COMPLETE INVARIANTS FOR HAMILTONIAN TORUS ACTIONS WITH TWO DIMENSIONAL QUOTIENTS

YAEL KARSHON AND SUSAN TOLMAN

We study torus actions on symplectic manifolds with proper moment maps in the case that each reduced space is two-dimensional. We provide a complete set of invariants for such spaces.

CONTENTS

1. Introduction

Let a torus $T \cong (S^1)^{\dim T}$ act on a symplectic manifold (M, ω) by symplectic transformations with moment map $\Phi \colon M \longrightarrow \mathfrak{t}^*$. We take the sign convention

$$(1.1) \qquad \iota(\xi_M)\omega = -d \langle \Phi, \xi \rangle .$$

Here, ξ is in the Lie algebra $\mathfrak{t}$ of T and ξ_M is the vector field on M induced by ξ. Assume that the T-action is effective[1] on each connected component of M. We call (M, ω, Φ) a **Hamiltonian T-manifold**.[2] If $\mathcal{T} \subseteq \mathfrak{t}^*$ is an open subset containing image Φ and the map $\Phi \colon M \longrightarrow \mathcal{T}$ is proper, then we call $(M, \omega, \Phi, \mathcal{T})$ a **proper Hamiltonian T-manifold**.

The **complexity** of (M, ω, Φ) is the difference $k = \frac{1}{2} \dim M - \dim T$; it is half the dimension of the reduced space $\Phi^{-1}(\alpha)/T$ at a regular value α in image Φ. For brevity, we call a complexity one proper Hamiltonian T-manifold $(M, \omega, \Phi, \mathcal{T})$ a **complexity one space** if M is connected and $\mathcal{T}$ is convex. A complexity one Hamiltonian T-manifold is **tall** if every reduced space is two dimensional, that is, if no reduced space is a single point.

The simplest example of a complexity one space is a compact symplectic surface (Σ, σ) with no group action. The next simplest example is the fiberwise circle action on a ruled surface. More generally, let (M, ω, Φ) be a *symplectic toric manifold*, that is, a compact complexity zero Hamiltonian T-manifold. Let $P \longrightarrow \Sigma$ be a principal T-bundle and let $\Theta \in \Omega^1(P, \mathfrak{t})$ be a connection one form. For sufficiently large

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[1]A group action is *effective* if only the identity element acts trivially.

[2]One sometimes allows ω to be degenerate. Here we do not allow this.

k, the form $\tilde{\omega} = k\sigma + \omega + d\langle\Theta, \Phi\rangle \in \Omega^2(P \times M)$ descends to a symplectic form on $P \times_T M$ with moment map $\tilde{\Phi}([p,m]) = \Phi(m)$. Then $(P \times_T M, \tilde{\omega}, \tilde{\Phi})$ is a tall complexity one space. For example, $\Sigma \times M$ is a tall complexity one space. Finally, given a tall complexity one space, its equivariant symplectic blow-up at any fixed point is also a tall complexity one space.

Symplectic toric manifolds are classified by their moment map images [**De**]. By Moser [**Mo**], compact symplectic surfaces are classified by their genus and total area. The next examples of complexity one spaces are compact symplectic four manifolds with Hamiltonian circle actions, which were classified by the first author [**K**], following earlier work by Ahara, Hattori, and Audin [**AH**, **Au1**, **Au2**]. In the algebraic category, complexity one actions (of possibly non-abelian groups) were classified by Timashëv [**T1**, **T2**]. Chiang [**C**] classified complexity one Hamiltonian actions of non-abelian groups on six-manifolds. Li [**L**] has obtained some classification results for certain Hamiltonian circle actions on six manifolds. However, a complete classification of complexity two Hamiltonian torus actions would entail a classification of four dimensional symplectic manifolds, which is not tractable. See [**KT**] for a more extensive list of related works.

Complexity one spaces are substantially more complicated than symplectic toric manifolds. Symplectic toric manifolds provide a useful source of examples and counter-examples. We hope that complexity one spaces will prove similarly useful, and that their greater complexity $\smile$ will enable them to demonstrate phenomena that do not occur on symplectic toric manifolds. For example, all symplectic toric manifolds are Kähler. However, there exist complexity one spaces with isolated fixed points and no invariant Kähler structure [**T**].

This paper is the second in a series of papers in which we study complexity one spaces. Our goal is to classify these spaces. This consists of two parts. First, uniqueness: we must determine whether or not two given spaces are equivariantly symplectomorphic. Second, existence: we must provide a list of all complexity one spaces.

In [**KT**] we obtained a *local uniqueness* result: we determined when two complexity one spaces are equivariantly symplectomorphic over small subsets of $\mathfrak{t}^*$.

In this paper we obtain a *global uniqueness* result: we provide invariants which determine when two tall complexity one spaces are equivariantly symplectomorphic.

In our next paper of this series we will obtain *global existence* results. This will enable us to construct examples.

While the complexity one assumption is absolutely vital to our results, the tall assumption is not. In fact, let $(M, \omega, \Phi, \mathcal{T})$ and $(M', \omega', \Phi', \mathcal{T})$ be any two complexity one spaces. By removing every moment map fiber which consists of only one orbit, we obtain tall complexity one spaces over an open subset of $\mathcal{T}$. We expect that the original manifolds will be equivariantly symplectomorphic exactly if these tall complexity one spaces are equivariantly symplectomorphic. However, the proof will require an additional ingredient: a variant of Smale's theorem on the diffeomorphisms of S^2.

Whereas many complexity one spaces that one encounters in nature are not tall, the "tall" case is sufficient for constructing interesting examples.

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2. Statement and proof of "Global Uniqueness"

We now describe the invariants of a tall complexity one space $(M, \omega, \Phi, \mathcal{T})$.

An **isomorphism** between two Hamiltonian T-manifolds is an equivariant symplectomorphism that respects the moment maps.

Recall that Liouville measure on M is given by integrating the volume form $\omega^n/n!$ with respect to the symplectic orientation. The Duistermaat–Heckman measure is the measure on $\mathcal{T}$ obtained as the pushforward of Liouville measure by the moment map.

The **stabilizer** of a point $x \in M$ is the closed subgroup $\mathrm{Stab}(x) = \{\lambda \in T \mid \lambda \cdot x = x\}$. The isotropy representation at x is the linear representation of $\mathrm{Stab}(x)$ on the tangent space $T_x M$. Points in the same orbit have the same stabilizer, and their isotropy representations are linearly symplectically isomorphic; this isomorphism class is the **isotropy representation** of the orbit. An orbit is **exceptional** if every nearby orbit in the same moment fiber $\Phi^{-1}(\alpha)$ has a strictly smaller stabilizer. Let

$$M_{\mathrm{exc}} \subset M/T$$

denote the set of exceptional orbits. The moment map induces a map $\overline{\Phi} \colon M_{\mathrm{exc}} \longrightarrow \mathcal{T}$ which is locally a proper embedding; this follows from the local normal form theorem.

Let M'_{exc} denote the set of exceptional orbits of another tall complexity one space. An **isomorphism** from M_{exc} to M'_{exc} is a homeomorphism that respects the moment maps and sends each orbit to an orbit with the same isotropy representations.

REMARK 2.1. Assume, for simplicity, that M is compact. The orbit type decomposition of M induces a stratification of M_{exc}. Given any stratum, N, the pre-image in M of its closure $\overline{N}$ is a compact toric variety (with the action of $T/\operatorname{Stab}(N)$). The moment map induces a diffeomorphism $\overline{\Phi}$ between $\overline{N}$ and the convex polytope $\overline{\Phi}(\overline{N}) \subseteq \mathfrak{t}^*$. If a stratum N' is contained in $\overline{N}$ then $\overline{\Phi}(\overline{N'})$ is a face of $\overline{\Phi}(\overline{N})$. So, topologically, M_{exc} is a union of convex polytopes in $\mathfrak{t}^*$, glued along faces, and $\overline{\Phi}\colon M_{\text{exc}} \longrightarrow \mathfrak{t}^*$ restricts to the inclusion map on each polytope. This partially ordered collection of polytopes is essentially equivalent to the notion of an X-ray as defined in [**T**].

To define our other invariants, we need the following result.

Proposition 2.2. *Let $(M, \omega, \Phi, \mathcal{T})$ be a tall complexity one space. There exists a closed oriented surface Σ and a map $f\colon M/T \longrightarrow \Sigma$ so that*

$$(\overline{\Phi}, f)\colon M/T \longrightarrow (\operatorname{image} \Phi) \times \Sigma$$

is a homeomorphism, where $\overline{\Phi}$ is induced by the moment map. Given two such maps f and f', there exists a homeomorphism $\xi\colon \Sigma' \longrightarrow \Sigma$ so that f is homotopic to $\xi \circ f'$ through maps which induce homeomorphisms $M/T \longrightarrow (\operatorname{image} \Phi) \times \Sigma$.

The **genus** of the complexity one space is the genus of Σ. Note that every nonempty reduced space $\Phi^{-1}(\alpha)/T$ is homeomorphic to Σ.

A **painting** is a map $f\colon M_{\text{exc}} \longrightarrow \Sigma$ such that the map

$$(\overline{\Phi}, f)\colon M_{\text{exc}} \longrightarrow \mathcal{T} \times \Sigma$$

is one to one, where M_{exc} is the set of exceptional orbits. Two paintings, $f\colon M_{\text{exc}} \longrightarrow \Sigma$ and $f'\colon M'_{\text{exc}} \longrightarrow \Sigma'$, are **equivalent** if there exist an isomorphism $i\colon M_{\text{exc}} \longrightarrow M'_{\text{exc}}$ and a homeomorphism $\xi\colon \Sigma \longrightarrow \Sigma'$ such that the compositions $\xi \circ f\colon M_{\text{exc}} \longrightarrow \Sigma'$ and $f' \circ i\colon M_{\text{exc}} \longrightarrow \Sigma'$ are homotopic *through paintings*.

Proposition 2.2 implies that there is a well-defined equivalence class of paintings associated to every tall complexity one space; just restrict f to M_{exc}.

We can now state our main theorem:

Theorem 1. *Let $(M, \omega, \Phi, \mathcal{T})$ and $(M', \omega', \Phi', \mathcal{T})$ be tall complexity one spaces. They are isomorphic if and only if they have the same Duistermaat-Heckman measure, the same genus, and equivalent paintings.*

REMARK 2.3. A compact symplectic 4-manifold M equipped with a Hamiltonian circle action is tall if and only if the maximum and minimum of the moment map are both attained on two dimensional surfaces, $\Sigma_{\max}$ and $\Sigma_{\min}$. Equivalently, it is tall exactly if it can be obtained from a ruled surface by a sequence of blowups [**K**].

By [**K**], the space is determined up to equivariant symplectomorphism by the moment map values, genus, and symplectic areas of these surfaces, together with the graph whose vertices correspond to isolated fixed points in M and are labeled by their moment map values and whose edges correspond to 2-spheres in M with non-trivial finite stabilizers and are labeled by the cardinalities of these stabilizers.

These invariants are equivalent to those of Theorem 1. First, by Proposition 2.2, the genus of M is the genus of $\Sigma_{\min}$ and $\Sigma_{\max}$. Second, M and M' have equivalent paintings exactly if M_{exc} is isomorphic to M'_{exc}; this follows from the fact that M_{exc} is a union of intervals. By the Guillemin-Lerman-Sternberg formula [**GLS**, Section 3.5], the graph determines the Duistermaat-Heckman measure up to an affine function; this function is determined by the moment map values and symplectic areas of $\Sigma_{\min}$ and $\Sigma_{\max}$. These, in turn, are determined by the Duistermaat-Heckman measure.

Proof of Proposition 2.2. By the convexity theorem for proper moment maps, the image of Φ is convex [**LMTW**]. In [**KT**, Corollary 9.8], we proved that $\overline{\Phi} \colon M/T \longrightarrow \operatorname{image} \Phi$ is topologically a locally trivial bundle whose fiber is a closed oriented surface Σ. Since the base is contractible and paracompact, the bundle is trivializable; see [**Hu**].

Any two trivializations, $(\overline{\Phi}, f)$ and $(\overline{\Phi}, f')$, differ by a family of homeomorphisms $\xi_\beta \colon \Sigma' \longrightarrow \Sigma$, parametrized by $\beta \in \operatorname{image} \Phi$, that is determined by $f(x) = \xi_{\overline{\Phi}(x)}(f'(x))$ for all x. Pick any $\alpha \in \operatorname{image} \Phi$. Let $f_t(x) = \xi_{(1-t)\overline{\Phi}(x)+t\alpha}(f'(x))$. Then $f_0 = f$, and $f_1 = \xi_\alpha \circ f'$. $\qquad\square$

We now prove our main theorem, using definitions and results that we develop later in the paper.

Proof of Theorem 1. By Proposition 3.3, it is enough to show that the quotients M/T and M'/T are Φ-diffeomorphic. (See Definition 3.2.)

Now apply Proposition 17.5. Let N and N' be painted surface bundles which are associated to M and M'. The quotients M/T and M'/T are Φ-diffeomorphic if and only if N and N' are isomorphic. (See the definitions in Sections 16 and 17.)

We can associate to N and N' *smooth* paintings $f\colon M_{\mathrm{exc}} \longrightarrow \Sigma$ and $f'\colon M'_{\mathrm{exc}} \longrightarrow \Sigma$, and these paintings are equivalent. Here we use Proposition 18.3 and our assumption that M and M' have equivalent paintings.

This implies that f and f' are *smoothly* equivalent, by Proposition 19.1. Then, N and N' are isomorphic, by Proposition 18.3. $\qquad\square$

REMARK 2.4. Fix a local homeomorphism $i\colon \mathcal{T} \longrightarrow \mathfrak{t}^*$. Consider a symplectic manifold (M, ω) with a proper map $\Phi\colon M \longrightarrow \mathcal{T}$ such that $i \circ \Phi$ is a moment map for a T action. Suppose that $\Delta := \Phi(M)$ is contractible and the fibers $\Phi^{-1}(\alpha)$ are connected and two dimensional. Then M/T is homeomorphic to $\Delta \times \Sigma$ as in Proposition 2.2, and Theorem 1 should remain true.

Part I: Passing to M/T

3. global structure of M/T.

The goal of this paper is to determine when two complexity one spaces are isomorphic. In this section we reduce this question to a simpler question about their quotients, M/T and M'/T. To state this precisely, we introduce some definitions from [**KT**].

Definition 3.1. Let a torus T act on oriented manifolds M and M' with T-invariant maps $\Phi\colon M \longrightarrow \mathfrak{t}^*$ and $\Phi'\colon M' \longrightarrow \mathfrak{t}^*$. A **$\Phi$-$T$-diffeomorphism** from (M, Φ) to (M, Φ') is an orientation preserving equivariant diffeomorphism $f\colon M \longrightarrow M'$ that satisfies $\Phi' \circ f = \Phi$. (Cf., [**KT**, Definition 3.1].)

Let a compact torus T act on a manifold N. The quotient N/T can be given the quotient topology and a natural differential structure, consisting of the sheaf of real-valued functions whose pullbacks to N are smooth. We say that a map $h\colon N/T \longrightarrow N'/T$ is **smooth** if it pulls back smooth functions to smooth functions; it is a **diffeomorphism** if it is smooth and has a smooth inverse. See [**Sch**]. If N and N' are oriented, the choice of an orientation on T determines orientations on the smooth part of N/T and N'/T. Whether or not a diffeomorphism $f\colon N/T \longrightarrow N'/T$ preserves orientation is independent of this choice.

Definition 3.2. Let $(M, \omega, \Phi, \mathcal{T})$ and $(M', \omega', \Phi', \mathcal{T})$ be complexity one Hamiltonian T-manifolds. A **Φ-diffeomorphism** from M/T to M'/T is an orientation preserving diffeomorphism $f\colon M/T \longrightarrow M'/T$ such that $\overline{\Phi}' \circ f = \overline{\Phi}$ and such that f and f^{-1} lift to Φ-T-diffeomorphisms in a neighborhood of each exceptional orbit. (Cf., [**KT**, Definition 4.1]) Here, $\overline{\Phi}$ and $\overline{\Phi}'$ are induced by the moment maps.

Proposition 3.3. *Two tall complexity one spaces are isomorphic if and only if their quotients are Φ-diffeomorphic and their Duistermaat-Heckman measures are the same.*

The conditions are clearly necessary. Proposition 2.2 implies that the restriction $H^2(M/T, \mathbb{Z}) \longrightarrow H^2(\Phi^{-1}(y)/T, \mathbb{Z})$ is one to one for every $y \in \text{image}\,\Phi$. Proposition 3.3 then follows from the results below, which we proved in [**KT**, Propositions 3.3 and 4.2]:

Proposition 3.4. *Let $(M, \omega, \Phi, \mathcal{T})$ and $(M', \omega', \Phi', \mathcal{T})$ be complexity one spaces with the same Duistermaat-Heckman measure. Assume that the restriction map $H^2(M/T, \mathbb{Z}) \longrightarrow H^2(\Phi^{-1}(y)/T, \mathbb{Z})$ is one to one for some regular value y of Φ. Then*

1) *There exists an isomorphism from M to M' if and only if there exists a Φ-T-diffeomorphism from M to M'.*
2) *There exists an Φ-T-diffeomorphism from M to M' if and only if there exists a Φ–diffeomorphism from M/T to M'/T.*

Part II: Abstract non-sense

In [**KT**], we gave invariants that determine the local pieces of a complexity one space. In this paper we explain how these pieces can be glued together.

This is analogous to classifying principle G-bundles over a manifold $\mathcal{T}$: locally they are trivial, and to determine them globally, one needs to determine how the local pieces are glued together. If G is abelian, this is very easy: the Čech cohomology, $\mathrm{H}^i(\mathcal{T}, G)$, is well defined for all $i \geq 0$, and G-bundles are classified by $\mathrm{H}^1(\mathcal{T}, G)$. Here, G also denotes the sheaf of smooth functions to the group G. When G is not abelian, the ith Čech cohomology is only defined for $i = 0$ and $i = 1$, and G-bundles are still classified by $\mathrm{H}^1(\mathcal{T}, G)$.

A proper Hamiltonian T-manifold $(M, \omega, \Phi, \mathcal{T})$ determines a sheaf of non-abelian groups over $\mathcal{T}$: associate to an open subset $U \subseteq \mathcal{T}$ the group of isomorphisms of the preimage $\Phi^{-1}(U)$. The first cohomology

of this sheaf classifies Hamiltonian T-manifolds that are locally isomorphic to $(M, \omega, \Phi, \mathcal{T})$, where "locally" means over small subsets of $\mathcal{T}$.

We prefer, instead, to allow *different* Hamiltonian T-manifolds over each U, so that the isomorphisms between them form a *groupoid*. Besides being more elegant, in that it does not single out one manifold above others, this machinery lets us glue pieces of manifolds without a-priori assuming that this can be done. We use this in Section 20, where we prove a technical result that will allow us, in subsequent papers, to determine the full list of complexity one spaces ("global existence"). In Sections 1–19, the reader may still choose to fix a distinguished space and work with groups instead of groupoids.

We define sheaves of groupoids and their cohomology in Sections 4 and 5. This straightforward extension of sheaves of abelian groups and Čech cohomology sets up a convenient formalism. These ideas are not new; closely related notions appear in the literature. See, for example, [**Br**, Chapter V].

In Sections 6–15 we apply this formalism to a series of sheaves, and show that they all have the same first cohomology. This reduces the classification of tall complexity one spaces to a classification of simpler objects, "painted surface bundles", which we classify directly in Sections 16–19.

4. Sheaves of groupoids

A **groupoid** is a category $\mathcal{A}$ where every arrow is invertible. We let $\mathrm{ob}\,\mathcal{A}$ denote the set of objects of $\mathcal{A}$. Given objects A and A', let $\hom_{\mathcal{A}}(A, A')$ denote the set of arrows with domain A and codomain A', and let $f\colon A \longrightarrow A'$ denote an element of $\hom_{\mathcal{A}}(A, A')$. A **homomorphism** of groupoids is a functor. Equivalently, given groupoids $\mathcal{A}$ and $\mathcal{B}$, a homomorphism $\psi\colon \mathcal{A} \longrightarrow \mathcal{B}$ consists of a map $\psi\colon \mathrm{ob}\,\mathcal{A} \longrightarrow \mathrm{ob}\,\mathcal{B}$ and for each A and A' in $\mathrm{ob}\,\mathcal{A}$ a map

$$\psi\colon \hom_{\mathcal{A}}(A, A') \longrightarrow \hom_{\mathcal{B}}(\psi(A), \psi(A'))$$

so that $\psi(\mathrm{id}_A) = \mathrm{id}_{\psi(A)}$ and $\psi(f) \circ \psi(f') = \psi(f \circ f')$.

