Japan J. Indust. Appl. Math., 26 (2009), 125–143 Area ⟨2⟩

Theory of an Interval Algebra and Its Application to Numerical Analysis

Teruo Sunaga

University of Tokyo

This is reprinted, with permission from the author and the publisher, from the original paper published in Research Association of Applied Geometry (RAAG) Memoirs, Vol. 2 (1958) pp. 29–46, published by Gakujutsu Bunken Fukyu-kai, Tokyo, Japan.

RAAG MEMOIRS
Vol. 2, 1958

Misc. II

Theory of an Interval Algebra and its Application to Numerical Analysis

By
Teruo SUNAGA*

[Received February 1958]

INTRODUCTION

THIS is the first report on our study of the Geometry of Numerals and presents Set-theoretical Topological Considerations and Practical Applications to Numerical Calculation of the Algebra and Calculus of "Interval Lattices" which we introduce in this article.

A parallel investigation, which is on the rigorous error estimation in connexion with inverting of matrices of higher order, has been carried out by J. von Neumann and H. H. Goldstine [1]. It points out an essential feature of this kind of problem, but its method is not so general as ours.

The motive of this study is as follows. As is clear from the recent theory of information and communication [2], its fundamental idea is that the animal and the machine have the same structure in regard to their organs of communication. For instance, the modern high speed automatic computer may be called brain machine in that there are many similar points between the human brain and the automatic computing machine. By a preliminary study stated in another article of these Memoirs, namely [3], the author has been led to the conviction that our scientific statement should essentially be based on the concept of finiteness. To him it also appears that there is a contradiction between the continuity concept and that of discreteness.

This paper is intended as a realization of what follows from the conviction.

* University of Tokyo, Tokyo, Japan.

CHAPTER I

INTERVAL CALCULUS

This chapter is dedicated to the basic exposition of the concept of interval. Its algebraic properties are investigated and various concepts of analysis such as function or differentiation, are considered from our standpoint.

1. Significance of interval

In order fully and effectively to utilize pure mathematics for the analysis of natural phenomena, we must be aware that there are many phases concerning which mathematics and reality do not perfectly agree. For example, neither one point on the real number axis is sufficient to represent a physical quantity, nor is any trace of a moving body described completely as a continuous function of time having no "breadth". In expressing numerical quantities by means of a finite number of digits we cannot express an irrational number, but only rational ones.

Let us consider the procedures of calculating $\sqrt{2}$. We first calculate up to 1 and then proceed to 1.4, 1.41, etc. Now we ask "What are the meanings of this sequence of numerals?" The numeral 1, appearing in the sequence of calculation, does not simply denote either a natural or a rational number. It implies that one of the figures from 0 to 9 will be obtained by the calculation of the following step. Namely, the figure 1 here denotes the interval [1, 2] which contains all the real numbers from

1 to 2. Similarly, the numeral 1.4 denotes the interval [1.4, 1.5], 1.41 denotes [1.41, 1.42], etc.

The concept indicated by $\sqrt{2}$ cannot be formed without assuming the series of intervals accompanying it and that can be said to be associated with a rule of calculating it out.

The above consideration leads us to the following conclusion.

i) The concept of an interval is more fundamental than that of a real number.

ii) To denote an interval, we need not necessarily use two rational numbers.

The numerical expression obtained by rounding off, i. e., by counting 5 and higher fractions and disregarding the rest, is a good example of denoting an interval by one rational number. For instance, such a numeral as

$$\langle 1.414 \rangle$$

should be regarded as denoting the interval

$$[1.4135,\ 1.4145].$$

That the concept of an interval is fundamental is not only so in the case of numerical calculation. It is better to use an interval instead of a real number to describe a physical quantity, treating the latter so as to have a certain "breadth", and say, "*A body exists in interval X when time is in interval T*", instead of saying, "*A body exists at point ξ when time is at point τ*".

The reader may think of such more familiar expressions as "*Statistically ......*" or "*The statistical values are ......*". But probable or stochastical numerical values are not different from our physical quantities and should also be described by intervals.

Thus, we might say that the interval concept is on a borderline linking pure mathematics with reality and pure analysis with applied analysis.

In the following sections of Chapter I we shall investigate the fundamental treatment of intervals with regard to their mutual relations, operations, functions, functionals, differentials, etc. Their application to numerical calculation will be described in Chapter II.

Although we have stated that the concept of an interval is more fundamental than that of a real number, we have no intention of discussing the theory of real numbers, and therefore we shall not make use of anything more than the commonplace knowledge of real numbers and continuous functions.

In the following description we shall use Greek letters α, β, ..., ξ, η, ... to denote real numbers.

