common divisor of c_j and $\#\Gamma_j^2$. Then there exists a pre-quantum line bundle at level k , determined by a choice of character $\phi \in \text{Hom}(\Gamma^{2h}, \text{U}(1))$, and for any $\mu \in \Lambda_k^*$, the Spin_c -index of the symplectic quotient $\mathcal{M}'(\Sigma)_\mu$ at μ is given by the formula,

$$(37') \quad \chi(\mathcal{M}'(\Sigma)_\mu) = \frac{(\#T_{k+c})^{h-1}}{\#\Gamma^{2h}} \sum_{\lambda \in \Lambda_k^*} \epsilon(\phi, \lambda) \frac{\#\Gamma_\lambda^{2h}}{|J(t_\lambda)|^{2h-2}} \chi_\mu(t_\lambda)^*.$$

Here $\epsilon(\phi, \lambda) = 1$ if ϕ restricts to the trivial homomorphism on Γ_λ^{2h} , and 0 otherwise. The sum over components leads to a simpler formula, which involves only a sum over weights corresponding to characters of G' . In the case $G' = \text{PSU}(p)$ (5.6) reduces to formulas of Pantev [2'] ($p = 2$), and Beauville [1'] ($p \geq 3$ prime).

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UNIVERSITY OF GENEVA
SECTION OF MATHEMATICS
2-4 RUE DU LIÈVRE
1211 GENÈVE 24
SWITZERLAND
E-mail address: `alekseev@math.unige.ch`

UNIVERSITY OF TORONTO
DEPARTMENT OF MATHEMATICS
100 ST GEORGE STREET
TORONTO, ONTARIO M5S3G3
CANADA
E-mail address: `mein@math.toronto.edu`

MATHEMATICS-HILL CENTER
RUTGERS UNIVERSITY

110 FRELINGHUYSEN ROAD
PISCATAWAY NJ 08854-8019
E-mail address: `ctw@math.rutgers.edu`