Definition 4.1. A **presheaf of groupoids** over a topological space $\mathcal{T}$ assigns

1) a groupoid $\mathcal{A}(U)$ to every open subset $U \subseteq \mathcal{T}$, and
2) a homomorphism $\mathcal{A}(\iota_U^V)\colon \mathcal{A}(U) \longrightarrow \mathcal{A}(V)$, called the **restriction map**, to every inclusion of open sets $V \subseteq U$.

These must satisfy
1) $\mathcal{A}(\iota_U^U) = \mathrm{id}_{\mathcal{A}(U)}$ for any open set U.
2) $\mathcal{A}(\iota_V^W) \circ \mathcal{A}(\iota_U^V) = \mathcal{A}(\iota_U^W)$ for any inclusions of open sets $W \subseteq V \subseteq U$.

For an object A and an arrow f in $\mathcal{A}(U)$, let $A|_V$ and $f|_V$ denote $\mathcal{A}(\iota_U^V)(A)$ and $\mathcal{A}(\iota_U^V)(f)$, respectively. Objects in $\mathcal{A}(\mathcal{T})$ are called **global objects**.

EXAMPLE 4.2. Given a Lie group G, define a presheaf as follows: the objects over $U \subset \mathcal{T}$ are principle G bundles over U, and the arrows are bundle isomorphisms. Here, and in all other examples in this paper, the restriction maps are given by restriction.

EXAMPLE 4.3. Let T be a torus and $\mathcal{T} \subseteq \mathfrak{t}^*$ an open subset. We may consider the following two presheaves. In both, the objects over $U \subseteq \mathcal{T}$ are complexity one Hamiltonian T-manifolds with proper moment maps to U. The arrows in the first presheaf are equivariant symplectomorphisms which respect the moment maps. The arrows in the second presheaf are Φ-diffeomorphisms between the quotient spaces M/T.

A sheaf is a presheaf where the arrows are determined by local data:

Definition 4.4. A **sheaf** over $\mathcal{T}$ is a presheaf $\mathcal{A}$ which satisfies the following two **sheaf axioms**. Let $\{W_\alpha\}$ be a collection of open subsets of $\mathcal{T}$. Let A and A' be objects in $\mathcal{A}(\cup W_\alpha)$.
1) If $f \colon A \longrightarrow A'$ and $g \colon A \longrightarrow A'$ are arrows such that $f|_{W_\alpha} = g|_{W_\alpha}$ for all α, then $f = g$.
2) Given a collection of arrows $f_\alpha \colon A|_{W_\alpha} \longrightarrow A'|_{W_\alpha}$ which are compatible on intersections, there exists an arrow $f \colon A \longrightarrow A'$ such that $f|_{W_\alpha} = f_\alpha$ for all α.

The second sheaf axiom says that arrows can be "glued". A sheaf $\mathcal{A}$ has **gluable objects** if for every collection $\{W_\alpha\}$ of open subsets of $\mathcal{T}$, objects $A_\alpha \in \mathcal{A}(W_\alpha)$, and transition maps $f_{\beta\alpha} \colon A_\alpha|_{W_\alpha \cap W_\beta} \longrightarrow A_\beta|_{W_\alpha \cap W_\beta}$, such that $f_{\alpha\alpha} = \mathrm{id}_{A_\alpha}$ and $f_{\gamma\beta} \circ f_{\beta\alpha} = f_{\gamma\alpha}$ on $W_\alpha \cap W_\beta \cap W_\gamma$, there exists an object $A \in \mathcal{A}(\cup W_\alpha)$ and isomorphisms $\theta_\alpha \colon A|_{W_\alpha} \longrightarrow A_\alpha$ for all α so that $\theta_\beta \circ \theta_\alpha^{-1} = f_{\beta\alpha}$ on $W_\alpha \cap W_\beta$.

The presheaves in Examples 4.2 and 4.3 are sheaves. The first sheaf in Example 4.3 has gluable objects; the second sheaf does not.

Definition 4.5. Let $\mathcal{A}$ and $\mathcal{B}$ be presheaves of groupoids over $\mathcal{T}$. A **map of presheaves** $\mathcal{F} \colon \mathcal{A} \longrightarrow \mathcal{B}$ assigns to every open subset $U \subseteq \mathcal{T}$

a homomorphism $\mathcal{F}(U)\colon \mathcal{A}(U) \longrightarrow \mathcal{B}(U)$ so that for every inclusion of open sets $V \subseteq U$ we have $\mathcal{B}(\iota_U^V) \circ \mathcal{F}(U) = \mathcal{F}(V) \circ \mathcal{A}(\iota_U^V)$.

EXAMPLE 4.6. Any equivariant symplectomorphism $M \longrightarrow M'$ descends to a Φ-diffeomorphisms $M/T \longrightarrow M'/T$. This gives a natural map from the first sheaf of Example 4.3 to the second sheaf of that example.

5. Čech cohomology for sheaves of groupoids

In this section we define the first cohomology of a sheaf of groupoids and give a geometric interpretation.

Definition 5.1. Fix a sheaf of groupoids $\mathcal{A}$ over $\mathcal{T}$ and an open cover $\mathfrak{U}$ of $\mathcal{T}$.

A **zero cochain** $a \in C^0(\mathfrak{U}, \mathcal{A})$ associates to each $U \in \mathfrak{U}$ an arrow a_U in $\mathcal{A}(U)$. A **one cochain** $\alpha \in C^1(\mathfrak{U}, \mathcal{A})$ associates to each $U \in \mathfrak{U}$ an object $A_U \in \mathcal{A}(U)$ and to each pair $U, V \in \mathfrak{U}$ an arrow $\alpha_{VU}\colon A_U|_{U \cap V} \longrightarrow A_V|_{U \cap V}$.

A **one cocycle** $\alpha \in Z^1(\mathfrak{U}, \mathcal{A})$ is a one cochain that is **closed**, meaning that $\alpha_{UU} = \mathrm{id}_{A_U}$ for all $U \in \mathfrak{U}$ and that $\alpha_{WV} \circ \alpha_{VU} = \alpha_{WU}$ holds on $U \cap V \cap W$ for every triple $U, V, W \in \mathfrak{U}$,

The groupoid of zero cochains acts on the set of one cocycles by

$$g_{VU} \mapsto (f_V|_{U \cap V}) \circ g_{VU} \circ (f_U|_{U \cap V})^{-1},$$

wherever this makes sense. The **first cohomology** is the set of equivalence classes under this action:

$$\mathrm{H}^1(\mathfrak{U}, \mathcal{A}) := Z^1(\mathfrak{U}, \mathcal{A})/C^0(\mathfrak{U}, \mathcal{A}).$$

An open cover $\mathfrak{V}$ is[3] a **refinement** of an open cover $\mathfrak{U}$ if every set $V \in \mathfrak{V}$ is a subset of a set $U \in \mathfrak{U}$. As in the abelian case, this induces a map in cohomology:

Lemma 5.2. *If $\mathfrak{V}$ is a refinement of $\mathfrak{U}$, we get a well defined map in cohomology* $\mathrm{H}^1(\mathfrak{U}, \mathcal{A}) \longrightarrow \mathrm{H}^1(\mathfrak{V}, \mathcal{A})$ *for any presheaf of groupoids $\mathcal{A}$.*

Proof. Choose any map $f\colon \mathfrak{V} \longrightarrow \mathfrak{U}$ such that $V \subset f(V)$. This map induces a map on one cocycles; given a cocycle α, simply restrict every object and every map from $f(V)$ to V. This clearly descends to a map on cohomology. If $f'\colon \mathfrak{V} \longrightarrow \mathfrak{U}$ is different map such that $V \subset f'(V)$, the resulting cocycles differ by the following zero cochain: associate to each $V \in \mathfrak{V}$ the restriction to V of the arrow $\alpha_{f(V)f'(V)}$. $\qquad\square$

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[3] $\mathfrak{V}$ is $\mathtt{\$\backslash mathfrak\ V\$}$.

Since the set of open covers is a directed set, this makes $\mathrm{H}^1(\mathfrak{U}, \mathcal{A})$ into a direct system of sets. The **Čech cohomology** of $\mathcal{T}$ with values in $\mathcal{A}$, denoted by $\mathrm{H}^1(\mathcal{T}, \mathcal{A})$, is defined to be the direct limit of this direct system.

The following lemma is easy to check.

Lemma 5.3. *A map of sheaves of groupoids $f \colon \mathcal{A} \longrightarrow \mathcal{B}$ induces a map of cohomology $f_* \colon \mathrm{H}^1(\mathcal{T}, \mathcal{A}) \longrightarrow \mathrm{H}^1(\mathcal{T}, \mathcal{B})$.*

Lemma 5.4. *Let $\mathcal{A}$ be a sheaf of groupoids over $\mathcal{T}$. A global object in $\mathcal{A}$ naturally determines a class $[A] \in \mathrm{H}^1(\mathcal{T}, \mathcal{A})$. Two objects A and A' are isomorphic if and only if $[A] = [A']$. If the sheaf has gluable objects, every class in $\mathrm{H}^1(\mathcal{T}, \mathcal{A})$ arises in this way.*

Proof. A global object A in $\mathcal{A}(\mathcal{T})$ maps to the one cocycle with object A and arrow id_A. If $f \colon A \longrightarrow A'$ is an isomorphism, then f is a zero cochain that interchanges the two cocycles. On the other hand, if A and A' map to the same cohomology class. Then there exist isomorphisms $f_U \colon A|_U \longrightarrow A'|_U$, for U in some open cover $\mathfrak{U}$, so that $f_U|_{U \cap V} = f_V|_{U \cap V}$. By the second sheaf axiom, this implies that A is isomorphic to A'. (See Definition 4.4.) Finally, the definition of gluable objects is chosen exactly so that the map $A \mapsto [A]$ is onto for such sheaves. $\qquad\square$

Part III: Sheaves of maps of M/T

We wish to determine whether two tall complexity one spaces are isomorphic. By Proposition 3.3, it is enough to determine whether their quotients are Φ-diffeomorphic. By the results of [**KT**], the quotient M/T is, topologically, a surface bundle over $\Phi(M)$. If this were true in the C^∞ category, it would be easy to determine whether two quotients are Φ-diffeomorphic. Unfortunately, however, the quotient M/T is naturally a manifold with corners on the complement of the exceptional orbits, but not on the exceptional orbits themselves. (See Lemma 9.4.)

To overcome this difficulty, we convert our problem to the problem of determining whether two cohomology classes are the same. We define the sheaf $\mathcal{Q}$ of Φ-diffeomorphisms. For each open subset $U \subseteq \mathcal{T}$, the objects in the groupoid $\mathcal{Q}(U)$ are the tall complexity one proper Hamiltonian T-manifolds over U; the arrows are Φ-diffeomorphisms between their quotients. By Lemma 5.4 and Proposition 3.3, two tall complexity one space are isomorphic if and only if they induce the same cohomology class in $H^1(\mathcal{T}, \mathcal{Q})$.

At first, this may not seem like a great improvement. However, in this part of the paper, we gradually transform the sheaf $\mathcal{Q}$ into one whose first cohomology we can compute. To do this we "correct" the smooth structure near the exceptional orbits so that M/T is a *smooth surface bundle*. (For more details, see Section 7.) This process is unnatural; it relies on a choice of "grommets".

6. Grommets

In this section, we define a new sheaf: Φ-diffeomorphisms with grommets.

We fix an inner product on $\mathfrak{t}$, once and for all. Let a closed subgroup $H \subseteq T$ act on $\mathbb{C}^n$ as a subgroup of $(S^1)^n$, with moment map $\Phi_H \colon \mathbb{C}^n \longrightarrow \mathfrak{h}^*$. There exists an invariant symplectic form on

$$Y = T \times_H \mathbb{C}^n \times \mathfrak{h}^0$$

with moment map

$$\Phi_Y([t, z, \nu]) = \alpha + \Phi_H(z) + \nu,$$

where $\alpha \in \mathfrak{t}^*$, $\mathfrak{h}^0 \subseteq \mathfrak{t}^*$ is the annihilator of the Lie algebra $\mathfrak{h}$, and we embed $\mathfrak{h}^*$ in $\mathfrak{t}^*$ using the metric. The space Y is called a **complexity one model**.

The *local normal form theorem* [**GS2**, **M**] implies that any orbit in a Hamiltonian T-manifold has a neighborhood which is isomorphic to a neighborhood of the orbit $\{[t, 0, 0]\}$ in some complexity one model Y. This model is determined uniquely up to permutations of the coordinates on $\mathbb{C}^n$; we call it the **local model** associated to the orbit.

We recall the following definition from [**KT**, Definition 8.1].

Definition 6.1. Let (M, ω, Φ) be a Hamiltonian T-manifold. A **grommet** is a Φ-T-diffeomorphism $\psi \colon D \longrightarrow M$ from an open subset D of a local model $Y = T \times_H \mathbb{C}^n \times \mathfrak{h}^0$ to an open subset of M.

Note that the domain D need not contain the orbit $\{[t, 0, 0]\}$. Therefore, a grommet can be restricted to any open subset.

Definition 6.2. A complexity one Hamiltonian T-manifold $(M, \omega, \Phi, \mathcal{T})$ is **grommetted** if it is equipped with grommets whose images are disjoint and cover the union of the exceptional orbits in M.

We think of this as attaching a grommet to the fabric of the manifold at every exceptional orbit. In real life, the fabric can flow however it wants away from the grommets, but at the grommet it can only spin;

this allows all the necessary freedom of movement but prevents the fabric from ripping at the points of stress. Similarly, our grommets are designed to give enough freedom so that we can approximate any map well, but still prevent us from having to cope with the stress of really dealing with what happens at the exceptional orbits.

We now define the sheaf $\hat{\mathcal{Q}}$ of Φ-diffeomorphisms with grommets. For each open set $U \subseteq \mathcal{T}$ the objects in the groupoid $\hat{\mathcal{Q}}(U)$ are the grommeted tall complexity one proper Hamiltonian T-manifolds over U; the arrows are Φ-diffeomorphisms between their quotients (which ignore their grommets).

The only difference between this sheaf $\hat{\mathcal{Q}}$ and the sheaf $\mathcal{Q}$ defined in the beginning of Part III is that the objects in $\hat{\mathcal{Q}}$ carry grommets. These sheaves have the same first cohomology.

Proposition 6.3. *The forgetful functor $\hat{\mathcal{Q}} \longrightarrow \mathcal{Q}$ induces an isomorphism in cohomology,*

$$\mathrm{H}^1(\mathcal{T}, \hat{\mathcal{Q}}) \cong \mathrm{H}^1(\mathcal{T}, \mathcal{Q}).$$

The proof of this proposition uses an abstract sheaf-theoretic lemma:

Lemma 6.4. *Let $i \colon \mathcal{A} \longrightarrow \mathcal{B}$ be a map of sheaves such that:*

1) *For any open subset $U \subset \mathcal{T}$ and objects $A, A' \in \mathcal{A}(U)$,*

$$i \colon \hom_{\mathcal{A}}(A, A') \longrightarrow \hom_{\mathcal{B}}(i(A), i(A'))$$

   is a bijection.
2) *For any open subset $U \subseteq \mathcal{T}$ and object $B \in \mathcal{B}(U)$, every point in U has a neighborhood $V \subseteq U$ and an object $A \in \mathcal{A}(V)$ so that $i(A)$ is isomorphic to $B|_V$.*

Then i induces an isomorphism

$$i_* \colon \mathrm{H}^1(\mathcal{T}, \mathcal{A}) \longrightarrow \mathrm{H}^1(\mathcal{T}, \mathcal{B}).$$

Proof. First we prove that i_* is onto. Let $\mathfrak{U}$ be a cover. A cocycle $\beta \in Z^1(\mathfrak{U}, \mathcal{B})$ associates to each $U \in \mathfrak{U}$ an object B_U over U. After passing to a refinement (which we still call $\mathfrak{U}$), assumption (2) guarantees that for each $U \in \mathfrak{U}$ there exists an object A_U so that $i(A_U)$ is isomorphic to B_U. By assumption (1), for each $U, V \in \mathfrak{U}$ there exists a unique $\alpha_{VU} \colon A_U|_{U \cap V} \longrightarrow A_V|_{U \cap V}$ so that $i(\alpha_{VU}) = \beta_{VU}$. Then α is a cocycle and $i(\alpha)$ is cohomologous to β.

Now we prove that i_* is one-to-one. It is enough to prove that the map $i_*\colon H^1(\mathfrak{U}, \mathcal{A}) \longrightarrow H^1(\mathfrak{U}, \mathcal{B})$ is one-to-one for every cover $\mathfrak{U}$. Suppose that α and α' are in $Z^1(\mathfrak{U}, \mathcal{A})$, and that $i(\alpha)$ and $i(\alpha')$ are cohomologous. Then there exists a zero cochain which associates to each $U \in \mathfrak{U}$ an arrow b_U so that $b_V^{-1} \circ i(\alpha_{VU}) \circ b_U = i(\alpha'_{VU})$ for all V and U in $\mathfrak{U}$. By assumption (1), for each $U \in \mathfrak{U}$, there exists a unique arrow a_U such that $i(a_U) = b_U$. Then $a_V^{-1} \circ \alpha_{VU} \circ a_U = \alpha'_{VU}$. $\qquad\square$

Proof of Proposition 6.3. This follows from Lemma 6.4 and from the fact that every complexity one proper Hamiltonian T-manifold can be locally grommeted (see [**KT**, Lemma 8.4]). $\qquad\square$

7. Φ-homeomorphisms

In this section, we list the sheaves that we need in this part of the paper. First, we must consider maps which have the following form:

Definition 7.1. Let M and M' be complexity one Hamiltonian T-manifolds. A **Φ-homeomorphism** between M/T and M'/T is a homeomorphism which sends each orbit to an orbit with the same local model, is a diffeomorphism off the set of exceptional orbits, respects the moment maps, and preserves the orientation of the moment map fibers.

Let us now define the sheaf $\hat{\mathcal{H}}$ of Φ-homeomorphisms (with grommets). For each open subset $U \subseteq \mathcal{T}$, the objects in the groupoid $\hat{\mathcal{H}}(U)$ are the grommeted tall complexity one proper Hamiltonian T-manifolds over U; the arrows are Φ-homeomorphisms between their quotients (which ignore the grommets).

We work with a sequence of subsheaves of $\hat{\mathcal{H}}$. Each subsheaf has same objects as $\hat{\mathcal{H}}$ does, but the arrows are Φ-homeomorphisms which satisfy additional conditions. The first cohomology of each sheaf is isomorphic to that of $\mathcal{Q}$.

We list the names and symbols for the sheaf $\hat{\mathcal{H}}$ and its relevant subsheaves:

Φ-homeomorphisms (with grommets)	$\hat{\mathcal{H}}$
Φ-diffeomorphisms (with grommets)	$\hat{\mathcal{Q}}$
locally rigid Φ-homeomorphisms	$\mathcal{RQ}$
local stretch maps	$\mathcal{E}$
locally sb-rigid Φ-homeomorphisms	$\mathcal{RP}$
sb-diffeomorphisms	$\hat{\mathcal{P}}$

In Section 6 we defined the sheaf $\hat{\mathcal{Q}}$ of Φ-diffeomorphisms with grommets; note that it is a subsheaf of the sheaf of Φ-homeomorphisms (with grommets). We next restrict to the subsheaf $\mathcal{RQ} \subset \hat{\mathcal{Q}}$ consisting of those Φ-homeomorphisms that are, roughly speaking, given by "rigid rotations" along the exceptional orbits. Then we extend to the sheaf $\mathcal{E}$ of maps that are given by "rotations and stretches" along the exceptional orbits. We then restrict again to a sheaf $\mathcal{RP}$ of "rigid rotations", except that these are defined differently, in such a way that they are smooth with respect to a new differential structure that makes M/T into a smooth manifold with corners. Finally, we extend to the sheaf $\hat{\mathcal{P}}$ of *all* maps that are smooth with respect to the new differential structure.

We define the rest of these sheaves in Sections 8–11. We also show that we have the inclusions

(7.2) $$\hat{\mathcal{Q}} \supseteq \mathcal{RQ} \subseteq \mathcal{E} \supseteq \mathcal{RP} \subseteq \hat{\mathcal{P}}.$$

In Sections 13–15 we show that each of these inclusions induces an isomorphism on the first cohomology.

8. Locally rigid Φ-homeomorphisms

In this section we define the second sheaf in our sequence of sheaves: locally rigid Φ-homeomorphisms.