2. The interval lattice

We shall first give the definition of an interval and then investigate its properties.

Definition 1. Interval: The set X of all ξ satisfying the condition

$$\alpha \leq \xi \leq \beta$$

is called the interval and is denoted by $[\alpha, \beta]$.

Intervals will be denoted by Roman letters in the following.

Example 2·1. In Fig. 1, we have interval A

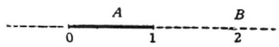

FIG. 1

which will be denoted by [0, 1] and also B which will be expressed by [2, 2]. Generally, a real number is regarded as a limiting case of an interval.

Definition 2. Inclusion: If each element ξ of interval X is always that of interval Y, i. e., if logically

$$\xi \in X \longrightarrow \xi \in Y,$$

then we write

$$X \longrightarrow Y \qquad (2\cdot1)$$

and say that X is included by Y.[1)]

Example 2·2. In Fig. 2, A is [1, 2] and B

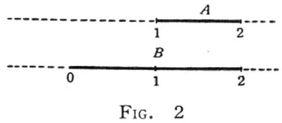

FIG. 2

is [0, 2]. Then we have

$$A \longrightarrow B.$$

1) Cf. §2 of reference [4].

When X is included by Y, we customarily write

$$X \subseteq Y.$$

But the relation of inclusion will be used in this paper so often that it will be more convenient to use symbol $\rightarrow$ than $\subseteq$.

We may say that numerical calculation is essentially to deal with the inclusion relation.

Definition 3. Coincidence: If $X \longrightarrow Y$ and $Y \longrightarrow X$, we say that X and Y coincide with each other and write

$$X \rightleftarrows Y. \qquad (2\cdot 2)$$

Example 2·3. In Fig. 1 the interval A coincides with itself, i. e.,

$$A \rightleftarrows A.$$

Theorem 1. *The system of the intervals forms a partially ordered set in the following sense.*

$$\left.\begin{array}{ll} \text{i)} & X \longrightarrow X. \\ \text{ii)} & \textit{If } X \longrightarrow Y \textit{ and } Y \longrightarrow Z, \textit{ then } X \longrightarrow Z. \\ \text{iii)} & \textit{If } X \longrightarrow Y \textit{ and } Y \longrightarrow X, \textit{ then } X \rightleftarrows Y. \end{array}\right\} \qquad (2\cdot 3)$$

These relations, especially ii), play an important rôle in numerical calculation.

Next, we shall define two dual operations by either of which each pair of intervals can be associated with a third interval.

Definition 4. Join: By the joint of X and Y we mean the least among all the intervals which include both X and Y and this interval is denoted by

$$X \vee Y. \qquad (2\cdot 4)$$

Example 2·4. In Fig. 3 the join of A and

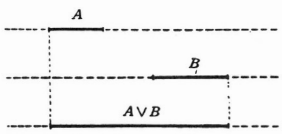

FIG. 3

B is explained graphically. In this case, if both A and B are reduced to real numbers, $A \vee B$ is the interval denoted by $[A, B]$, i. e.,

$$A \vee B \rightleftarrows [A, B].$$

Example 2·5. We have

i) $\pi \longrightarrow 3.141 \vee 3.142.$

ii) Let $<1.414>$ be the value rounded off, then
$$<1.414> \rightleftarrows 1.4135 \vee 1.4145.$$

iii) $1 \vee 2 \longrightarrow 0 \vee 2$, and $0 \vee 2 \longrightarrow 0 \vee 3.$
Hence
$$1 \vee 2 \longrightarrow 0 \vee 3.$$

iv) $[0, 1] \vee [2, 2.5] \rightleftarrows [0, 2.5] \rightleftarrows 0 \vee 2.5.$

Definition 5. Meet: By the meet of X and Y we mean the greatest among all the intervals which are included by both X and Y and this interval is denoted by

$$X \wedge Y. \qquad (2\cdot 5)$$

Example 2·6. In Fig. 4 the meet of A and

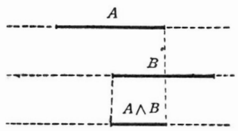

FIG. 4

B is explained graphically. In this case, if A and B do not intersect each other, then

$$A \wedge B \rightleftarrows \emptyset.$$

Example 2·7. We have

i) $[0, 3] \wedge [2, 4] \rightleftarrows [2, 3] \rightleftarrows 2 \vee 3,$

ii) $[-1, 1] \wedge [2, 3] \rightleftarrows \emptyset.$

Theorem 2. *The system of the intervals forms a lattice in the following sense* [5]*:*

$$\left.\begin{array}{ll} \text{i)} & X \vee X \rightleftarrows X, \quad X \wedge X \rightleftarrows X; \\ \text{ii)} & X \vee Y \rightleftarrows Y \vee X, \quad X \wedge Y \rightleftarrows Y \wedge X; \\ \text{iii)} & (X \vee Y) \vee Z \rightleftarrows X \vee (Y \vee Z), \\ & (X \wedge Y) \wedge Z \rightleftarrows X \wedge (Y \wedge Z); \\ \text{iv)} & (X \vee Y) \wedge X \rightleftarrows X, \quad (X \wedge Y) \vee X \rightleftarrows X. \end{array}\right\} (2\cdot 6)$$

These relations are called the idempotent, the commutative, the associative and the absorptive law respectively.