Let an h dimensional closed subgroup $H \subseteq T$ act on $\mathbb{C}^{h+1}$ through an inclusion map

$$\rho = (\rho_0, \ldots, \rho_h) \colon H \longrightarrow (S^1)^{h+1}$$

so that the origin is an exceptional orbit. Let $R_\rho \subset U(h+1)$ denote the group of unitary transformations that commute with the H action. Up to permutation there are only two possibilities: either the ρ_i are all different, or they are all different except that $\rho_0 = \rho_1$. In the former case,

$$R_\rho = (S^1)^{h+1} \subset U(h+1).$$

In the latter case,

$$R_\rho = U(2) \times (S^1)^{h-1} \subset U(h+1).$$

The group

$$R_Y := T \times_H R_\rho$$

acts on the complexity one model $Y := T \times_H \mathbb{C}^{h+1} \times \mathfrak{h}^0$. Define

$$\overline{R}_Y = R_Y / T = R_\rho / H.$$

The action of R_Y on Y descends to an action of $\overline{R}_Y$ on Y/T through the short exact sequence $1 \longrightarrow T \longrightarrow R_Y \longrightarrow \overline{R}_Y \longrightarrow 1$.

REMARK 8.1. In *tall* complexity one models, the ρ_i are always different, so that $R_\rho = (S^1)^{h+1}$ and $\overline{R}_Y \cong S^1$. However, in this section we allow the general case because it does not require much extra work and will be useful in subsequent papers.

We would like to use $\overline{R}_Y$ to define "locally rigid Φ-homeomorphisms" between grommeted complexity one Hamiltonian T-manifolds. However, even if the manifolds are isomorphic, the domains of the given grommets might lie in different models, whereas the elements of R_Y are maps from a model to itself. We solve this problem by passing to *sub-grommets*. We define these in the next several pages.

Lemma 8.2. *Let* $Y = T \times_H \mathbb{C}^{h+1} \times \mathfrak{h}^0$ *be a complexity one model. Let* $E = \{[t, y, \mu]\}$ *be an exceptional orbit, where* $y_i = 0$ *exactly if* $0 \le i \le k$.
The local model associated to E is $Y_E = T \times_K \mathbb{C}^{k+1} \times \mathfrak{k}^0$, *where*

$$(8.3) \qquad K = H \cap \left((S^1)^{k+1} \times \{1\}^{h-k} \right)$$

acts on $\mathbb{C}^{k+1}$ *as the restriction of the H action on $\mathbb{C}^{h+1}$ to the first $k+1$ coordinates.*

Proof. Clearly, K is the stabilizer of E. Let $\mathcal{O} = H \cdot y$. Then $T_y\mathcal{O} = \{0\}^{k+1} \times V$, where V is an $(h - \dim K)$-dimensional isotropic subspace of $\mathbb{C}^{h-k}$. The symplectic slice is

$$(T_y\mathcal{O})^\omega / T_y\mathcal{O} \cong \mathbb{C}^{k+1} \times \mathbb{C}^m,$$

where $m = \dim K - k$ and K acts trivially on $\mathbb{C}^m$. If the orbit E is exceptional, then $m = 0$. $\qquad\qquad \square$

Definition 8.4. Let $Y = T \times_H \mathbb{C}^{h+1} \times \mathfrak{h}^0$ be a complexity one model. Let $Y_E = T \times_K \mathbb{C}^{k+1} \times \mathfrak{k}^0$ be the local model associated to an exceptional orbit $E = \{[t, y, \mu]\} \subset Y$, where $y_i = 0$ exactly if $0 \le i \le k$.
    A **canonical inclusion** is a Φ-T-diffeomorphism

$$\Lambda \colon D_E \longrightarrow Y = T \times_H \left(\mathbb{C}^{k+1} \times \mathbb{C}^{h-k} \right) \times \mathfrak{h}^0$$

such that

$$(8.5) \qquad \Lambda\left([t, z, \nu]\right) = \left[t, z, f([t, z, \nu]), \alpha([t, z, \nu])\right],$$

where $D_E \subseteq Y_E$ is an open subset, and $f \colon D_E \longrightarrow \mathbb{R}_{>0}^{h-k}$ and $\alpha \colon D_E \longrightarrow \mathfrak{h}^0$ are R_{Y_E} invariant functions.

The composition of two canonical inclusions is a canonical inclusion.

Lemma 8.6. *Let Y be a complexity one model, and let Y_E be the local model associated to an exceptional orbit $E \subset Y$.*

There exists a a canonical inclusion $\Lambda \colon D_E \longrightarrow Y$ on some neighborhood D_E of $\{[t, 0, 0]\}$ in Y_E. On any open subset D_E of Y_E there exists at most one canonical inclusion $\Lambda \colon D_E \longrightarrow Y$.

Proof. Here, we use the notation of Definition 8.4.

Let $\eta_j \in \mathfrak{h}^*$ denote the weights for the H action on $\mathbb{C}^{h+1}$. A moment map for Y is

$$(8.7) \qquad \Phi_Y([t, w, \alpha]) = \frac{1}{2} \sum_{j=0}^{h} \eta_j |w_j|^2 + \alpha.$$

The weights for the K action on $\mathbb{C}^{k+1}$ are $\iota^* \eta_j$, for $0 \le j \le k$, where ι^* is the dual to the inclusion map $\iota \colon \mathfrak{k} \longrightarrow \mathfrak{h}$. The corresponding moment map for the local model Y_E is

$$
\begin{aligned}
\Phi_E([t, z, \nu]) \;&=\; \Phi_Y(E) + \frac{1}{2} \sum_{j=0}^{k} \iota^* \eta_j |z_j|^2 + \nu. \\
(8.8) \\
&= \frac{1}{2} \sum_{j=k+1}^{h} \eta_j |y_j|^2 + \mu + \frac{1}{2} \sum_{j=0}^{k} \iota^* \eta_j |z_j|^2 + \nu.
\end{aligned}
$$

Given *any* pair of smooth T invariant functions, $f \colon D_E \longrightarrow \mathbb{R}_{>0}^{h-k}$ and $\alpha \colon D_E \longrightarrow \mathfrak{h}^0$, we can define a smooth T-equivariant map $\Lambda \colon D_E \longrightarrow Y$ by the formula (8.5). This map satisfies $\Phi_Y \circ \Lambda = \Phi_E$ if and only if

$$
\begin{aligned}
(8.9) \quad & \frac{1}{2} \left(\sum_{j=0}^{k} \eta_j |z_j|^2 + \sum_{j=k+1}^{h} \eta_j f_j^2 \right) + \alpha \\
& = \frac{1}{2} \sum_{j=k+1}^{h} \eta_j |y_j|^2 + \mu + \frac{1}{2} \sum_{j=1}^{k} \iota^* \eta_j |z_j|^2 + \nu.
\end{aligned}
$$

The metric induces a decomposition

$$\mathfrak{t}^* = \mathfrak{k}^* \oplus \left(\mathfrak{h}^* \cap \mathfrak{k}^0 \right) \oplus \mathfrak{h}^0.$$

By Lemma 8.2, the weights η_j, for $k < j \le h$, lie in $\mathfrak{h}^* \cap \mathfrak{k}^0$. Hence, the $\mathfrak{k}^*$ components of the left and right hand sides of (8.9) automatically agree; they are both equal to $\frac{1}{2} \sum_{j=0}^{k} \iota^* \eta_j |z_j|^2$ Therefore, Equation (8.9)

is equivalent to the equations

$$(8.10) \quad \frac{1}{2} \sum_{j=k+1}^{h} \eta_j f_j^2 = \frac{1}{2} \sum_{j=k+1}^{h} \eta_j |y_j|^2 + \frac{1}{2} \sum_{j=0}^{k} (\iota^* \eta_j - \eta_j)|z_j|^2 + \pi(\mu + \nu)$$

in $\mathfrak{h}^* \cap \mathfrak{k}^0$ and

$$\alpha = \mu + \nu - \pi(\mu + \nu)$$

in $\mathfrak{h}^0$, where π is the projection from $\mathfrak{k}^0$ to $\mathfrak{k}^0 \cap \mathfrak{h}^*$.

We find f_i^2 by solving the system (8.10) of linear equations. The solution exists and is unique because the coefficient vectors η_j, for $j = k+1, \ldots, h$, are a basis of $\mathfrak{h}^* \cap \mathfrak{k}^0$. Since $f_j^2 = |y_j|^2 > 0$ when $z = 0$ and $\nu = 0$, we can take smooth positive square roots of f_j^2 near $[t, 0, 0]$. Finally, the functions f_j are R_{Y_E}-invariant because the equation (8.10) is invariant and the solution is unique.

It is clear that the resulting map Λ preserves the orientation on each fiber. To show that Λ is a diffeomorphism, it is enough to show that Λ is a submersion.

Consider the map $H \longrightarrow (S^1)^{h-k}$ obtained by the inclusion into $(S^1)^{h+1}$ followed by the projection to the last $h - k$ coordinates. The kernel of this map is K. Since $\dim K = k$, the map must be onto. That is, we have a short exact sequence

$$1 \longrightarrow K \longrightarrow H \longrightarrow (S^1)^{h-k} \longrightarrow 1.$$

This implies that the natural inclusion $\mathbb{R}_+^{h-k} \longrightarrow (\mathbb{C}^\times)^{h-k}$ gives rise to a Φ-T-diffeomorphism

$$Y \supseteq T \times_H \left(\mathbb{C}^{k+1} \times (\mathbb{C}^\times)^{h-k} \right) \times \mathfrak{h}^0 \cong T \times_K \mathbb{C}^{k+1} \times \mathbb{R}_+^{h-k} \times \mathfrak{h}^0.$$

In these coordinates, Λ has the form $[t, z, \nu] \mapsto [t, z, f, \mu + \nu - \pi(\mu + \nu)]$. It is easy to check that this is a submersion. $\square$

Definition 8.11. Let M be a grommeted complexity one Hamiltonian T-manifold. Let $\Lambda \colon D_E \longrightarrow Y$ be a canonical inclusion whose image is contained in the domain of a grommet $\psi \colon D \longrightarrow M$. Define $\psi_E \colon D_E \longrightarrow M$ by $\psi_E = \psi \circ \Lambda$. We call the induced map $\overline{\psi_E} \colon D_E/T \longrightarrow M/T$ a **sub-grommet**.

Definition 8.12. Let M and M' be grommeted complexity one Hamiltonian T-manifolds. A Φ-homeomorphism $f \colon M/T \longrightarrow M'/T$ is **locally rigid** if for every exceptional orbit $E \in M/T$ and any pair of sub-grommets

$$\overline{\psi} \colon D/T \longrightarrow M/T \quad \text{and} \quad \overline{\psi}' \colon D/T \longrightarrow M'/T$$

whose images contain E and $f(E)$, there exists a smooth function $R\colon \mathfrak{t}^* \longrightarrow \overline{R}_Y$ such that

$$(8.13) \qquad\qquad (\overline{\psi}')^{-1} \circ f \circ \overline{\psi}(y) = R(\overline{\Phi}_Y(y)) \cdot y$$

on some neighborhood of $\overline{\psi}^{-1}(E)$. Here, both sub-grommets have the same domain $D/T \subset Y/T$, and $\overline{\Phi}_Y$ is induced by the moment map on Y.

REMARK 8.14. By Lemma 8.6, for any exceptional orbit E we can always find sub-grommets $\overline{\psi}\colon D/T \longrightarrow M/T$ and $\overline{\psi}'\colon D/T \longrightarrow M'/T$, with the same domain, whose images contain E and $f(E)$. Since a canonical inclusion induces an $\overline{R}_Y$-equivariant map on quotients, Equation (8.13) holds for either every such $\overline{\psi}$ and $\overline{\psi}'$ or for no such $\overline{\psi}$ and $\overline{\psi}'$.

The following lemma is straightforward.

Lemma 8.15. *Every locally rigid Φ-homeomorphism is a Φ-diffeomorphism.*

We now define the sheaf $\mathcal{RQ}$ of locally rigid Φ-homeomorphisms. For each open subset $U \subseteq \mathcal{T}$, the objects in the the groupoid $\mathcal{RQ}(U)$ are the grommeted tall complexity one proper Hamiltonian T-manifolds over U; the arrows are the locally rigid Φ-homeomorphisms between their quotients.

Our main claim, which we prove in Section 13, is

Proposition 8.16. *The inclusion $\mathcal{RQ} \subset \hat{\mathcal{Q}}$ induces an isomorphism*

$$\mathrm{H}^1(\mathcal{T}, \mathcal{RQ}) \cong \mathrm{H}^1(\mathcal{T}, \hat{\mathcal{Q}}).$$

9. Local stretch maps

In this section we define the third sheaf in our sequence: local stretch maps.

Consider a tall complexity one model $Y = T \times_H \mathbb{C}^{h+1} \times \mathfrak{h}^0$ with moment map Φ_Y.

Lemma 9.1. *There exists a unique monomial $P\colon \mathbb{C}^{h+1} \longrightarrow \mathbb{C}$, called the **defining monomial**, such that*

$$(9.2) \qquad\qquad P(z) = \prod_{j=0}^{h} z_j^{\xi_j},$$

with $\xi_j \geq 0$ for all j, such that the following sequence is exact:

$$(9.3) \qquad 1 \longrightarrow H \overset{\rho}{\hookrightarrow} (S^1)^{h+1} \overset{P}{\longrightarrow} S^1 \longrightarrow 1.$$

Proof. See [**KT**, Lemma 5.8]. $\qquad\qquad\square$

The **trivializing homeomorphism** is the map

$$F = (\overline{\Phi}_Y, \overline{P}) \colon Y/T \longrightarrow (\text{image } \Phi_Y) \times \mathbb{C},$$

where $\overline{\Phi}_Y$ is induced from the moment map and $\overline{P}$ is induced from the defining monomial. (See [**KT**, Definitions 5.12 and 6.4].)

Lemma 9.4. *The trivializing homeomorphism F is a homeomorphism, and it is a diffeomorphism off the set of exceptional orbits.*

Proof. See [**KT**, Lemmas 6.2 and 7.1]. $\qquad\qquad\square$

A grommet on a tall complexity one Hamiltonian T-manifold induces a "coordinate chart" on M/T.

Definition 9.5. Let $\psi \colon D \longrightarrow M$ be a grommet with $D \subseteq Y$, and let $F \colon Y/T \longrightarrow \text{image } \Phi_Y \times \mathbb{C}$ be the trivializing homeomorphism. Let $B = F(D) \subset \text{image } \Phi_Y \times \mathbb{C}$ and define $\varphi \colon B \longrightarrow M/T$ by

$$\varphi := \overline{\psi} \circ F^{-1}.$$

Then φ is a homeomorphism onto an open subset of M/T; it is the **surface bundle grommet** associated to ψ.

We can now give our main definition.

Definition 9.6. Let M and M' be grommeted tall complexity one Hamiltonian T-manifolds. A Φ-homeomorphism $f \colon M/T \longrightarrow M'/T$ is a **local stretch map** if for every exceptional orbit $E \in M/T$ and the pair of associated surface bundle grommets $\varphi \colon B \longrightarrow M/T$ and $\varphi' \colon B' \longrightarrow M'/T$ whose images contain E and $f(E)$, there exists a function $R \colon \mathfrak{t}^* \longrightarrow S^1$ and an S^1-invariant function $\lambda \colon \mathfrak{t}^* \times \mathbb{C} \longrightarrow \mathbb{R}_{>0}$ such that

$$(9.7) \qquad (\varphi')^{-1} \circ f \circ \varphi(\alpha, z) = (\alpha, R(\alpha)\lambda(\alpha, z) \cdot z),$$

on some neighborhood of $\varphi^{-1}(E) \subseteq \mathfrak{t}^* \times \mathbb{C}$.

We need the following lemma.

Lemma 9.8. *Let $\Lambda \colon D_E \longrightarrow Y$ be a canonical inclusion for $D_E \subset Y_E$. Let $F \colon Y/T \longrightarrow \text{image } \Phi_Y \times \mathbb{C}$ and $F_E \colon Y_E/T \longrightarrow \text{image } \Phi_{Y_E} \times \mathbb{C}$ be*

the trivializing homeomorphisms. Then there exists an S^1 invariant function $\lambda\colon D_E \longrightarrow \mathbb{R}_{>0}$ such that

$$F \circ \overline{\Lambda} \circ F_E^{-1}(\alpha, z) = (\alpha, \lambda(\alpha, z) \cdot z),$$

for all $(\alpha, z) \in F_E(D_E/T) \subset \mathfrak{t}^* \times \mathbb{C}$.

Proof. By Lemma 8.2, the defining monomial P_E for Y_E consists of the first $k + 1$ factors the the defining monomial P for Y. Hence, for $u = [t, z, \nu] \in Y_E$, we have

$$P_E(u) = P_E([t, z, \nu]) = \prod_{j=0}^{k} z_j^{\xi_j}$$

and

$$P(\Lambda(u)) = P([t, z, f, \tilde{\mu}]) = \prod_{j=0}^{k} z_j^{\xi_j} \prod_{j=k+1}^{h} f_j^{\xi_j}$$

for some $\xi_0, \ldots, \xi_h$. Thus, the lemma holds with

$$\lambda(\alpha, z) = \prod_{j=k+1}^{h} f_j(F_E^{-1}(\alpha, z))^{\xi_j}.$$

$\square$

Carefully unwinding the definitions, this leads to the following lemma.

Lemma 9.9. *Every locally rigid Φ-homeomorphism is a local stretch map.*

We introduce the sheaf $\mathcal{E}$ of local stretch maps. For each open subset $U \subseteq \mathcal{T}$, the objects in the groupoid $\mathcal{E}(U)$ are the grommeted tall complexity one proper Hamiltonian T-manifolds M_U; the arrows are the local stretch maps between their quotients.

Our main claim, which we prove in Section 14, is

Proposition 9.10. *The inclusion $\mathcal{RQ} \subset \mathcal{E}$ induces an isomorphism*

$$\mathrm{H}^1(\mathcal{T}, \mathcal{RQ}) \cong \mathrm{H}^1(\mathcal{T}, \mathcal{E}).$$

10. Locally sb-rigid Φ-homeomorphisms

In this section we define the fourth sheaf in our sequence: locally sb-rigid Φ-homeomorphisms. Here, "sb" stands for "surface bundle".

Definition 10.1. Let M and M' be grommeted tall complexity one proper Hamiltonian T-manifolds. A Φ-homeomorphism $f\colon M/T \longrightarrow M'/T$ is **locally sb-rigid** if for every exceptional orbit $E \in M/T$ and the pair of associated surface bundle grommets $\varphi\colon B \longrightarrow M/T$ and $\varphi'\colon B' \longrightarrow M'/T$ whose images contain E and $f(E)$ (see Definition 9.5), there exists a smooth function $R\colon \mathfrak{t}^* \longrightarrow S^1$ such that

$$(10.2) \qquad (\varphi')^{-1} \circ f \circ \varphi(\alpha, z) = (\alpha, R(\alpha) \cdot z)$$

on some neighborhood of $\varphi^{-1}(E) \subseteq \mathfrak{t}^* \times \mathbb{C}$.

From this and Definition 9.6 we immediately get the following result:

Lemma 10.3. *Every locally sb-rigid Φ-homeomorphism is a local stretch map.*

We now define the sheaf $\mathcal{RP}$ of locally sb-rigid Φ-homeomorphisms. For each open set $U \subseteq \mathcal{T}$, the objects in the groupoid $\mathcal{RP}(U)$ are the grommeted tall complexity one proper Hamiltonian T-manifolds over U; the arrows are the locally sb-rigid Φ-homeomorphisms between their quotients.

Our main claim, which we prove in Section 14, is

Proposition 10.4. *The inclusion $\mathcal{RP} \longrightarrow \mathcal{E}$ induces an isomorphism*

$$\mathrm{H}^1(\mathcal{T}, \mathcal{RP}) \cong \mathrm{H}^1(\mathcal{T}, \mathcal{E}).$$

11. sb-diffeomorphisms

In this section we define the fifth sheaf in our sequence: sb-diffeomorphisms. Again, "sb" stands for "surface bundle".

Definition 11.1. Let M and M' be grommeted tall complexity one Hamiltonian T-manifolds. A Φ-homeomorphism $f\colon M/T \longrightarrow M'/T$ is an **sb-diffeomorphism** if for every pair of associated surface bundle grommets $\varphi\colon B \longrightarrow M/T$ and $\varphi'\colon B' \longrightarrow M'/T$, the composition

$$g = (\varphi')^{-1} \circ f \circ \varphi$$

is a diffeomorphism.

The definition clearly implies

Lemma 11.2. *If a Φ-homeomorphism is locally sb-rigid then it is an sb-diffeomorphism.*

We now define the sheaf $\hat{\mathcal{P}}$ of sb-diffeomorphisms. For each open subset $U \subseteq \mathcal{T}$, the objects in the groupoid $\hat{\mathcal{P}}(U)$ are the grommeted tall complexity one proper Hamiltonian T-manifolds over U; the arrows are the sb-diffeomorphisms between their quotients.

Our main claim, which we prove in Section 15, is

Proposition 11.3. *The inclusion $\mathcal{RP} \subset \hat{\mathcal{P}}$ induces an isomorphism*

$$\mathrm{H}^1(\mathcal{T}, \mathcal{RP}) \cong \mathrm{H}^1(\mathcal{T}, \hat{\mathcal{P}}).$$

Part IV: Rigidification

In this evil part of the paper we prove the main propositions stated in the previous part: Propositions 8.16, 9.10, 10.4, and 11.3. In Section 12 we prove a technical lemma, which we use four times, in Sections 13, 14, and 15, to show that each of the inclusions of sheaves

$$\hat{\mathcal{Q}} \supseteq \mathcal{RQ} \subseteq \mathcal{E} \supseteq \mathcal{RP} \subseteq \hat{\mathcal{P}}$$

induces an isomorphism on H^1. The reader may choose to skip to the next part of the paper (Section 16) on first reading.

Checking the assumptions of the technical lemma involves *isotopies* of maps on complexity one quotients. In previous sections we defined what it means for a map between complexity one quotients to be a Φ-homeomorphism, a Φ-diffeomorphism, a locally rigid Φ-homeomorphism, a local stretch map, an sb-diffeomorphism, or a locally sb-rigid Φ-homeomorphism. In Sections 13–15 we will need to extend these notions to "isotopies" and to being "rigid at a point". Also, we will need to have similar notions on "surface bundle models". We start with the most general definition:

Definition 11.4. Let $(M, \omega, \Phi, \mathcal{T})$ and $(M', \omega', \Phi', \mathcal{T})$ be complexity one Hamiltonian T-manifolds. A map F from an open subset of $[0, 1] \times M/T$ to an open subset of M'/T is an **isotopy of Φ-homeomorphisms** if $f_t(\cdot) = F(t, \cdot)$ is a Φ-homeomorphism for each t and F is smooth on the complement of the exceptional orbits (as a map between manifolds with corners).

REMARK 11.5. We may consider a single Φ-homeomorphism as an isotopy which is independent of the parameter t.

Note that if $f_t(\cdot) = F(t, \cdot)$ is an isotopy of Φ-homeomorphisms from M/T to M'/T and E is an exceptional orbit in M then the exceptional orbit $f_t(E)$ remains the same as t varies continuously.

12. Reducing from sheaf to sheaf

In this section we prove a lemma which guarantees that the first cohomology of two different sheaves agree if the sheaves satisfy certain technical conditions. Let $\mathcal{A}$ and $\mathcal{B}$ be sheaves of groupoids over $\mathcal{T}$ with the same objects and with the arrows in $\mathcal{A}$ forming subsets of the arrows in $\mathcal{B}$. The inclusion maps $\mathcal{A}(U) \hookrightarrow \mathcal{B}(U)$ induce a map from $\mathrm{H}^1(\mathfrak{U}, \mathcal{A})$ to $\mathrm{H}^1(\mathfrak{U}, \mathcal{B})$. We describe a condition which guarantees that this map is a bijection.

Let us begin with a simple analogue for sheaves of abelian groups. Consider a manifold M and two abelian Lie groups $H \subset G$. Denote the sheaves of smooth functions to H and to G by $\mathbf{H}$ and $\mathbf{G}$, respectively. Suppose that H is a smooth deformation retract of G; for example, $H = S^1$ and $G = \mathbb{C}^*$. Then the natural map between the cohomology groups $\mathrm{H}^i(M, \mathbf{H})$ and $\mathrm{H}^i(M, \mathbf{G})$ is an isomorphism for all $i \geq 1$.

One proof relies on the the fact that these sheaves satisfy the following condition: if we are given an open set Z, a smooth function $\beta \colon Z \longrightarrow G$, and a pair of open sets X and Y such that $\overline{X} \cap \overline{Y} = \emptyset$, we can define a new smooth function $\beta' \colon Z \longrightarrow G$ which satisfies the following conditions:

1) For all $x \in X \cap Z$, $\beta'(x)$ is in H.
2) For all $y \in Y \cap Z$, $\beta'(y) = \beta(y)$.
3) For all $u \in Z$, if $\beta(u)$ is in H, then $\beta'(u)$ is also in H.

This fact is easy to show: By assumption, there exists a smooth map $F \colon G \times [0,1] \longrightarrow G$ so that $F(g,0) = g$ for all $g \in G$, that $F(g,1) \in H$ for all $g \in G$, and that $F(h,t) = h$ for all $h \in H$ and $t \in [0,1]$. Since $\overline{X} \cap \overline{Y} = \emptyset$, we can find a smooth function $\lambda \colon M \longrightarrow [0,1]$ such that $\lambda(x) = 1$ for all $x \in X$, and $\lambda(y) = 0$ for all $y \in Y$. Now, simply define $\beta'(u) = F(\beta(u), \lambda(u))$.

Our main technical lemma, which appears below, is a generalization of the fact that this condition is itself enough to prove that for all $i \geq 1$ the cohomologies $\mathrm{H}^i(M, \mathbf{G})$ and $\mathrm{H}^i(M, \mathbf{H})$ are isomorphic.

Lemma 12.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be sheaves of groupoids on $\mathcal{T}$ with the same objects and such that the arrows in $\mathcal{A}$ are subsets of the arrows in $\mathcal{B}$.*

Suppose that for every open cover $\mathfrak{U} = \{U_i\}$ and cocycle $\beta \in \check{Z}^1(\mathfrak{U}, \mathcal{B})$ there exists an open cover $\{U_i'\}$ such that $\overline{U_i'} \subset U_i$ for all i and such that the following holds:

Take any U_i and U_j in $\mathfrak{U}$; let N and N' be the restriction to $Z := U_i \cap U_j$ of the objects associated to U_i and U_j by β. Let X and Y be any open sets such that $\overline{X} \cap \overline{Y} = \emptyset$, let $W = U_1' \cup \cdots \cup U_{p-1}'$ for any integer p, and let $f \colon N \longrightarrow N'$ be any $\mathcal{B}$ arrow. Then there exists a $\mathcal{B}$-arrow $f' \colon N \longrightarrow N'$ with the following properties.
1) *The restriction of f' to $X \cap Z$ is in $\mathcal{A}$.*
2) *The restrictions of f' and f to $Y \cap Z$ coincide.*
3) *If the restriction of f to $W \cap Z$ lies in $\mathcal{A}$, then the restriction of f' to $W \cap Z$ also lies in $\mathcal{A}$.*

Then the inclusion map $i \colon \mathcal{A} \longrightarrow \mathcal{B}$ induces an isomorphism

$$i_* \colon \mathrm{H}^1(\mathcal{T}, \mathcal{A}) \xrightarrow{\cong} \mathrm{H}^1(\mathcal{T}, \mathcal{B}).$$

Proof. It is enough to show that

$$(12.2) \qquad i_* \colon \mathrm{H}^1(\mathfrak{U}, \mathcal{A}) \longrightarrow \mathrm{H}^1(\mathfrak{U}, \mathcal{B})$$

is an isomorphism for every countable cover $\mathfrak{U} = \{U_i\}_{i=1}^{\infty}$ such that each U_i intersects only a finite number of U_j's.

Proof that the map (12.2) is onto.
Let $\beta \in Z^1(\mathfrak{U}, \mathcal{B})$ be a one cocycle. We want to find a zero cochain, $b \in C^0(\mathfrak{U}, \mathcal{B})$, such that $b_V \circ \beta_{VU} \circ b_U^{-1}$ is in $\mathcal{A}$ for all $U, V \in \mathfrak{U}$. Let $\{U_i'\}$ be a cover as in the statement of the lemma.

By induction, it suffices to prove that for any one cocycle $\beta \in Z^1(\mathfrak{U}, \mathcal{B})$ such that the restriction of β_{VU} to $(U_1' \cup \cdots \cup U_{p-1}') \cap (U \cap V)$ is in $\mathcal{A}$ for all $U, V \in \mathfrak{U}$, we can find a zero cochain $b \in C^0(\mathfrak{U}, \mathcal{B})$ such that the restriction of $b_V \circ \beta_{VU} \circ b_U{}^{-1}$ to $(U_1' \cup \cdots \cup U_p') \cap (U \cap V)$ is in $\mathcal{A}$ for all $U, V \in \mathfrak{U}$, and such that b_V is the identity for all $V \in \mathfrak{U}$ such that $U_p \cap V = \emptyset$. Since each U_i intersects only a finite number of U_j's, the last condition ensures that for each pair $U, V \in \mathfrak{U}$ the arrow β_{VU} stabilizes after a finite number of steps.

Let $\beta \in Z^1(\mathfrak{U}, \mathcal{B})$ be a one cocycle such that the restriction of β_{VU} to $(U_1' \cup \cdots \cup U_{p-1}') \cap (U \cap V)$ is in $\mathcal{A}$ for all $U, V \in \mathfrak{U}$. Let Y be an open set such that $Y \cup U_p = \mathcal{T}$ and $\overline{Y} \cap \overline{U_p'} = \emptyset$. Let N_U denote the object associated by β to U, for each $U \in \mathfrak{U}$. By the assumption of the lemma, for every $V \in \mathfrak{U}$ there exists a $\mathcal{B}$-arrow $\beta'_{VU_p} \colon N_{U_p}|_{U_p \cap V} \longrightarrow N_V|_{U_p \cap V}$ with the following properties:
1) The restriction of β'_{VU_p} to $U_p' \cap V$ is in $\mathcal{A}$.
2) The restrictions of β'_{VU_p} and β_{VU_p} to $Y \cap (U_p \cap V)$ coincide.
3) The restriction of β'_{VU_p} to $(U_1' \cup \cdots \cup U_{p-1}') \cap (U_p \cap V)$ is in $\mathcal{A}$.

By item (2), we may define $b_V \colon N_V \longrightarrow N_V$ by $b_V = \beta'_{VU_p} \circ \beta^{-1}_{VU_p}$ on $U_p \cap V$ and by $b_V = \mathrm{id}$ on $Y \cap V$. As required, if $U_p \cap V = \emptyset$ then $b_V = \mathrm{id}$.

Now we claim that the restriction of $b_V \circ \beta_{VU} \circ b_U^{-1}$ to $(U'_1 \cup \cdots \cup U'_p) \cap (U \cap V)$ is in $\mathcal{A}$, for any $U, V \in \mathfrak{U}$.

By the definition of b, the restriction of $b_V \circ \beta_{VU} \circ b_U^{-1}$ to $U_p \cap (U \cap V)$ is given by $(\beta'_{VU_p} \circ \beta^{-1}_{VU_p}) \circ \beta_{VU} \circ (\beta_{UU_p} \circ \beta'_{UU_p}{}^{-1}) = \beta'_{VU_p} \circ \beta'_{UU_p}{}^{-1}$. The restrictions of β'_{VU_p} and β'_{UU_p} to $(U'_1 \cup \cdots \cup U'_p) \cap U_p \cap U \cap V$ both lie in $\mathcal{A}$, by items (1) and (3). Therefore, the restrictions of $b_V \circ \beta_{VU} \circ b_U^{-1}$ to $(U'_1 \cup \cdots \cup U'_p) \cap U_p \cap (U \cap V)$ is also in $\mathcal{A}$.

The restriction of $b_V \circ \beta_{VU} \circ b_U^{-1}$ to $Y \cap (U \cap V)$ coincides with the restriction of β_{VU} itself. By the induction hypothesis, its restriction to $(U'_1 \cup \cdots \cup U'_{p-1}) \cap Y \cap (U \cap V)$ is in $\mathcal{A}$.

Thus, the restriction of $b_V \circ \beta_{VU} \circ b_U^{-1}$ to the set $(U_1 \cup \cdots \cup U'_p) \cap U_p \cap (U \cap V)$ and to the set $(U'_1 \cup \cdots \cup U'_{p-1}) \cap Y \cap (U \cap V)$ lie in $\mathcal{A}$. Because the union of these sets is $(U'_1 \cup \cdots \cup U'_p) \cap (U \cap V)$, we are done.

Proof that the map (12.2) is one to one.
Let $\alpha \in Z^1(\mathfrak{U}, \mathcal{A})$ be a one cocycle, and let $b \in C^0(\mathfrak{U}, \mathcal{B})$ be a zero cochain such that $b_V \circ \alpha_{VU} \circ b_U^{-1}$ is in $\mathcal{A}$ for all $U, V \in \mathfrak{U}$. We want to find a zero cochain $a \in C^0(\mathfrak{U}, \mathcal{A})$ such that $a_V \circ \alpha_{VU} \circ a_U^{-1} = b_V \circ \alpha_{VU} \circ b_U^{-1}$ for all $U, V \in \mathfrak{U}$. Let $\{U'_i\}$ be a cover associated to $i(\alpha) \in \check{Z}^1(\mathfrak{U}, \mathcal{B})$ as in the statement of the lemma.

By induction, it suffices to prove that if we are given a zero cocycle $b \in C^0(\mathfrak{U}, \mathcal{B})$ such that $b_V \circ \alpha_{VU} \circ b_U^{-1}$ is in $\mathcal{A}$ for all $U, V \in \mathfrak{U}$ and the restriction of b_V to $(U'_1 \cup \ldots \cup U'_{p-1}) \cap V$ is in $\mathcal{A}$ for all $V \in \mathfrak{U}$, then we can find a zero cochain $b' \in C^0(\mathfrak{U}, \mathcal{B})$ such that $b'_V \circ \alpha_{VU} \circ b'_U{}^{-1} = b_V \circ \alpha_{VU} \circ b_U^{-1}$ for all $U, V \in \mathfrak{U}$, the restriction of b'_V to $(U'_1 \cup \cdots \cup U'_p) \cap V$ is in $\mathcal{A}$ for all $V \in \mathfrak{U}$, and such that if $V \cap U_p = \emptyset$ then $b'_V = b_V$. Again, let Y be such that $Y \cup U_p = \mathcal{T}$ and $\overline{Y} \cap \overline{U'_p} = \emptyset$.

Let N_U denote the object associated by $i(\alpha)$ to U, for each $U \in \mathfrak{U}$. By the assumptions of the lemma, with $U_i = U_j = U_p$, there exists a $\mathcal{B}$-arrow $b'_{U_p} \colon N_{U_p} \longrightarrow N_{U_p}$ with the following properties:

1) The restriction of b'_{U_p} to U'_p is in $\mathcal{A}$.
2) The restrictions of b'_{U_p} and b_{U_p} to $Y \cap U_p$ coincide.
3) The restriction of b'_{U_p} to $(U'_1 \cup \ldots \cup U'_{p-1}) \cap U_p$ is in $\mathcal{A}$.

By item (2), for every $V \in \mathfrak{U}$ we may define a $\mathcal{B}$-arrow $b_V' \colon N_V \longrightarrow N_V$ by $b_V' = b_V$ over $Y \cap V$ and $b_V' = b_V \circ \alpha_{VU_p} \circ b_{U_p}^{-1} \circ b_{U_p}' \circ \alpha_{VU_p}^{-1}$ on $U_p \cap V$. As required, if $V \cap U_p = \emptyset$, then $b_V' = b_V$.

Now we claim that $b_V' \circ \alpha_{VU} \circ b_U'^{-1} = b_V \circ \alpha_{VU} \circ b_U^{-1}$ over $U \cap V$ for every $U, V \in \mathfrak{U}$. Over $U \cap V \cap Y$ we have $b_V' = b_V$ and $b_U' = b_U$, hence, $b_V' \circ \alpha_{VU} \circ b_U'^{-1} = b_V \circ \alpha_{VU} \circ b_U^{-1}$. Over $U \cap V \cap U_p$ we have $b_V' \circ \alpha_{VU} \circ b_U'^{-1} = (b_V' \circ \alpha_{VU_p} \circ b_{U_p}'^{-1}) \circ (b_U' \circ \alpha_{UU_p} \circ b_{U_p}'^{-1})^{-1}$ which, by the definition of b_U' and b_V', is equal to $(b_V \circ \alpha_{VU_p} \circ b_{U_p}^{-1}) \circ (b_U \circ \alpha_{UU_p} \circ b_{U_p}^{-1})^{-1} = b_V \circ \alpha_{VU} \circ b_U^{-1}$. Since $U_p \cup Y = \mathcal{T}$, this proves that claim.

Finally, we claim that b_V' is in $\mathcal{A}$ over $(U_1' \cup \ldots \cup U_p') \cap V$, for every $V \in \mathfrak{U}$. Over $(U_1' \cup \cdots \cup U_p') \cap V \cap U_p$ we have $b_V' = (b_V \circ \alpha_{VU_p} \circ b_{U_p}^{-1}) \circ b_{U_p}' \circ \alpha_{VU_p}^{-1}$, the first and third factors are in $\mathcal{A}$ by assumption, and the second is in $\mathcal{A}$ by items (1) and (3). Hence, their product is in $\mathcal{A}$. Over $(U_1' \cup \cdots \cup U_p') \cap V \cap Y$, b_V' is equal to b_V. Since $U_p' \cap Y = \emptyset$, this set is the same as $(U_1' \cup \cdots \cup U_{p-1}') \cap V \cap Y$, on which b_V is in $\mathcal{A}$ by the induction hypothesis. Since $U_p \cup Y = \mathcal{T}$, we are done. $\qquad\square$

13. Rigidification of Φ-diffeomorphisms

In this section we prove that the sheaf $\hat{\mathcal{Q}}$ of Φ-diffeomorphisms has the same first cohomology as the subsheaf $\mathcal{RQ}$ of locally rigid Φ-homeomorphisms.

13.1. Definitions. Consider two complexity one Hamiltonian T-manifolds M and M'. (As a special case we allow complexity one models.) Let $\mathcal{W} \subseteq [0,1] \times M/T$ be an open subset, and let

$$G \colon \mathcal{W} \longrightarrow M'/T$$

be an isotopy of Φ-homeomorphisms. (See Definition 11.4.) Let $g_t(\cdot) = G(t, \cdot)$.

Definition 13.1. G is an **isotopy of Φ-diffeomorphisms** if near each exceptional orbit (t_0, E) it lifts to a smooth map $\tilde{G}$ from an open subset of $[0,1] \times M$ to M' such that $\tilde{G}(t, \cdot)$ is a Φ-T-diffeomorphism with its image for all t.

Definition 13.2. G is **rigid** at an exceptional orbit (t_0, E) if for any pair of sub-grommets $\overline{\psi} \colon D/T \longrightarrow M/T$ and $\overline{\psi}' \colon D/T \longrightarrow M'/T$ whose images contain E and $g_{t_0}(E)$ there exists a smooth function $R \colon [0,1] \times \mathfrak{t}^* \longrightarrow \overline{R}_Y$ such that

$$\overline{\psi}'^{-1} \circ g_t \circ \overline{\psi}(y) = R(t, \overline{\Phi}_Y(y)) \cdot y$$

on some neighborhood of $(t_0, \psi^{-1}(E))$. Note that both sub-grommets must have the same domain $D/T \subseteq Y/T$.

Note that a single Φ-homeomorphism $g\colon W \longrightarrow M'/T$, where W is an open subset of M/T, is locally rigid if and only if it is rigid at every exceptional orbit in W. (See Definition 8.12 and Remark 11.5.)

Definition 13.3. Let Y be a complexity one model. Let $\mathcal{W} \subseteq [0,1] \times Y$ be an open subset. A map

$$(13.4) \qquad\qquad G\colon \mathcal{W} \longrightarrow Y$$

is an **isotopy** of Φ-T-diffeomorphisms if it is smooth and if $g_t(\cdot) = G(t, \cdot)$ is a Φ-T-diffeomorphism (onto its image) for every $t \in [0,1]$. (Here, we have the same model Y in the domain and range.) The isotopy G is **rigid** at an exceptional orbit (t_0, E) if it descends to an isotopy of Φ-diffeomorphisms

$$\overline{G}\colon \mathcal{W}/T \longrightarrow Y/T$$

which is rigid at (t_0, E). (See Definition 13.2.)

Lemma 13.5. *An isotopy of Φ-T-diffeomorphisms* (13.4) *is rigid at an exceptional orbit (t_0, E) if and only if there exists a smooth T-invariant function*

$$S\colon \mathcal{W} \longrightarrow R_Y$$

such that $G(t, y) = S(t, y) \cdot y$ on some neighborhood of (t_0, E) and such that the composition of S with the projection map $R_Y \longrightarrow \overline{R}_Y$ is equal to the pullback via Φ_Y of a smooth function from $[0,1] \times \mathfrak{t}^*$ to $\overline{R}_Y$.