One can readily verify them through the

definition of meet and join.

The law iv) is the same as

$$X \longrightarrow X \vee Y, \quad X \wedge Y \longrightarrow X. \qquad (2\cdot7)$$

Example 2·8. We have

i) $(1 \vee 2) \vee 3 \rightleftarrows 1 \vee (2 \vee 3) \rightleftarrows 1 \vee 2 \vee 3 \rightleftarrows 1 \vee 3.$
In this case, the order of operation does not matter.

ii) $[0, 2] \wedge [1, 3] \longrightarrow [0, 2].$
This is an example of the absorption law.

Now that we have seen that the system of intervals is a lattice, we can investigate the properties of the system as a lattice from an algebraical point of view.

For example, let $P(X, Y, \cdots)$ be a lattice polynomial, i. e., a formula composed of some elements $X, Y, \cdots$ which are associated with one another by symbols $\vee$ and $\wedge$. Then

$$P(X, Y, \cdots) \longrightarrow P(X', Y', \cdots),$$

provided that

$$X \longrightarrow X', \; Y \longrightarrow Y', \; \cdots.$$

Example 2·9. We have

$$1 \longrightarrow [0, 2] \text{ and } 3 \longrightarrow [3, 4],$$

hence

$$1 \vee 3 \longrightarrow [0, 2] \vee [3, 4].$$

In fact, the right-hand side is

$$[0, 2] \vee [3, 4] \rightleftarrows 0 \vee 4.$$

Therefore it includes the left-hand side.

3. Arithmetical operations

We shall define here arithmetical operations on intervals and investigate their relations to the lattice operations.

Definition 6. i) *Addition:* By the sum of X and Y we mean the interval consisting of the set Z of all

$$\xi + \eta \quad (\xi \in X, \; \eta \in Y),$$

and we write

$$X + Y \rightleftarrows Z. \qquad (3\cdot1)$$

ii) *Subtraction:* By the difference between X and Y we mean the interval consisting of the set Z of all

$$\xi - \eta \quad (\xi \in X, \; \eta \in Y),$$

and we write

$$X - Y \rightleftarrows Z. \qquad (3\cdot2)$$

iii) *Multiplication:* By the product of X and Y we mean the interval consisting of the set Z of all

$$\xi \eta \quad (\xi \in X, \; \eta \in Y),$$

and we write

$$XY \rightleftarrows Z. \qquad (3\cdot3)$$

iv) *Division:* By the quotient of X to Y, provided that zero does not fall in Y, we mean the interval consisting of the set Z of all

$$\xi / \eta \quad (\xi \in X, \; \eta \in Y),$$

and we write

$$X / Y \rightleftarrows Z. \qquad (3\cdot4)$$

It should be noted that these four operations are mutually independent. For instance, the interval X which satisfies the following relation

$$A + X \rightleftarrows B,$$

is generally different from the interval

$$B - A.$$

In this respect our system differs from that of ordinary numbers.

Example 3·1. We have

i) $(1 \vee 2) + (3 \vee 6) \rightleftarrows 4 \vee 8,$
ii) $(5 \vee 6) - (5 \vee 6) \rightleftarrows -1 \vee 1,$
iii) $(2 \vee 3) \times (-6 \vee 4) \rightleftarrows -18 \vee 12,$
iv) $(4 \vee 8) \div (1 \vee 2) \rightleftarrows 2 \vee 8.$

Here, the order of operations is as follows:

$$\times, \; \div; \; +, \; -; \; \vee, \; \wedge; \; \rightleftarrows, \; \longrightarrow.$$

Definition 7. Abbreviations:

i) $-X$ is the abbreviation of $0 - X$;
ii) X^{-1} is the abbreviation of $1/X$.