Proof. By definition, there exists a smooth function $R\colon [0,1] \times \mathfrak{t}^* \longrightarrow \overline{R}_Y$ so that $\overline{G}(t, y) = R(t, \overline{\Phi}_Y(y)) \cdot y$ on some neighborhood of (t_0, E), where $\overline{G}\colon \mathcal{W}/T \longrightarrow Y/T$ is the map induced by G. Let $\tilde{R}\colon [0,1] \times \mathfrak{t}^* \longrightarrow R_Y$ be a smooth map which is a lift of R on some neighborhood of $(t_0, \Phi_Y(E))$. For each (t, y) in a neighborhood of (t_0, E), the values $G(t, y)$ and $\tilde{R}(t, \Phi_Y(y)) \cdot y$ are in the same T-orbit. This implies that these maps differ on this neighborhood by a smooth T-invariant map from $\mathcal{W}$ to T, by a theorem in [**HS**] (see [**KT**, Theorem 4.12]). We let S be the product of this map with $\tilde{R} \circ \Phi_Y$. $\qquad\square$

13.2. Rigidification on $\mathbb{C}^{h+1}$. Let an h dimensional group H act on $\mathbb{C}^{h+1}$ through an inclusion map $\rho = (\rho_0, \ldots, \rho_h)\colon H \longrightarrow (S^1)^{h+1}$. Let L_ρ denote the group of $\mathbb{R}$-linear automorphisms of $\mathbb{C}^{h+1}$ that preserve orientation, commute with the H-action, and preserve the moment map $\Phi_H\colon \mathbb{C}^{h+1} \longrightarrow \mathfrak{h}^*$. Let $R_\rho \subset L_\rho$ denote the subgroup of unitary transformations that commute with the H-action, as in Section 8.

Lemma 13.6. R_ρ is an H-equivariant smooth strong deformation retract of L_ρ.

Proof. Let $\eta_j = d\rho_j \in \mathfrak{h}^*$ denote the weights for the action. Note that $\eta_j \neq 0$ if and only if $\rho_j(H) = S^1$.

Let us first assume that ρ_0 is equal to either ρ_1 or to ρ_1^{-1}. The complexity one assumption then implies that $\rho_j(H) = S^1$ and hence $\eta_j \neq 0$ for all j. It also implies that ρ_i is different from both ρ_j and ρ_j^{-1}, and hence they define non-isomorphic real representations of H on $\mathbb{C} = \mathbb{R}^2$, for all $1 \leq i < j \leq h$. The group of $\mathbb{R}$-linear transformations of $\mathbb{C}^{h+1}$ that commute with the H-action is, by Schur's lemma,

$$(13.7) \qquad\qquad A_1 \times \ldots \times A_h,$$

where A_1 consists of the $\mathbb{R}$-linear transformations of $\mathbb{C}^2$ which commute with the action of H by (ρ_0, ρ_1) and where $A_j = \mathbb{C}^\times$ for $2 \leq j \leq h$. We now have two sub-cases.

1) Suppose that $\rho_0 = \rho_1$. Then $A_1 = \mathrm{GL}(2, \mathbb{C})$. The moment map is

$$\Phi_H = \frac{1}{2} \left(\eta_1 \left(|z_0|^2 + |z_1|^2 \right) + \sum_{j=2}^{h} \eta_j |z_j|^2 \right).$$

The subgroup of (13.7) consisting of those elements that preserve (orientation and) the moment map is $L_\rho = U(2) \times (S^1)^{h-1}$. Thus, $L_\rho = R_\rho$.

2) Suppose that $\rho_0 = \rho_1^{-1}$. We apply the $\mathbb{R}$-linear transformation

$$(w_0, w_1, \ldots, w_h) = (\overline{z_0}, z_1, \ldots, z_h).$$

In these new coordinates, H acts by $(\rho_1, \rho_1, \rho_2, \ldots, \rho_h)$, and $A_1 = \mathrm{GL}(2, \mathbb{C})$. The moment map is

$$\Phi_H = \frac{1}{2} \left(\eta_1 \left(-|w_0|^2 + |w_1|^2 \right) + \sum_{j=2}^{h} \eta_j |z_j|^2 \right).$$

The subgroup of (13.7) consisting of those elements that preserve (orientation and) the moment map is $L_\rho = U(1,1) \times (S^1)^{h-1}$. The group $R_\rho = (S^1)^{h+1}$ of rigid maps is a strong deformation retract of L_ρ.

Up to permutation, the only other case is that ρ_i is different from both ρ_j and ρ_j^{-1} for all $i \neq j$. The group of real linear transformations of $\mathbb{C}^{h+1}$ that commute with the H action is, by Schur's lemma,

$$(13.8) \qquad\qquad A_0 \times \ldots \times A_h,$$

where each A_i is the commutator of $\rho_i(H)$ in $\mathrm{GL}(2,\mathbb{R})$. The group of rigid maps is $R_\rho = (S^1)^{h+1}$. We again distinguish between two subcases:

1) Assume that $\rho_0(H)$ is finite. The complexity one condition then implies that $\rho_j(H) = S^1$, so $A_j = \mathbb{C}^\times$, for all $1 \le j \le h$. We have $\eta_0 = 0$, and the moment map is

$$\Phi_H = \frac{1}{2} \sum_{j=1}^{h} \eta_j |z_j|^2$$

with $\eta_j \ne 0$ for $1 \le j \le h$.

If $\rho_0(H) \not\subseteq \{1, -1\}$, then $A_1 = \mathbb{C}^\times$. If $\rho_0(H) \subseteq \{1, -1\}$, then $A_1 = \mathrm{GL}(2,\mathbb{R})$. The subgroup of (13.8) consisting of those elements that preserve the orientation and the moment map is $L_\rho = \mathbb{C}^\times \times (S^1)^h$ or $L_\rho = \mathrm{GL}^+(2,\mathbb{R}) \times (S^1)^h$, respectively. In either case, $R_\rho = (S^1)^{h+1}$ is a strong deformation retract of L_ρ.

2) Up to permutation, the only other case is where $\rho_i(H) = S^1$ for all i. Then the group (13.8) is $(\mathbb{C}^\times)^{h+1}$. The moment map is

$$\Phi_H = \frac{1}{2} \sum_{j=0}^{h} \eta_j |z_j|^2$$

with $\eta_j \ne 0$ for all j. The subgroup of (13.8) consisting of those elements of $(\mathbb{C}^\times)^{h+1}$ that preserve (the orientation and) the moment map is $L_\rho = (S^1)^{h+1}$. Thus $L_\rho = R_\rho$.

$\square$

13.3. Rigidification on a local model. Consider a complexity one model

$$Y = T \times_H \mathbb{C}^{h+1} \times \mathfrak{h}^0$$

with moment map $\Phi_Y \colon Y \longrightarrow \mathfrak{t}^*$.

Definition 13.9. We let $\mathbb{R}_+ = \{s \in \mathbb{R} \mid s \ge 0\}$ act on $Y = T \times_H \mathbb{C}^{h+1} \times \mathfrak{h}^0$ by

$$\mu_s([\lambda, z, \nu]) = [\lambda, sz, \nu], \quad s \in \mathbb{R}_+.$$

Lemma 13.10. *Fix a smooth function* $s \colon [0, 1] \longrightarrow [0, 1]$ *such that* $s(t) = 0$ *for t in an open set containing* $[\frac{1}{2}, 1]$ *and* $s(0) = 1$. *To a T-invariant open subset* $W \subseteq Y$ *we associate an open subset* $\mathcal{W} \subseteq [0, 1] \times Y$ *by*

$$\mathcal{W} := \{(t, w) \mid \mu_{s(t)}(w) \in W\}.$$

Let $g \colon W \longrightarrow W' \subset Y$ be any Φ-T-diffeomorphism. Then there exists an isotopy of Φ-T-diffeomorphisms, $G \colon \mathcal{W} \longrightarrow Y$, with the following properties. Denote $g_t(\cdot) = G(t, \cdot)$.

1) *$g_0 = g$.*
2) *g_1 is rigid at every exceptional orbit E.*
3) *If g is rigid at $\mu_{s(t)}(E)$ then G is rigid at (t, E), for any exceptional orbit E and any t.*

(See Definitions 13.3 and 13.9.)

Proof. Write $W = T \times_H V$, where $V \subseteq \mathbb{C}^{h+1} \times \mathfrak{h}^0$ is open and H-invariant. Since g is a Φ-T-diffeomorphism, it locally has the form

$$(13.11) \qquad g([\lambda, z, \nu]) = [\tau(z, \nu) \cdot \lambda, f(z, \nu), \nu],$$

where $\tau \colon V \longrightarrow T$ is smooth and H-invariant, and where $f \colon V \longrightarrow \mathbb{C}^{h+1}$ is smooth, and for each $\nu \in \mathfrak{h}^0$, the map $f(\cdot, \nu)$ is an H-equivariant diffeomorphism between open subsets of $\mathbb{C}^{h+1}$ that preserves the orientation and the moment map Φ_H. Because the origin is exceptional, $f(0, \nu) = 0$ for all ν.

Let $f_0(\nu) \colon \mathbb{C}^{h+1} \longrightarrow \mathbb{C}^{h+1}$ be the $\mathbb{R}$-linear map obtained as the derivative of $f(\cdot, \nu)$ at the origin. For $0 \le t \le \frac{1}{2}$, define

(13.12)

$$g_t([\lambda, z, \nu]) = \begin{cases} \left[\tau(s(t)z, \nu) \cdot \lambda, \frac{1}{s(t)} f(s(t)z, \nu), \nu \right] & \text{if } 0 < s(t) \le 1 \\ [\tau(0, \nu) \cdot \lambda, f_0(\nu)(z), \nu] & \text{if } s(t) = 0. \end{cases}$$

The map $f_0(\nu)$ belongs to the group L_ρ of $\mathbb{R}$-linear automorphisms of $\mathbb{C}^{h+1}$ that preserve orientation, commute with the H-action, and preserve the moment map Φ_H. Lemma 13.6 gives a smooth family of H-equivariant maps $D_\sigma \colon L_\rho \longrightarrow L_\rho$, for $0 \le \sigma \le 1$, such that $D_0 = \mathrm{id}$, image $D_1 = R_\rho$, and $D_\sigma|_{R_\rho} = \mathrm{id}|_{R_\rho}$ for all σ, where R_ρ is the group of rigid maps of $\mathbb{C}^{h+1}$.

Let $\sigma \colon [\frac{1}{2}, 1] \longrightarrow [0, 1]$ be a smooth function such that $\sigma(t) = 0$ for t near $\frac{1}{2}$ and $\sigma(1) = 1$. For $\frac{1}{2} \le t \le 1$, define

$$(13.13) \qquad g_t([\lambda, z, \nu]) = [\tau(0, \nu) \cdot \lambda, D_{\sigma(t)}(f_0(\nu))(z), \nu].$$

Note that whereas τ and f in (13.11) can be chosen in different ways, and can only be chosen locally, g_t only depends on g and is therefore well defined.

Moreover, if g is rigid at $\mu_{s(t_0)}(E)$, then there exists a T-invariant smooth function $S \colon Y \longrightarrow R_Y$ so that $g(y) = S(y) \cdot y$ on some neighborhood of $\mu_{s(t_0)}(E)$, and so that the composition of S with the projection

from R_Y to $\overline{R}_Y$ is the pull-back of a smooth function on $\mathfrak{t}^*$. (See Lemma 13.5.) On a neighborhood of (t_0, E), g_t is given by multiplication by $S(\mu_{s(t)}(y))$. For $0 \leq t \leq \frac{1}{2}$ this is a straightforward computation. For $\frac{1}{2} \leq t \leq 1$ this follows from the fact that the deformation D_σ fixes R_ρ. Finally, by the formulas for μ_s and for Φ_Y, the function $(t, y) \mapsto S(\mu_{s(t)}(y))$ satisfies the properties in Lemma 13.5. $\qquad\square$

13.4. Rigidification locally on M/T.

Lemma 13.14. *Let $(M, \omega, \Phi, \mathcal{T})$ and $(M', \omega', \Phi', \mathcal{T})$ be grommeted complexity one proper Hamiltonian T-manifolds. Let $\psi\colon D/T \longrightarrow M/T$ and $\psi'\colon D/T \longrightarrow M'/T$ be sub-grommets with the same domain $D \subseteq Y$. Suppose that D/T is contractible. Let $C \subset Y$ be a set of exceptional orbits whose closure in Y is contained in D. Let $X \subseteq D$ be a set of exceptional orbits such that $\mu_s(X) \subseteq X$ for all $0 \leq s \leq 1$. (See Definition 13.9.)*

For any Φ-diffeomorphism $f\colon M/T \longrightarrow M'/T$ that sends $\psi(C/T)$ to $\psi'(C/T)$ there exists an isotopy of Φ-diffeomorphisms $F\colon [0,1] \times M/T \longrightarrow M'/T$ with the following properties. Denote $f_t = F(t, \cdot)$.

1) *$f_0 = f$.*
2) *f_1 is rigid at every exceptional orbit in $\psi(C/T)$.*
3) *$f_t = f$ outside $\psi(D/T) \cap f^{-1}(\psi'(D/T))$.*
4) *If f is rigid at every orbit in $\psi(X/T)$, then F is rigid at every orbit in $[0,1] \times \psi(X/T)$.*

(See Definitions 13.1 and 13.2.)

REMARK 13.15. In fact, the isotopy F depends "smoothly" on the function f. This feature is relevant for the study of the space of automorphisms of a complexity one space. A similar situation occurs in Section 15 but not in Section 14.

Proof of Lemma 13.14. Once and for all, we choose a T-invariant smooth function

$$\rho\colon Y \longrightarrow \mathbb{R}$$

such that

$$\text{support}(\rho) \subset D \quad \text{and} \quad \rho|_V \equiv 1$$

for some open neighborhood V of C, such that ρ is a pullback of a smooth function on $\mathfrak{t}^*$ on some neighborhood of the exceptional orbits.

Define open subsets of Y by

$$D_f := \{b \in D \mid f(\psi(b)) \in \psi'(D/T)\}, \quad \text{and}$$
$$D'_f := \{b' \in D \mid \psi'(b') \in f(\psi(D/T))\}.$$

Because D/T is contractible, $\psi^{-1} \circ f \circ \psi$ lifts to a Φ-T-diffeomorphism $g \colon D \longrightarrow Y$. In fact, it is enough to assume that $H^2(D/T, \mathbb{Z}) = 0$. This is explained in our proof of Lemma 4.11 in [**KT**], following techniques of [**HS**] and [**BM**].

Our maps fit into a commutative diagram:

$$
\begin{array}{ccccccccc}
Y \supset & D & \supset D_f & \xrightarrow{\ g\ } & D'_f \subset & D & \subset Y \\
& \overline{\psi} \downarrow & & & & \downarrow \overline{\psi'} & \\
& M/T & & \xrightarrow{\ f\ } & & M'/T. &
\end{array}
$$

Moreover, $C \subset D_f \cap D'_f$ and $\overline{g}|_{C/T} = \mathrm{id}|_{C/T}$.

We now apply Lemma 13.10 with $W = D_f$ to obtain an isotopy $G(t, \cdot) = g_t(\cdot)$,

$$
g_t \colon \mu_{s(t)}^{-1} D_f \longrightarrow \mu_{s(t)}^{-1} D'_f.
$$

Let ξ_t be the vector field which generates this isotopy. This means that ξ_t is a vector field defined on $\mu_{s(t)}^{-1} D'_f$ for each t, so that

$$
\frac{dg_t}{dt}(u) = \xi_t \circ g_t(u)
$$

for each $u \in \mu_{s(t)}^{-1} D_f$ (as an equality in $T_{g_t(u)} Y$). Let ξ_t^{cutoff} be the vector field on $D'_f \cap \mu_{s(t)}^{-1} D'_f$ given by

$$
(13.16) \quad \xi_t^{\mathrm{cutoff}}(u) := \rho(u)\, \rho(g^{-1}(u))\, \rho(\mu_{s(t)}(u))\, \rho(g^{-1}(\mu_{s(t)}(u)))\, \xi_t(u).
$$

First, note that the support of ξ_t^{cutoff} in Y is contained in the open set $D'_f \cap \mu_{s(t)}^{-1} D'_f$, so ξ_t^{cutoff} extends to a smooth vector field ξ_t^{cutoff} on all of Y. Construct an isotopy $h_t \colon D_f \longrightarrow Y$ by solving the ordinary differential equation

$$
\frac{dh_t}{dt} = \xi_t^{\mathrm{cutoff}} \circ h_t
$$

with initial condition

$$
h_0 = g.
$$

Since our cut-off functions are constant on orbits, and since g_t is T equivariant, h_t is also T equivariant. Similarly, h_t respects the moment maps.

Second, note that there exists a closed subset of D'_f such that the support of ξ_t^{cutoff} is contained in this set for all t. Therefore, each h_t is a diffeomorphism from D_f onto D'_f, which coincides with g on a neighborhood of the boundary of D_f in Y. Using the sub-grommets ψ and ψ', the isotopy h_t can be plugged back into M/T to give an isotopy

of Φ-diffeomorphisms $f_t \colon M/T \longrightarrow M'/T$ such that $f_t = f$ outside the image of D_f and such that $\overline{\psi}'^{-1} \circ f_t \circ \overline{\psi} = \overline{h}_t$.

Third, note that ξ_t^{cutoff} coincides with ξ_t on the set $V'_f \cap \mu_{s(t)}^{-1} V'_f$, where $V'_f = \{v' \in V \mid \psi'(v') \in f(\psi(V))\}$ (and V is an open neighborhood of C where $\rho = 1$). The intersection of these sets for all $t \in [0,1]$ has a non-empty interior that contains C. Hence, there exists an open neighborhood of C in D on which $h_t = g_t$ for all t. Hence, $\overline{h}_1 = \overline{\psi}'^{-1} \circ f_1 \circ \overline{\psi}$ is rigid on C/T.

Finally, if f is rigid at every orbit in $\psi(X/T)$, then g is rigid at every orbit in X. Since $\mu_s(X) \subseteq X$ for all $0 \leq s \leq 1$, it follows from the third item of Lemma 13.10 that G is rigid at every orbit in $[0,1] \times X$. From Lemma 13.5 it follows that the time dependent vector field ξ_t has the following property. On a neighborhood of every exceptional orbit, ξ_t is induced by a smooth T-invariant map from $[0,1] \times Y$ to the Lie algebra of R_Y, whose projection to the Lie algebra of $\overline{R}_Y$ is the pullback of a smooth function $[0,1] \times \mathfrak{t}^* \longrightarrow \overline{R}_Y$. This implies that ξ_t^{cutoff} has the same property (because the cut-off functions are pullbacks from $\mathfrak{t}^*$ near exceptional orbits). Again by Lemma 13.5, it follows that h_t is rigid at every orbit in $[0,1] \times X$. $\qquad\square$

13.5. Rigidification globally on M/T. Let $(M, \omega, \Phi, \mathcal{T})$ be a complexity one Hamiltonian T-manifold. For each point $\alpha \in \mathcal{T}$, let $\mathfrak{t}_\alpha$ be the subspace of $\mathfrak{t}$ spanned by all the infinitesimal stabilizers to points in $\Phi^{-1}(\alpha) \subset M$. Define an affine space $A_\alpha = \alpha + \mathfrak{t}_\alpha^0$. Let $p_\alpha \colon \mathfrak{t}^* \longrightarrow A_\alpha$ be the orthogonal projection determined by the fixed metric on $\mathfrak{t}^*$.

Definition 13.17. An open subset V of $\mathcal{T}$ is **orthogonal to the skeleton** if and only if for each $\alpha \in \mathfrak{t}^*$ there exists a neighborhood on which V coincides with a set of the form $p_\alpha^{-1}(V')$ for some open subset $V' \subseteq A_\alpha$.

REMARK 13.18. For every open cover $\{U_i\}$ of $\mathcal{T}$ there exists an open cover $\{U'_i\}$ such that $\overline{U'_i} \subset U_i$ and U'_i is orthogonal to the skeleton for all i.

Proposition 13.19. *Let $(M, \omega, \Phi, \mathcal{T})$ and $(M', \omega', \Phi', \mathcal{T})$ be tall grommeted complexity one proper Hamiltonian T-manifolds. Let*

$$f \colon M/T \longrightarrow M'/T$$

be a Φ-diffeomorphism. Let $W \subseteq \mathcal{T}$ be an open subset which is orthogonal to the skeleton. There exists an isotopy of Φ-diffeomorphisms

$F \colon [0,1] \times M/T \longrightarrow M'/T$ with the following properties. Denote $f_t = F(t, \cdot)$.

1) $f_0 = f$.
2) f_1 is locally rigid.
3) If f is rigid at every exceptional orbit in $\Phi^{-1}(W)/T$, then F is rigid at every exceptional orbit in $[0,1] \times \Phi^{-1}(W)/T$.

(See Definitions 13.17 13.1 and 13.2.)

Proof. Define the **level** of an exceptional orbit to be the dimension of its stabilizer. Let l be an integer. Suppose that f is rigid at all exceptional orbits of level $> l$. Let W_l be the set of exceptional orbits of level l at which f is rigid. Let K_l be the set of exceptional orbits of level l at which f is not rigid. Note that K_l is closed.

For each orbit E in K_l there exist sub-grommets $\overline{\psi}_E \colon D_E/T \longrightarrow M/T$ and $\overline{\psi}'_E \colon D_E/T \longrightarrow M'/T$, such that $\overline{\psi}_E([1,0,0]) = E$, the domain D_E/T is contractible, and f sends the exceptional orbits in $\overline{\psi}_E(D_E/T)$ to the exceptional orbits in $\overline{\psi}'_E(D_E/T)$. Moreover, because W is orthogonal to the skeleton, we can choose the domains such that the set X_E of exceptional orbits in $D_E \cap \Phi_E^{-1}(W)$ satisfies $\mu_s(X_E) \subseteq X_E$ for all $0 \le s \le 1$. (See Definition 13.9.) Since K_l is closed, we can choose a collection of such sub-grommets

$$\overline{\psi}_i \colon D_i/T \longrightarrow M/T \quad \text{and} \quad \overline{\psi}'_i \colon D_i/T \longrightarrow M'/T,$$

with domains $D_i \subset Y_i$ for $i = 1, \ldots, N \le \infty$, and such that the images $\overline{\psi}_i(D_i/T)$ form a locally finite cover of K_l. For each i, choose a subset C_i of D_i/T consisting of exceptional orbits of level l whose closure in Y_i/T is still contained in D_i/T, and such that the images $\overline{\psi}_i(C_i)$ still cover K_l.

By induction on i, we construct a sequence of Φ-diffeomorphisms

$$f_i \colon M/T \longrightarrow M'/T$$

such that f_i is rigid at all points of

$$\Phi^{-1}(W) \cup W_l \cup \bigcup_{j \le i} \overline{\psi}_j(C_j)$$

and a sequence of isotopies

$$F_i \colon [0,1] \times M/T \longrightarrow M'/T$$

such that $F_{i-1}(0, \cdot) = F_i(1, \cdot) = f_i(\cdot)$ for all i.

For $i = 0$, set $f_0 = f$. Given f_{i-1}, let

$$X_i = \Phi_i^{-1}(W) \cup \psi_i^{-1}(W_l \cup \bigcup_{j<i} C_j).$$

Notice that any subset of D_i that consists of orbits of level l is fixed by the $\mathbb{R}_+$ action μ_s on D_i (see Definition 13.9). It follows that $\mu_s(X_i) \subseteq X_i$ for all $0 \leq s \leq 1$. Apply Lemma 13.14 with the sub-grommet $\overline{\psi}_i$ and the subsets C_i and X_i to get an isotopy F_i, and set $f_i(\cdot) := F(1, \cdot)$.

Each point $x \in M/T$ has a neighborhood U and an $n = n(x)$ such that $F_j(s, y) =: f_\infty(y)$ is independent of j and s for all $j \geq n$ and $y \in U$. Therefore, there exists an isotopy of Φ-diffeomorphisms

$$F \colon [0,1] \times M/T \longrightarrow M/T$$

such that $F(0, \cdot) = f(\cdot)$, $F(1-\frac{1}{i+1}) = f_i(\cdot)$ for all i, and $F(1, \cdot) = f_\infty(\cdot)$. Note that $F(1, \cdot)$ is rigid at all exceptional orbits of level $\geq l$. The result now follows by induction on l. $\square$

We are ready to prove Proposition 8.16. We recall its statement:

Proposition 8.16. *The inclusion $\mathcal{RQ} \subset \hat{\mathcal{Q}}$ induces an isomorphism*

$$\mathrm{H}^1(\mathcal{T}, \mathcal{RQ}) \cong \mathrm{H}^1(\mathcal{T}, \hat{\mathcal{Q}}).$$

Proof. The proposition follows from Lemma 12.1 once we show that the assumptions of this lemma are satisfied.

Let $\mathfrak{U} = \{U_i\}$ be any cover and $\beta \in \check{Z}^1(\mathfrak{U}, \hat{\mathcal{Q}})$ be any one cocycle. We can choose an open cover $\{U_i'\}$ such that $\overline{U_i'} \subset U_i$ for each i and such that $U_i' \cap U$ is an orthogonal set with respect to M_U for each $U \in \mathfrak{U}$, where M_U is the complexity one manifold associated to U by β.

Take any U_i and U_j in $\mathfrak{U}$. Let M and M' be the restriction to $Z = U_i \cap U_j$ of M_{U_i} and M_{U_j}. Take any open sets X and Y such that $\overline{X} \cap \overline{Y} = \emptyset$, and let $W = U_1' \cup \ldots \cup U_{p-1}'$, for any integer p. Let $f \colon M/T \longrightarrow M'/T$ be a Φ-diffeomorphism.

Let $\rho \colon Z \longrightarrow [0,1]$ be a smooth function which vanishes on $Y \cap Z$ and is equal to one on $X \cap Z$.

Let f_t be the isotopy obtained from Proposition 13.19. Then $f'(x) = f_{\rho(\Phi(x))}(x)$ fulfills the requirements (1)–(3) of Lemma 12.1. $\square$

14. Rigidification of local stretch maps

In this section we prove that the sheaf of local stretch maps has the same first cohomology as the subsheafs of locally rigid Φ-homeomorphisms and of locally sb-rigid Φ-homeomorphisms.

14.1. Definitions.

Definition 14.1. Let $Y = T \times_H \mathbb{C}^{h+1} \times \mathfrak{h}^0$ be a tall complexity one model with moment map Φ_Y. Its **associated painted surface bundle model** is the polyhedral subset

$$Z = (\text{image } \Phi_Y) \times \mathbb{C} \subseteq \mathfrak{t}^* \times \mathbb{C},$$

and, for each exceptional orbit E in Y, a label on the point $p = F(E)$ of Z, consisting of the isotropy representation at E. Such a point is said to be **painted** by its label; the set of labeled points in Z is the **paint**.

REMARK. We think of Z as a bundle over $(\text{image } \Phi_Y)$ with fiber $\mathbb{C}$.

Let Y and Y' be tall complexity one models, Z and Z' their associated surface bundle models, and $F \colon Y \longrightarrow Z$ and $F' \colon Y' \longrightarrow Z'$ the trivializing homeomorphisms. (See Section 9.) Let $W \subseteq Y/T$ and $\tilde{W} \subseteq Z$ be open subsets such that $F(W) = \tilde{W}$. Let $G \colon [0,1] \times W \longrightarrow Y'/T$ be an isotopy of Φ-homeomorphisms and $\tilde{G} \colon [0,1] \times \tilde{W} \longrightarrow Z'$ be such that $\tilde{g}_t = F' \circ g_t \circ F^{-1}$, where $g_t(\cdot) = G(t, \cdot)$ and $\tilde{g}_t(\cdot) = \tilde{G}(t, \cdot)$.

Definition 14.2. G and $\tilde{G}$ are **isotopies of stretch maps** if there exists a function $R \colon [0,1] \times \Phi(W) \longrightarrow S^1$ and an S^1-invariant function $\lambda \colon [0,1] \times \tilde{W} \longrightarrow \mathbb{R}_{>0}$ such that

$$\tilde{g}_t(\alpha, z) = (\alpha, R(t,\alpha)\lambda(t,\alpha,z) \cdot z)$$

for $(\alpha, z) \in \tilde{W} \subseteq \mathfrak{t}^* \times \mathbb{C}$.

Definition 14.3. $\tilde{G}$ is **sb-rigid** at a painted point (t_0, p) if there exists a smooth function $R \colon [0,1] \times \mathfrak{t}^* \longrightarrow S^1$ such that

$$\tilde{g}_t(\beta, z) = (\beta, R(t,\beta) \cdot z)$$

on some neighborhood of $(t_0, p) \in [0,1] \times Z = [0,1] \times (\text{image } \Phi_Y) \times \mathbb{C}$.

Now, let $(M, \omega, \Phi, \mathcal{T})$ and $(M', \omega', \Phi', \mathcal{T})$ be grommeted tall complexity one Hamiltonian T-manifolds. Let

$$F \colon [0,1] \times M/T \longrightarrow M'/T$$

be an isotopy of Φ-homeomorphisms. Let $f_t(\cdot) = F(t, \cdot)$.

Definition 14.4. F is an **isotopy of local stretch maps** if for every exceptional orbit $E \in M/T$ and the pair of surface bundle grommets $\varphi \colon B \longrightarrow M/T$ and $\varphi' \colon B' \longrightarrow M'/T$ whose images contain E and $f_{t_0}(E)$, the composition $\varphi'^{-1} \circ f_t \circ \varphi$ is an isotopy of stretch maps on some neighborhood of $[0,1] \times E$. (See Definition 14.2.)

Note that Definition 14.4 is consistent with Definition 9.6 for an isotopy that is independent of the parameter t.

Definition 14.5. F is **sb-rigid** at an exceptional orbit (t_0, E) if, for the pair of surface bundle grommets $\varphi \colon B \longrightarrow M/T$ and $\varphi' \colon B' \longrightarrow M'/T$ whose images contain E and $f_{t_0}(E)$, the composition $\varphi'^{-1} \circ f_t \circ \varphi$ is sb-rigid on some neighborhood of (t_0, E). (See Definition 14.3.)

Note that a single Φ-homeomorphism $f \colon W \longrightarrow M'/T$, where W is an open subset of M/T, is locally sb-rigid if and only if it is sb-rigid at every exceptional orbit in W. (See Definition 10.1 and Remark 11.5.)

14.2. Rigidification on a local model. Let Y and Y' be tall complexity one models, Z and Z' their associated surface bundle models, and $F \colon Y \longrightarrow Z$ and $F' \colon Y' \longrightarrow Z'$ the trivializing homeomorphisms. (See Section 9.)

Lemma 14.6. *Let W be an open subset of Y/T. For any stretch map $g \colon W \longrightarrow Y'/T$ there exists an isotopy of stretch maps $G \colon [0,1] \times W \longrightarrow Y'/T$ with the following properties. let $g_t(\cdot) = G(t, \cdot)$.*

1) *$g_0 = g$.*
2) *g_1 is locally rigid.*
3) *If g is rigid at E, then G is rigid at (t, E) for any exceptional orbits E and any $t \in [0,1]$.*

(See Definitions 14.2 and 13.2.)

Proof. First, we define an S^1-invariant continuous function $\tilde{\lambda} \colon F(W) \longrightarrow \mathbb{R}_{>0}$ such that for every pair of canonical inclusions $\Lambda \colon D_E \longrightarrow Y$ and $\Lambda' \colon D_E \longrightarrow Y'$ with the same domain we have

$$(14.7) \qquad F' \circ \overline{\Lambda}' \circ \overline{\Lambda}^{-1} \circ F^{-1}(\alpha, z) = (\alpha, \tilde{\lambda}(\alpha, z) \cdot z)$$

on some neighborhood of the exceptional orbits in $F(W \cap \overline{\Lambda}(D_E/T))$.

By Lemma 9.8, for every such pair of canonical inclusions there exists an S^1 invariant function $\tilde{\lambda}$ satisfying (14.7). These functions agree on the intersections of their domains, by Lemma 8.6 and because compositions of canonical inclusions are canonical inclusions. Hence, we

can define $\tilde{\lambda}$ on a neighborhood of the exceptional orbits in $F(W)$. We extend it arbitrarily to all of $F(W)$.

By assumption there exist $\lambda\colon F(W) \longrightarrow \mathbb{R}_{>0}$ and $R\colon \Phi(W) \longrightarrow S^1$ such that
$$F' \circ g \circ F^{-1}(\alpha, z) = (\alpha, R(\alpha)\lambda(\alpha, z) \cdot z).$$
Define g_t by

(14.8) $$F' \circ g_t \circ F^{-1}(\alpha, z) = (\alpha, R(\alpha)\lambda_t(\alpha, z) \cdot z),$$

where $\lambda_t = (1 - t)\lambda + t\tilde{\lambda}$.

The function λ_t is smooth on the complement of the exceptional orbits, because so are the functions λ and $\tilde{\lambda}$. The maps $z \mapsto \lambda_t(\alpha, z) \cdot z$ have positive derivative everywhere because so do the maps $z \mapsto \lambda(\alpha, z) \cdot z$ and $z \mapsto \tilde{\lambda}(\alpha, z) \cdot z$. It follows that $\{g_t\}$ is an isotopy of Φ-homeomorphisms, and, by (14.8), an isotopy of stretch maps. □

14.3. Rigidification on M/T.

Proposition 14.9. *Let $(M, \omega, \Phi, \mathcal{T})$ and $(M', \omega', \Phi', \mathcal{T})$ be grommeted tall complexity one proper Hamiltonian T-manifolds. For any local stretch map $f\colon M/T \longrightarrow M'/T$ there exists an isotopy of local stretch maps $F\colon [0, 1] \times M/T \longrightarrow M'/T$ with the following properties. Let $f_t(\cdot) = F(t, \cdot)$.*

1) *$f_0 = f$.*
2) *f_1 is locally rigid.*
3) *If f is rigid at an exceptional orbit E, then F is rigid at (t, E) for all t.*

(See Definitions 14.4 and 13.2.)

Proof. We may assume that for each grommet $\psi\colon D \longrightarrow M$ there exists a unique grommet $\psi'\colon D' \longrightarrow M'$ such that f sends the exceptional orbits in $\psi(D)$ to the exceptional orbits in $\psi'(D')$.

Let $W \subseteq D/T$ be a neighborhood of the exceptional orbits on which the composition $g := \psi'^{-1} \circ f \circ \psi$ is a stretch map. We apply Lemma 14.6 to obtain an isotopy of stretch maps
$$g_t\colon W \longrightarrow Y'/T$$
such that g_1 is locally rigid.

Let ξ_t be the vector field on the complement of the exceptional orbits in $g_t(W)$ for each t, such that
$$\frac{dg_t}{dt} = \xi_t \circ g_t.$$

Note that there exists a neighborhood of the exceptional orbits in D' which is contained in $g_t(W)$ for all t. Choose a function

$$\rho \colon (Y'/T)|_{\mathcal{T}} \longrightarrow [0, 1]$$

whose support is contained in $g_t(W) \cap D'/T$ for all t, such that $\rho \equiv 1$ on a neighborhood of the exceptional orbits, and such that the restriction of Φ_Y to the support of ρ is proper.

The vector field

$$\xi_t^{\mathrm{cutoff}} = \rho(u) \cdot \xi_t(u).$$

extends to a smooth vector field on the complement of the exceptional orbits in $(Y'/T)|_{\mathcal{T}}$, supported in D'/T. Our goal is to find maps

$$h_t \colon W \longrightarrow (Y'/T)|_{\mathcal{T}}$$

which satisfy the ordinary differential equation

(14.10)
$$\frac{dh_t}{dt} = \xi_t^{\mathrm{cutoff}} \circ h_t$$

with initial condition $h_0 = g$.

Let $U' \subset (Y'/T)|_{\mathcal{T}}$ be an open neighborhood of the exceptional orbits on which $\rho(\cdot) \equiv 1$. Let $U \subset W$ be an open neighborhood of the exceptional orbits such that $g_t(U) \subseteq U'$ for all $t \in [0, 1]$. If $x \in U$ then $g_t(x)$ solves (14.10).

Let $V \subset W$ be an open set whose closure does not contain any exceptional orbits and such that $V \cup U = W$. Let $V' \subset (Y'/T)|_{\mathcal{T}}$ be an open set such that $\overline{V'}$ does not contain any exceptional orbits and such that $g_t(V) \subseteq V'$ for all $t \in [0, 1]$. Let $\eta_t = \tilde{\rho} \cdot \xi_t^{\mathrm{cutoff}}$ where $\tilde{\rho} \equiv 1$ on $\overline{V'}$ and $\tilde{\rho} \equiv 0$ on a neighborhood of the exceptional orbits. Because $L_{\eta_t} \Phi_{Y'} = 0$ and the restriction of $\Phi_{Y'}$ to the support of η_t in $(Y' \smallsetminus Y'_{\mathrm{exc}})|_{\mathcal{T}}$ is proper, there exists a solution $\tilde{h}_t$ to the differential equation $\frac{d\tilde{h}_t}{dt} = \eta_t \circ \tilde{h}_t$ with initial condition $\tilde{h}_0 = g$, defined on the complement of the exceptional orbits in W.

Because $\eta_t = \xi_t^{\mathrm{cutoff}}$ on V' and the restriction of g_t to $V \cap U$ takes values in V', the restrictions to $V \cap U$ of $\tilde{h}_t$ and g_t coincide. Hence, we can define $h_t := g_t$ on U and $h_t := \tilde{h}_t$ on V, and this solves (14.10).

Because h_t coincides with g near the boundary of W in D/T, we can plug it back into the manifold M/T to get an isotopy of local stretch maps with the required properties. $\square$

We recall the statement of Proposition 9.10.

Proposition 9.10. *The inclusion $\mathcal{RQ} \subset \mathcal{E}$ induces an isomorphism*

$$\mathrm{H}^1(\mathcal{T}, \mathcal{RQ}) \cong \mathrm{H}^1(\mathcal{T}, \mathcal{E}).$$

Proposition 9.10 follows from Proposition 14.9 in exactly the same way that Proposition 8.16 followed from Proposition 13.19, (except that the U_i'''s no longer need to be orthogonal to the skeleton).

14.4. sb-Rigidification.

Lemma 14.11. *Let W be an open subset of Z. For any stretch map*

$$g \colon W \longrightarrow Z'$$

there exists an isotopy of stretch maps

$$G \colon [0, 1] \times W \longrightarrow Z'$$

with the following properties. Let $g_t(\cdot) = G(t, \cdot)$.

1) *$g_0 = g$.*
2) *g_1 is locally sb-rigid.*
3) *If g is sb-rigid at p, then G is sb-rigid at (t, p), for any painted point p and any $t \in [0, 1]$.*

(See Definitions 14.2 and 14.3.)

Proof. Let $\lambda \colon \mathfrak{t}^* \times \mathbb{C}$ and $R \colon \mathfrak{t}^* \longrightarrow S^1$ be such that

$$g(\alpha, z) = (\alpha, R(\alpha)\lambda(\alpha, z) \cdot z).$$

Define

$$g_t(\alpha, z) = (\alpha, R(\alpha)\lambda_t(\alpha, z) \cdot z)$$

where $\lambda_t = (1-t)\lambda + t \cdot 1$. As in the proof of Lemma 14.6, $G(t, \cdot) = g_t(\cdot)$ is an isotopy of stretch maps. $\qquad\square$

Proposition 14.12. *Let $(M, \omega, \Phi, \mathcal{T})$ and $(M', \omega', \Phi', \mathcal{T})$ be grommeted tall complexity one proper Hamiltonian T-manifolds. For any local stretch map $f \colon M/T \longrightarrow M'/T$ there exists an isotopy of local stretch maps $F \colon [0, 1] \times M/T \longrightarrow M'/T$ with the following properties. Let $f_t(\cdot) = F(t, \cdot)$.*

1) *$f_0 = f$.*
2) *f_1 is locally sb-rigid.*
3) *If f is sb-rigid at an exceptional orbit E, then $\{f_t\}$ is sb-rigid at (t, E) for all t.*

(See Definitions 14.4 and 14.5.)

Proof. Proposition 14.12 follows from Lemma 14.11 in the same way that Proposition 14.9 followed from Lemma 14.6. $\square$

We recall the statement of Proposition 10.4:

Proposition 10.4. *The inclusion $\mathcal{RP} \longrightarrow \mathcal{E}$ induces an isomorphism*

$$\mathrm{H}^1(\mathcal{T}, \mathcal{RP}) \cong \mathrm{H}^1(\mathcal{T}, \mathcal{E}).$$

Proposition 10.4 follows from Proposition 14.12 in the same way that Proposition 9.10 followed from Proposition 14.9.

15. Rigidification of sb-diffeomorphisms

In this section we prove that the sheaf of sb-diffeomorphisms has the same first cohomology as the subsheaf of locally sb-rigid Φ-homeomorphisms.

15.1. Definitions.

Definition 15.1. Let Z be a painted surface bundle model and $\mathcal{W} \subseteq [0, 1] \times Z$ an open subset. A map

$$G \colon \mathcal{W} \longrightarrow Z'$$

is an **isotopy of sb-diffeomorphisms** if it sends each painted point to a painted point with the same label, is smooth, and each $g_t(\cdot) = G(t, \cdot)$ is a diffeomorphism with its image.

Definition 15.2. Let M and M' be grommeted tall complexity one Hamiltonian T-manifolds. An isotopy of Φ-homeomorphisms

$$F \colon [0, 1] \times M/T \longrightarrow M'/T$$

is an **isotopy of sb-diffeomorphisms** if for every pair of associated surface bundle grommets $\varphi \colon B \longrightarrow M/T$ and $\varphi' \colon B' \longrightarrow M'/T$ the composition $\varphi'^{-1} \circ f_t \circ \varphi$ is an isotopy of sb-diffeomorphisms. (See Definition 15.1.)

15.2. Rigidification on a local model. Let Y and Y' be tall complexity one models, and let Z and Z' be their associated surface bundle models.

Let $Y = T \times_H \mathbb{C}^{h+1} \times \mathfrak{h}^0$ and $\Phi_Y([t, z, \nu]) = \alpha + (\Phi_H(z), \nu)$, where the splitting $\mathfrak{t}^* = \mathfrak{h}^* \times \mathfrak{h}^0$ is obtained from the metric on $\mathfrak{t}$. Then image Φ_Y is the product of the affine space $A_\alpha = \alpha + \mathfrak{h}^0$ and the set image Φ_H, so that

(15.3) $$Z = A_\alpha \times \text{image } \Phi_H \times \mathbb{C}.$$

Definition 15.4. We define an $\mathbb{R}_+$-action on (15.3) by

$$\mu_s(q, \beta, z) = (q, s\beta, sz) \quad \text{for } s \in \mathbb{R}_+.$$

REMARK 15.5. Definition 15.4 does not correspond to Definition 13.9 under the trivializing homeomorphism $F \colon Y/T \longrightarrow Z$.

Lemma 15.6. *Fix a smooth function* $s \colon [0,1] \longrightarrow [0,1]$ *such that* $s(t) = 0$ *for t in an open set containing* $[\frac{1}{2}, 1]$ *and* $s(0) = 1$. *To an open subset* $W \subseteq Z$ *we associate an open subset* $\mathcal{W} \subseteq [0,1] \times Z$ *by*

$$\mathcal{W} := \{(t, w) \mid \mu_{s(t)}(w) \in W\}.$$

Let $g \colon W \longrightarrow W' \subseteq Z$ *be any sb-diffeomorphism. Then there exists an isotopy of sb-diffeomorphisms,* $G \colon \mathcal{W} \longrightarrow Z$, *with the following properties. Denote* $g_t = G(t, \cdot)$.

1) $g_0 = g$.
2) g_1 *is locally sb-rigid.*
3) *If g is sb-rigid at* $\mu_{s(t)}(p)$ *then G is sb-rigid at* (t, p), *for any painted point p and any t.*

(See Definitions 15.1 and 14.3.)

Proof. Since g is an sb-diffeomorphism, we have

$$g(q, \beta, z) = (q, \beta, h(q, \beta, z)),$$

where $h \colon W \longrightarrow \mathbb{C}$ is smooth, and each map $h(q, \beta, \cdot)$ is an orientation preserving diffeomorphism between open subsets of $\mathbb{C}$ that fixes the origin if $(q, \beta, 0)$ is painted; in particular, $h(q, 0, 0) = 0$.

Let

$$h_0(q) \colon \mathfrak{h}^* \times \mathbb{C} \longrightarrow \mathbb{C}$$

be the $\mathbb{R}$-linear map obtained as the derivative of $h(q, \cdot, \cdot)$ at the origin. The map $h_0(q)$ belongs to the group of $\mathbb{R}$-linear maps from $\mathfrak{h}^* \times \mathbb{C}$ to $\mathbb{C}$ of the form

$$B + A$$

where $A \colon \mathbb{C} \longrightarrow \mathbb{C}$ is in $\mathrm{GL}_2^+(\mathbb{R}^2)$ and $B \colon \mathfrak{h}^* \longrightarrow \mathbb{C}$ is any linear map such that $B(\Phi_H(E)) = 0$ for each exceptional orbit E in $\mathbb{C}^{h+1}$. This group, which we denote L, strongly deformation retracts to the subgroup with $B = 0$, and, further, to the circle subgroup R consisting of maps of the form

$$(\beta, z) \mapsto \lambda z$$

for some $\lambda \in S^1$. Choose a smooth family of maps $D_\sigma \colon L \longrightarrow L$, for $0 \le \sigma \le 1$, such that $D_0 = \mathrm{id}$, image $D_1 = R$, and $D_\sigma|_R = \mathrm{id}|_R$ for all σ.

We first linearize. For $0 \le t \le \frac{1}{2}$, let $s = s(t)$, and define

$$g_t(q, \beta, z) = \begin{cases} (q, \beta, \frac{1}{s}h(q, s\beta, sz)) & \text{if } 0 < s \le 1 \\ (q, \beta, h_0(q)(\beta, z)) & \text{if } s = 0. \end{cases}$$

We now rigidify. Let $\sigma \colon [\frac{1}{2}, 1] \longrightarrow [0, 1]$ be a smooth function such that $\sigma(t) = 0$ for t near $\frac{1}{2}$ and $\sigma(1) = 1$. For $\frac{1}{2} \le t \le 1$, define

$$g_t(q, \beta, z) = (q, \beta, D_{\sigma(t)}(h_0(q))(\beta, z)).$$

$\square$

15.3. Rigidification locally on M/T.
To rigidify, we need to work with grommets in M and M' whose domains are the same. The notion of "sub-grommets" from Section 8 is not good for this purpose, as the "canonical inclusions" are not sb-diffeomorphisms. The "surface bundle grommets" of Definition 9.5 are not good either, because their domains are prescribed. We introduce the notion of "surface bundle sub-grommets".

Definition 15.7. Let M be a grommeted tall complexity one proper Hamiltonian T-manifold. Let $\varphi \colon B \longrightarrow M/T$ be an associated surface bundle grommet (Definition 9.5). Let E be a painted point in B with associated surface bundle model Z_E. Note that B and Z_E are both subsets of $\mathfrak{t}^* \times \mathbb{C}$ which contain the point $E = (\pi(E), 0)$. Any sufficiently small neighborhood B_E of E in Z_E is an open subset of B. In this case, we call the restriction

$$\varphi_E = \varphi|_{B_E} \colon B_E \longrightarrow M/T$$

a **surface bundle sub-grommet** on M.

Lemma 15.8. *Let $(M, \omega, \Phi, \mathcal{T})$ and $(M', \omega', \Phi', \mathcal{T})$ be grommeted tall complexity one proper Hamiltonian T-manifolds. Consider two associated surface bundle sub-grommets, $\varphi \colon B \longrightarrow M/T$ and $\varphi' \colon B \longrightarrow M'/T$, which have the same domain $B \subset Z = F(Y)$. Let $C \subset Z|_{\mathcal{T}}$ be a set of painted points whose closure in $Z|_{\mathcal{T}}$ is contained in B. Let $X \subset B$ be a set of painted points such that $\mu_s(X) \subseteq X$ for all $0 \le s \le 1$.*

For any sb-diffeomorphism $f \colon M/T \longrightarrow M'/T$ that sends $\varphi(C)$ to $\varphi'(C)$, there exists an isotopy of sb-diffeomorphisms $F \colon [0, 1] \times M/T \longrightarrow M'/T$ with the following properties. Denote $f_t = F(t, \cdot)$.

1) *$f_0 = f$.*
2) *f_1 is locally sb-rigid.*
3) *$f_t(s) = f(s)$ for all s outside $\varphi(B) \cap f^{-1}\varphi'(B)$.*

4) *If f is sb-rigid at every exceptional orbit in $\varphi(X)$, then F is sb-rigid at every exceptional orbit in $[0,1] \times \varphi(X)$.*

(See Definition 15.4, 15.2, and 14.5.)

Proof. Once and for all, we choose a smooth function

$$\rho \colon Z \longrightarrow \mathbb{R}$$

such that

$$\operatorname{support}(\rho) \subset B \quad \text{and} \quad \rho|_V \equiv 1$$

for some open neighborhood V of C, such that ρ is a pullback of a smooth function on $\mathfrak{t}^*$ on some neighborhood of the painted points.

Define open subsets of Z by

$$B_f := \{b \in B \mid f(\varphi(b)) \in \varphi'(B)\}, \quad \text{and}$$
$$B'_f := \{b' \in B \mid \varphi'(b') \in f(\varphi(B))\}.$$

Our maps fit into a commutative diagram:

$$
\begin{array}{ccccccc}
Z \supset & B & \supset B_f & \xrightarrow{g} & B'_f \subset & B & \subset Z \\
 & \varphi \downarrow & & & & \downarrow \varphi' & \\
 & M/T & & \xrightarrow{f} & & M'/T &
\end{array}
$$

Moreover, $C \subset B_f \cap B'_f$ and $g|_C = \operatorname{identity}|_C$.

We now apply Lemma 15.6 with $W = B_f$ to obtain an isotopy

$$g_t \colon \mu_{s(t)}^{-1} B_f \longrightarrow \mu_{s(t)}^{-1} B'_f.$$

Let ξ_t be the vector field which generates this isotopy. This means that ξ_t is a vector field defined on $\mu_{s(t)}^{-1} B'_f$ for each t, such that

$$\frac{dg_t}{dt}(u) = \xi_t \circ g_t(u)$$

for each $u \in \mu_{s(t)}^{-1} B_f$ (as an equality in $T_{g_t(u)} Z$). Let ξ_t^{cutoff} be the vector field on $B'_f \cap \mu_{s(t)}^{-1} B'_f$ given by

$$(15.9) \quad \xi_t^{\text{cutoff}}(u) := \rho(u)\, \rho(g^{-1}(u))\, \rho(\mu_{s(t)}(u))\, \rho(g^{-1}(\mu_{s(t)}(u)))\, \xi_t(u).$$

As in the proof of Lemma 13.14, we can construct an isotopy $h_t \colon B_f \longrightarrow Z$ such that

$$\frac{dh_t}{dt} = \xi_t^{\text{cutoff}} \circ h_t$$

and

$$h_0 = g,$$

and plug this back into M/T. This gives an isotopy of sb-diffeomorphisms $f_t \colon M/T \longrightarrow M'/T$ which satisfies the conditions of the lemma. $\quad\square$

15.4. Rigidification globally on M/T.

Proposition 15.10. *Let $(M, \omega, \Phi, \mathcal{T})$ and $(M', \omega', \Phi', \mathcal{T})$ be grommeted tall complexity one proper Hamiltonian T-manifolds. Let*

$$f \colon M/T \longrightarrow M'/T$$

be an sb-diffeomorphism. Let $W \subseteq \mathcal{T}$ be an open subset which is orthogonal to the skeleton. There exists an isotopy of sb-diffeomorphisms $F \colon [0,1] \times M/T \longrightarrow M'/T$ with the following properties. Denote $f_t = F(t, \cdot)$.

1) *$f_0 = f$.*
2) *f_1 is locally sb-rigid.*
3) *If f is sb-rigid at each exceptional orbit in W, then F is sb-rigid at each exceptional orbit in $[0,1] \times W$.*

Proof. This follows from Lemma 15.8 in the same way that Proposition 13.19 followed from Lemma 13.14. ☐

We recall the statement of Proposition 11.3:

Proposition 11.3. *The inclusion $\mathcal{RP} \subset \hat{\mathcal{P}}$ induces an isomorphism*

$$\mathrm{H}^1(\mathcal{T}, \mathcal{RP}) \cong \mathrm{H}^1(\mathcal{T}, \hat{\mathcal{P}}).$$

Proposition 11.3 follows from Proposition 15.10 in the same way that Proposition 8.16 followed from Proposition 13.19.

Part V: Paintings

16. Surface bundles

Let $(M, \omega, \Phi, \mathcal{T})$ be a tall complexity one space. The map $\overline{\Phi} \colon M/T \longrightarrow \mathcal{T}$ is topologically a fiber bundle over $\Phi(M)$ whose fibers are surfaces. (See Proposition 2.2.) Because $\Phi(M)$ is a polyhedral subset of $\mathcal{T}$, the complexity one quotient M/T is, topologically, a manifold with corners. However, smoothly this is only true outside the exceptional orbits. In this section we introduce *painted surface bundles* over $\Phi(M)$. Just like M/T, a painted surface bundle comes with a subset that is "painted" by isotropy data. Unlike M/T, a painted surface bundle is a manifold with corners *everywhere*. It is relatively easy to determine whether two surface bundles are isomorphic, and this will enable us to determine whether two complexity one quotients are Φ-diffeomorphic.

Definition 16.1. A **skeleton** over an open set $\mathcal{T} \subset \mathfrak{t}^*$ is a topological space S whose points are labeled by representations of subgroups of T, together with a proper map $\pi \colon S \longrightarrow \mathcal{T}$. This data must be locally modeled on the set of exceptional orbits of a complexity one space in the following sense. For each point $s \in S$, there exists a tall complexity one model $Y = T \times_H \mathbb{C}^{h+1} \times \mathfrak{h}^0$ with exceptional orbits $Y_{\mathrm{exc}} \subset Y/T$, and a homeomorphism Ψ from a neighborhood of s to an open subset of Y_{exc} which respects the labels and such that $\overline{\Phi}_Y \circ \Psi = \pi$, where $\Phi_Y \colon Y \longrightarrow \mathfrak{t}^*$ is the moment map.

EXAMPLE 16.2. The set M_{exc} of exceptional orbits in a tall complexity one space is naturally a skeleton.

Note that if (S, π) is a skeleton over $\mathcal{T}$, then the map $\pi \colon S \longrightarrow \mathcal{T}$ is a local embedding. Also, given an open subset $U \subset \mathcal{T}$, the restriction $S|_U := S \cap \pi^{-1}(U)$ is a skeleton over U.

If (S, π) and (S', π') are skeletons, an **isomorphism** from S to S' is a homeomorphism $i \colon S \longrightarrow S'$ that sends each point to a point with the same isotropy data and such that $\pi = \pi' \circ i$.

Let (S, π) be a skeleton. A function $\varphi \colon S \longrightarrow \mathbb{R}$ is **smooth** if for each point $s \in S$ there exists a neighborhood U of s in S and a neighborhood W of $\pi(s)$ in $\mathcal{T}$ and a smooth function $\tilde{\varphi} \colon W \longrightarrow \mathbb{R}$ such that $\tilde{\varphi}(\pi(x)) = \varphi(x)$ for all $x \in U$. More generally, if X is a manifold (with corners), a map from S to X is **smooth** if the pull-back of every smooth function on X is a smooth function on S.

Definition 16.3. Let $\mathcal{T} \subseteq \mathfrak{t}^*$ be an open subset. A **painted surface bundle** over $\mathcal{T}$ is a manifold-with-corners N,[4] together with a proper map $\pi \colon N \longrightarrow \mathcal{T}$ whose nonempty fibers are smooth oriented surfaces, and a "painted" subset $P \subset N$, whose points are labeled by representations of subgroups of T, subject to the following conditions. First, every point in N has a neighborhood U and a diffeomorphism $U \cong \pi(U) \times (\text{a disk})$ which carries π to the projection map to $\pi(U)$, Second, P is a skeleton, and the inclusion map from P to N is smooth.

Definition 16.4. An **isomorphism** between painted surface bundles is a diffeomorphism which respects the maps to $\mathfrak{t}^*$, the orientation on the fibers, and the paint.

---

[4] A manifold with corners structure on N is provided by an atlas consisting of homeomorphisms $X_i \colon U_i \longrightarrow \Omega_i$, where $\{U_i\}$ is an open cover of N and Ω_i is an open subset of $\mathbb{R}^n_+$, such that the transition maps $X_j \circ X_i^{-1}$ are smooth. Here, a function on a subset of $\mathbb{R}^n$ is **smooth** if it extends to a smooth function on an open subset of $\mathbb{R}^n$. See [**DH**].

Definition 16.5. Let $(M, \omega, \Phi, \mathcal{T})$ be a grommeted tall complexity one proper Hamiltonian T-manifold. The **associated painted surface bundle** consists of the following data:

1) The topological manifold-with-corners $N_M = M/T$, together with the map $\pi \colon N_M \longrightarrow \mathcal{T}$ that is induced by the moment map, and the orientation on each fiber of π obtained from the symplectic orientation of the reduced space $\Phi^{-1}(\alpha)/T$.

2) The manifold-with-corners structure on N_M that is given by the following coordinate charts. Choose arbitrary grommets whose images cover the complement of the exceptional orbits in M and are contained in this complement. For each given grommet and each chosen grommet, take the associated surface bundle grommet. (See Definition 9.5.)

3) The subset P of N_M consisting of the exceptional orbits, together with a label for each $p \in P$ consisting of the isotropy representation of the corresponding exceptional orbit in M. We call this information the **paint**.

The fact that the coordinate charts in item (2) give a well defined smooth structure on M/T follows from the facts that the smooth structures given by the different grommets coincide outside the set of exceptional orbits (see Lemma 9.4), and that each exceptional orbit lies in the image of only one grommet. The fact that we get a manifold with corners follows from [**KT**, Lemmas 4.7 and 7.1].

Note that a map from M/T to M'/T is an sb-diffeomorphism if and only if it is an isomorphism of the associated painted surface bundles. Here, M and M' are grommeted tall complexity one proper Hamiltonian T-manifolds.

Definition 16.6. A painted surface bundle N over $\mathcal{T}$ is **legal** if there exists a cover $\{W_\alpha\}$ of $\mathcal{T}$ and for each α there exists a grommeted tall complexity one proper Hamiltonian T-manifold M_α whose associated painted surface bundle is isomorphic to $N|_{W_\alpha}$.

We introduce the sheaf $\mathcal{P}$ of isomorphisms of surface bundles. To each subset $U \subset \mathcal{T}$ we associate a groupoid $\mathcal{P}(U)$. The objects in $\mathcal{P}(U)$ are the legal painted surface bundles over U. The arrows are the isomorphisms of painted surface bundles.

There is a natural inclusion map from $\hat{\mathcal{P}}$ to $\mathcal{P}$, which associates to each grommeted complexity one space its associated painted surface bundle and to each sb-diffeomorphism the corresponding isomorphism of painted surface bundles.

Lemma 16.7. *The map* $\hat{\mathcal{P}} \longrightarrow \mathcal{P}$ *induces an isomorphism in cohomology*

$$\mathrm{H}^1(\mathcal{T}, \hat{\mathcal{P}}) \cong \mathrm{H}^1(\mathcal{T}, \mathcal{P}).$$

Proof. This follows immediately from Lemma 6.4. $\qquad\square$

17. Global objects: the exciting transition

We are now ready, at last, to leave the world of sheaves of groupoids, and return to to the case where we have a single global object: a painted surface bundle.

First, we extend Definition 7.1 as follows:

Definition 17.1. A **Φ-homeomorphism** between a complexity one quotient and a painted surface bundle is a homeomorphism which respects the paint, is a diffeomorphism off the paint, respects the maps to $\mathcal{T}$, and preserves orientations on the level sets of these maps.

EXAMPLE 17.2. Suppose that M is grommeted. The identity map from M/T to the associated surface bundle is a Φ-homeomorphism.

Let $\mathcal{Q}$ be the sheaf of Φ-diffeomorphisms as defined in Section 6, and let $\mathcal{P}$ be the sheaf of isomorphisms of painted surface bundles as defined in Section 16.

By combining Lemmas 6.3, 8.16. 9.10, 10.4, 11.3, and 16.7, we get an isomorphism

(17.3) $$\mathrm{H}^1(\mathcal{T}, \mathcal{Q}) \cong \mathrm{H}^1(\mathcal{T}, \mathcal{P}).$$

By Lemma 5.4, every legal painted surface bundle N determines an element $[N]$ of $\mathrm{H}^1(\mathcal{T}, \mathcal{P})$. Similarly, every tall complexity one proper Hamiltonian T-manifold M determines an element $[M]$ of $\mathrm{H}^1(\mathcal{T}, \mathcal{Q})$.

Definition 17.4. Let $(M, \omega, \Phi, \mathcal{T})$ be a tall complexity one proper Hamiltonian T-manifold. An **associated painted surface bundle** is a legal painted surface bundle N such that $[M] \in \mathrm{H}^1(\mathcal{T}, \mathcal{P})$ and $[N] \in \mathrm{H}^1(\mathcal{T}, \mathcal{Q})$ correspond under the isomorphism (17.3).

Proposition 17.5. *Let* $(M, \omega, \Phi, \mathcal{T})$ *be a tall complexity one proper Hamiltonian T-manifold. Up to isomorphism, there exists a unique associated painted surface bundle N, and it is Φ-homeomorphic to M/T.*

If $(M', \omega', \Phi, \mathcal{T})$ *is another tall complexity one proper Hamiltonian T-manifold, and N' is a painted surface bundle associated to M', then M/T and M'/T are Φ-diffeomorphic if and only if N and N' are isomorphic.*

Proof. Let $c_M \in \mathrm{H}^1(\mathcal{T}, \mathcal{P})$ be the class that corresponds to $[M]$ under the isomorphism (17.3). Because the sheaf $\mathcal{P}$ has gluable objects, every element of $\mathrm{H}^1(\mathcal{T}, \mathcal{P})$ comes from a global object, and this object is unique up to isomorphism. (See Lemma 5.4.) In particular, there exists a legal painted surface bundle N with $[N] = c_M$, and it is unique up to isomorphism.

By Lemma 5.4, M/T and M'/T are Φ-diffeomorphic and only if $[M] = [M']$ in $\mathrm{H}^1(\mathcal{T}, \mathcal{Q})$. Similarly, N and N' are isomorphic if and only if $[N] = [N']$ in $\mathrm{H}^1(\mathcal{T}, \mathcal{P})$. Because the map (17.3) that sends $[M]$ to $[N]$ and $[M']$ to $[N']$ is an isomorphism, $[M] = [M']$ if and only if $[N] = [N']$.

It remains to prove that if $[M]$ corresponds to $[N]$ then M/T is Φ-homeomorphic to N. Let $\mathfrak{U}$ be a countable cover of $\mathcal{T}$ so that $M|_U$ can be grommeted for all $U \in \mathfrak{U}$, and so that each open set in $\mathfrak{U}$ intersects only finitely many other open sets. Let $a \in Z^1(\mathfrak{U}, \hat{\mathcal{Q}})$ be a cocycle such that for each $U \in \mathfrak{U}$ the object a_u is $M|_U$ with some choice of grommets, and the arrows are the identity maps. Clearly, under the isomorphism $\mathrm{H}^1(\mathcal{T}, \hat{\mathcal{Q}}) \longrightarrow \mathrm{H}^1(\mathcal{T}, \mathcal{Q})$, $[a]$ maps to $[M]$.

Let $[b] \in \mathrm{H}^1(\mathfrak{U}, \hat{\mathcal{P}})$ be the image of $[a]$ under the isomorphism of $\mathrm{H}^1(\mathfrak{U}, \hat{\mathcal{Q}})$ with $\mathrm{H}^1(\mathfrak{U}, \hat{\mathcal{P}})$. Each of the isomorphisms composed to construct this isomorphism was induced by an inclusion of two sheaves, each of which is a subsheaf of the sheaf $\hat{\mathcal{H}}$ of Φ-homeomorphisms defined in Section 6. Therefore, $[a]$ and $[b]$ descend to the same class in $\mathrm{H}^1(\mathfrak{U}, \hat{\mathcal{H}})$. Thus, for each $U \in \mathfrak{U}$, there exists a Φ-homeomorphism from $M/T|_U$ to the object b_U, such that for each pair $U, V \in \mathfrak{U}$, the associated arrow is $\beta_{UV} = f_U \circ f_V^{-1}$.

Under the inclusion $\mathrm{H}^1(\mathfrak{U}, \hat{\mathcal{P}}) \longrightarrow \mathrm{H}^1(\mathfrak{U}, \mathcal{P})$, the class $[b]$ maps to $[N]$. Hence, there exists Φ-homeomorphisms $g_u \colon b_U \longrightarrow N|_U$ such that $g_U \circ g_V^{-1} = \mathrm{id}$ for each pair $U, V \in \mathfrak{U}$. Composing with the Φ-homeomorphisms $M/T|_U \longrightarrow b_U$, we get Φ-homeomorphisms from $M/T|_U$ to $N|_U$ which coincide on overlaps. $\qquad \square$

18. Smooth Paintings

When $\mathcal{T}$ is convex, we can replace our painted surface bundle with a simpler object: a smooth painting.

Definition 18.1. Let (S, π) be a skeleton and let Σ be a closed oriented surface. A **painting** is a map $f \colon S \longrightarrow \Sigma$ such that the map $(\pi, f) \colon S \longrightarrow \mathcal{T} \times \Sigma$ is one to one. Paintings $f \colon S \longrightarrow \Sigma$ and $f' \colon S' \longrightarrow$

Σ' are **equivalent** if there exists an isomorphism $i\colon S \longrightarrow S'$ and a homeomorphism $\xi\colon \Sigma \longrightarrow \Sigma'$ such that the compositions $\xi\circ f\colon S \longrightarrow \Sigma'$ and $f' \circ i\colon S \longrightarrow \Sigma'$ are homotopic through paintings.

One similarly defines smooth paintings and smooth equivalence of paintings.

EXAMPLE 18.2. Each tall complexity one space naturally determines a painting $f\colon M_{\mathrm{exc}} \longrightarrow \Sigma$ up to equivalence; see Section 2.

Proposition 18.3. *Let $\mathcal{T} \subseteq \mathfrak{t}^*$ be an open set. To every painted surface bundle (N, P, π) over $\mathcal{T}$ such that $\mathrm{image}\,\pi$ is convex and $\pi\colon N \longrightarrow \mathrm{image}\,\pi$ is open, one can associate a smooth equivalence class of paintings $f\colon P \longrightarrow \Sigma$ such that two such surface bundles (N, P, π) and (N', P', π') are isomorphic if and only if they have smoothly equivalent paintings and $\mathrm{image}\,\pi = \mathrm{image}\,\pi'$.*

Moreover, if $(M, \omega, \Phi, \mathcal{T})$ is a tall complexity one proper Hamiltonian T-manifold and (N, P, π) is an associated painted surface bundle, then the associated paintings $M_{\mathrm{exc}} \longrightarrow \Sigma$ and $P \longrightarrow \Sigma$ are equivalent. (See Section 2.)

The proof of the proposition relies on a "trivialization" of the surface bundles:

Lemma 18.4. *Let $\mathcal{T} \subseteq \mathfrak{t}^*$ be an open set. Let (N, P, π) be a painted surface bundle over $\mathcal{T}$ such that $\mathrm{image}\,\pi$ is convex and $\pi\colon N \longrightarrow \mathrm{image}\,\pi$ is open. Then there exists a closed oriented surface Σ and a smooth map $f\colon N \longrightarrow \Sigma$ such that the map*

$$(\pi, f)\colon N \longrightarrow (\mathrm{image}\,\pi) \times \Sigma$$

is a diffeomorphism.

Moreover, given any two such maps f and f' there exists a diffeomorphism $\xi\colon \Sigma \longrightarrow \Sigma$ such that f is smoothly homotopic to $\xi \circ f'$ through maps which induce diffeomorphisms from N to $(\mathrm{image}\,\pi) \times \Sigma$.

Proof. By the definition of a painted surface bundle, every point in N has a neighborhood U and a diffeomorphism $U \cong \pi(U) \times$ (a disc) which carries π to the projection map to $\pi(U)$. Because $\pi\colon N \longrightarrow \mathrm{image}\,\pi$ is open, $\pi(U)$ is an open subset of $\mathrm{image}\,\pi$. This implies that every point in $\mathrm{image}\,\pi$ has a neighborhood V in $\mathrm{image}\,\pi$ and a diffeomorphism $\pi^{-1}(V) \cong V \times \Sigma$ where Σ is a closed oriented smooth surface which carries the map π to the projection map $V \times \Sigma \longrightarrow V$. The proof is similar to the proof of Ehresmann's lemma (that a proper submersion

is a fibration). Because image π is contractible, the bundle $\pi\colon N \longrightarrow$ image π is trivial. The second part of the lemma is proved exactly like Proposition 2.2. $\qquad\square$

Proof of Proposition 18.3. Let f be as in Lemma 18.4. By restricting f to the skeleton, we get a smooth painting, determined up to smooth equivalence. Isomorphic painted surface bundles give isomorphic smooth paintings.

We need to show that if two painted surface bundles give rise to smoothly equivalent paintings and have the same image in $\mathcal{T}$, then they are isomorphic.

Given a skeleton (S, π), let $f_t\colon S \longrightarrow \Sigma$ be a smooth homotopy through paintings.

Let $N = (\text{image}\,\pi) \times \Sigma$, and let π be the natural projection to $\mathfrak{t}^*$. Let $\hat{f}_t\colon S \longrightarrow N$ be given by $\hat{f}_t(x) = (\pi(x), f_t(x))$.

Since $\hat{f}_t$ is smooth and one-to one, for each t and each $x \in S$ there exists a vector field X_t on N, defined near $\hat{f}_t(x)$, such that

(18.5) $$\frac{d}{dt}\hat{f}_t(x) = X_t|_{\hat{f}_t(x)}.$$

Note that X_t is tangent to the fibers of π. Using a partion of unity, one can obtain globally defined vector fields X_t on $(\text{image}\,\pi) \times \Sigma$ which are tangent to the fibers of π and such that (18.5) holds. We then integrate these vector fields to a family of diffeomorphisms $g_t\colon (\text{image}\,\pi) \times \Sigma \longrightarrow (\text{image}\,\pi) \times \Sigma$ such that $g_0 = \text{id}$ and $\frac{d}{dt}g_t = X_t \circ g_t$. Each g_t preserves the fibers of π, and $g_t(f(x)) = f_t(x)$ for all $x \in S$. In particular, $g_1\colon N \longrightarrow N$ is an isomorphism that respects the π and such that $g_1(\hat{f}_0(x)) = f_1(x)$ for all $x \in S$.

The last claim follows from the fact that, by Proposition 17.5, M/T and N are Φ homeomorphic. $\qquad\square$

19. Eliminating the smooth structure

The final step is to show that, instead of working with smooth paintings, we can simply work with (continuous) paintings.

Proposition 19.1. *Let (S, π) be a skeleton, let Σ be a closed oriented surface, and consider paintings $f\colon S \longrightarrow \Sigma$. Every painting is equivalent to a smooth painting, and if two smooth paintings are equivalent, then they are smoothly equivalent.*

Proof. Embed Σ into $\mathbb{R}^3$. Choose an ϵ tubular neighborhood of Σ, and let $r\colon U \longrightarrow \Sigma$ be the associated normal retract.

We begin with the first claim. Let $f\colon S \longrightarrow \Sigma$ be a painting. Choose a continuous function $\epsilon\colon S \longrightarrow \mathbb{R}^+$ so that

$$\epsilon(s) < \epsilon \text{ and } \epsilon(s) < \frac{1}{4}||f(s) - f(s')|| \,\forall\, s' \in S \ni \pi(s') = \pi(s).$$

For all $s \in S$, choose a neighborhood $V_s \subset S$ such that

$$||f(y) - f(s)|| < \epsilon(y) \,\forall\, y \in V_s.$$

Let $\{U_\alpha\}$ be a locally finite refinement of $\{V_s\}$ with index assignment $\alpha \mapsto s(\alpha)$. Let λ_α be a smooth partition of unity subordinate to U_α.
Define $h\colon S \longrightarrow \mathbb{R}^3$ by $h(y) = \sum_\alpha \lambda_\alpha(y)f(s(\alpha))$; then

$$||h(y) - f(y)|| \leq \epsilon(y).$$

Since $\epsilon(y) < \epsilon$, we can define $g_t\colon S \longrightarrow \Sigma$

$$g_t(y) = r((1-t)f(y) + t\, h(y)) \,\forall t \in [0,1]$$

Clearly, $g_0 = f$ and g_1 is smooth. Moreover, $||g_t(y) - f(y)|| \leq 2\epsilon(y)$. Therefore, for any y and y' in S such that $\pi(y) = \pi(y')$,

$$||g_t(y) - g_t(y')|| \geq ||f(y) - f(y')|| - ||g_t(y) - f(y)|| - ||g_t(y') - f(y')|| > 0.$$

Therefore, g_t is a painting.

We now prove the second claim. Let A denote $\{0,1\} \times S$. Let $f\colon I \times S \longrightarrow \Sigma$ be a homotopy of paintings such that the restriction of f to A is smooth.

Choose a continuous function $\epsilon\colon I \times S \longrightarrow \mathbb{R}^+$ so that

$$\epsilon(t,s) < \epsilon \text{ and } \epsilon(t,s) < \frac{1}{4}||f(t,s) - f(t,s')|| \,\forall\, s' \in S \ni \pi(s') = \pi(s).$$

For all $x \in I \times S$, choose a neighborhood $V_x \subset I \times S$ and a function $h_x\colon V_x \longrightarrow \Sigma$ such that

- If $x \in A$, then h_x is a smooth local extension of $f|_{A \cap V_x}$.
- If $x \notin A$, then $V_x \cap A = \emptyset$ and $h_x(y) = f(x)$.
- For all $y \in V_x$, $||f(y) - f(x)|| < \epsilon(y)$ and $||f(y) - h_x(y)|| < \epsilon(y)$.

Let $\{U_\alpha\}$ be a locally finite refinement of $\{V_x\}$ with index assignment $\alpha \mapsto x(\alpha)$. Let λ_α be a smooth partition of unity subordinate to U_α.
Define $h\colon I \times S \longrightarrow \mathbb{R}^3$ by $h(y) = \sum_\alpha \lambda_\alpha(y)h_{x(\alpha)}(y)$; then

$$||h(y) - f(y)|| \leq \epsilon(y).$$

Since $\epsilon(y) < \epsilon$, we can define $g\colon [0,1] \times S \longrightarrow \Sigma$, by $g(y) = r(h(y))$.

Clearly, $g|_A = f|_A$ and g is smooth. Moreover, $||g(y) - f(y)|| \leq 2\epsilon(y)$. Therefore, for any $y = (t, s)$ and $y' = (t, s')$ in $[0, 1] \times S$ such that $\pi(s) = \pi(s')$, then

$$||g(y) - g(y')|| \geq ||f(y) - f(y')|| - ||g(y) - f(y)|| - ||g(y') - f(y')|| > 0.$$

Therefore, g is a homotopy through paintings. $\qquad\square$

20. Global existence up to Φ-diffeomorphisms

In this paper, we have shown that certain invariants determine a tall complexity one space up to isomorphism. In our next paper, we will construct complexity one spaces out of these invariants. For future reference, we give here one step in this direction: given a skeleton (S, π) and a closed convex subset $\Delta \subset \mathcal{T}$, we show that if they locally come from complexity one spaces, then any painting $f \colon S \longrightarrow \Sigma$ can be realized by gluing these spaces by Φ-diffeomorphisms.

Proposition 20.1. *Let $\mathcal{T}$ be an open subset of $\mathfrak{t}^*$ and $\Delta \subset \mathcal{T}$ a convex closed subset. Let $f \colon S \longrightarrow \Sigma$ be a painting, where Σ is a closed oriented surface of genus g and (S, π) is a skeleton over $\mathcal{T}$. Let $\mathfrak{U}$ be an open cover of $\mathcal{T}$. For each $U \in \mathfrak{U}$, let $(M_U, \omega_U, \Phi_U, U)$ be a connected complexity one proper Hamiltonian T-manifold of genus g over U, so that* image $\Phi_U = U \cap \Delta$ *and so that the set of exceptional orbits $(M_U)_{\mathrm{exc}}$ is isomorphic to the restriction $S|_U := S \cap \pi^{-1}(U)$.*

Then, after possibly refining the open cover, one can associate to each U and V in $\mathfrak{U}$ a Φ-diffeomorphisms $h_{VU} \colon M_U/T|_{U \cap V} \longrightarrow M_V/T|_{U \cap V}$ such that $h_{WV} \circ h_{VU} = h_{WU}$, and such that the following holds.

If $(M, \omega, \Phi, \mathcal{T})$ is a complexity one space such that for every $U \in \mathcal{U}$ there exists a Φ-T-diffeomorphism $\lambda_U \colon M|_U \longrightarrow M_U$ so that h_{VU} is the map induced by the composition $\lambda_V \circ (\lambda_U)^{-1}$, then the painting associated to M is equivalent to f.

The proof of Proposition 20.1 will use the fact that, locally, a painted surface bundle is uniquely determined by its skeleton, its image, and its genus:

Lemma 20.2 (Local uniqueness of surface bundles)**.** *Let (N, P, π) and (N', P', π') be two painted surface bundles over an open neighborhood of a point α in $\mathfrak{t}^*$. Suppose that they have the same genus, equivalent skeletons, and the same image in $\mathfrak{t}^*$. Then there exists a neighborhood U of α such that the restrictions $(N|_U, P|_U, \pi|_U)$ and $(N'|_U, P'|_U, \pi'|_U)$ are isomorphic.*

Proof. Let $f\colon P \longrightarrow \Sigma$ be a smooth painting associated to N as in Proposition 18.3. By the local normal form theorem, there exists a neighborhood U of α so that π identifies each component of $P|_U$ with a subset of $\mathfrak{t}^*$ which is star shaped about α. Hence, $f|_U$ is smoothly homotopic to a trivial painting; on each component, simply define $f_t(x) = f(t\alpha + (1-t)x)$. Applying a similar argument to N', $N|_U$ and $N'|_U$ have smoothly equivalent paintings. By Proposition 18.3, $N|_U$ and $N'|_U$ are isomorphic. $\qquad\square$

We can now prove our main proposition.

Proof of Proposition 20.1. By Proposition 19.1, after possibly passing to an equivalent painting, we may assume that the painting $f\colon S \longrightarrow \Sigma$ is smooth. Let $N = \Delta \times \Sigma$, let $\pi\colon N \longrightarrow \mathcal{T}$ be the projection to the first coordinate, and let $P \subset N$ be the subset defined by $P = \{(\pi(s), f(s)) \mid s \in S\}$, with the labels inherited from S. For each $U \in \mathcal{U}$, since M_U is tall there are non-exceptional orbits in each nonempty moment map fiber. Hence, the local normal form theorem, together with the stability of the moment map image, imply that image Φ_U is a manifold with corners. (See Lemma 4.7 in [**KT**].) Since, by assumption, image $\Phi_U = U \cap \Delta$, this implies that N is a manifold with corners. By Definition 16.3, (N, P, π) is a painted surface bundle.

Let $\alpha \in \mathcal{T}$ be any point. Let (N', P', π') be a painted surface bundle associated to $(M_U, \omega_U, \Phi_U, U)$, as in Proposition 17.5, where $\alpha \in U \in \mathcal{U}$. By Lemma 20.2 there exists a neighborhood W of α such that $(N|_W, P|_W, \pi|_W)$ is isomorphic to $(N'|_W, P'|_W, \pi'|_W)$, so that $(N|_W, P|_W, \pi|_W)$ is a surface bundle associated to $(M_U|_W, \omega_U|_W, \Phi_U|_W, W)$. By Definition 16.6, the surface bundle (N, P, π) is legal.

Hence, by Lemma 5.4, it gives rise to an element in the first cohomology $H^1(\mathcal{T}, \mathcal{P})$, where $\mathcal{P}$ is the sheaf of isomorphisms of legal surface bundles defined at the end of Section (16). By the isomorphism (17.3), this, in turn, comes from a cohomology class in $H^1(\mathcal{T}, \mathcal{Q})$. Let $\{h_{\alpha\beta}\}$ be a cocycle that represents this class.

Let $(M, \omega, \Phi, \mathcal{T})$ be as stated in the proposition. By the definitions, the element of $H^1(\mathcal{T}, \mathcal{P})$ associated to M is exactly the one represented by the cocycle $\{h_{\alpha\beta}\}$. By Definition 17.4, this means that (N, P, π) is an associated surface bundle to (M, ω, Φ). By Proposition 17.5, this implies that M/T is Φ-homeomorphic to N. Therefore, the paintings associated to M/T and N are equivalent. $\qquad\square$

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INST. OF MATHEMATICS, THE HEBREW UNIVERSITY OF JERUSALEM, ISRAEL, AND: DEPT. OF MATHEMATICS, THE UNIVERSITY OF TORONTO, TORONTO, ONTARIO M5S 3G3, CANADA
E-mail address: karshon@math.toronto.edu

DEPT. OF MATHEMATICS, THE UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN, 1409 W. GREEN STREET, URBANA, IL 61801, U.S.A
E-mail address: stolman@math.uiuc.edu