Theorem 3.

i) $-(\alpha_1 \vee \alpha_2) \rightleftarrows -\alpha_1 \vee -\alpha_2.$

ii) $(\alpha_1 \vee \alpha_2)^{-1} \rightleftarrows \frac{1}{\alpha_1} \vee \frac{1}{\alpha_2}.$

In ii), α_1 *and* α_2 *are assumed to have the same sign.*

Proof:

i) $-(\alpha_1 \vee \alpha_2) \rightleftarrows 0-(\alpha_1 \vee \alpha_2) \rightleftarrows -\alpha_1 \vee -\alpha_2.$

ii) $(\alpha_1 \vee \alpha_2)^{-1} \rightleftarrows \frac{1}{\alpha_1 \vee \alpha_2} \rightleftarrows \frac{1}{\alpha_1} \vee \frac{1}{\alpha_2}.$

Theorem 4.

$$\left.\begin{aligned} &X-Y \rightleftarrows X+(-Y), \\ &X/Y \rightleftarrows XY^{-1}. \end{aligned}\right\} \quad (3\cdot 5)$$

From this theorem it follows that subtraction and division are reduced to addition and multiplication respectively.

Theorem 5.

$$\left.\begin{aligned} &X+Y \rightleftarrows Y+X, \\ &(X+Y)+Z \rightleftarrows X+(Y+Z), \\ &XY \rightleftarrows YX, \\ &(XY)Z \rightleftarrows X(YZ), \\ &X(Y+Z) \longrightarrow XY+XZ. \end{aligned}\right\} \quad (3\cdot 6)$$

The last relation differs from that of ordinary numbers and an example is as follows.

Example 3·2. If

$$X \rightleftarrows 1 \vee 2,$$

then

$$\begin{aligned} X^2-X &\rightleftarrows (1 \vee 2)^2-(1 \vee 2) \\ &\rightleftarrows (1 \vee 4)-(1 \vee 2) \rightleftarrows -1 \vee 3 \end{aligned}$$

and

$$\begin{aligned} (X-1)X &\rightleftarrows \{(1 \vee 2)-1\}(1 \vee 2) \\ &\rightleftarrows (0 \vee 1)(1 \vee 2) \rightleftarrows 0 \vee 2. \end{aligned}$$

Hence, it follows that

$$(X-1)X \longrightarrow X^2-X.$$

From this example it follows that the calculation of[1]

$$A_n x^n + A_{n-1} x^{n-1} + \cdots\cdots + A_1 x + A_0,$$

is better carried out by applying the following form (Horner's method)

$$(\cdots((A_n X + A_{n-1})X + A_{n-2})X + \cdots + A_1)X + A_0.$$

Theorem 6.

$$\left.\begin{aligned} &\text{i)} \quad X+(Y \vee Z) \rightleftarrows X+Y \vee X+Z, \\ &\text{ii)} \quad X(Y \vee Z) \rightleftarrows XY \vee XZ. \end{aligned}\right\} \quad (3\cdot 7)$$

Proof:

i) $Y \longrightarrow Y \vee Z$ and $Z \longrightarrow Y \vee Z$,

hence

$$X+Y \longrightarrow X+(Y \vee Z),$$
$$X+Z \longrightarrow X+(Y \vee Z).$$

Therefore

$$\begin{aligned} &X+Y \vee X+Z \\ &\longrightarrow X+(Y \vee Z) \vee X+(Y \vee Z) \\ &\longrightarrow X+(Y \vee Z). \end{aligned}$$

Inversely, if

$$\xi \in X, \ \eta \in Y \text{ and } \zeta \in Z,$$

then

$$\xi+(\eta \vee \zeta) \rightleftarrows \xi+\eta \vee \xi+\zeta \longrightarrow X+Y \vee X+Z.$$

Therefore

$$X+(Y \vee Z) \longrightarrow X+Y \vee X+Z.$$

From these relations, we get

$$X+(Y \vee Z) \rightleftarrows X+Y \vee X+Z.$$

Similarly for ii).

Example 3·3. We have

i) $(\alpha_1 \vee \alpha_2)(\beta_1 \vee \beta_2) \rightleftarrows \alpha_1(\beta_1 \vee \beta_2) \vee \alpha_2(\beta_1 \vee \beta_2)$
$\rightleftarrows \alpha_1\beta_1 \vee \alpha_1\beta_2 \vee \alpha_2\beta_1 \vee \alpha_2\beta_2.$

This relation gives the law of multiplication, i.e.,

$$\begin{aligned} &(\alpha_1 \vee \alpha_2)\ (\beta_1 \vee \beta_2) \\ &\rightleftarrows (\text{the least among } \alpha_i\beta_j) \\ &\qquad \vee (\text{the greatest among } \alpha_i\beta_j). \end{aligned}$$

The other operations can be performed analogously.

ii) $\langle 1.414 \rangle \rightleftarrows 1.414-5\times 10^{-4} \vee 1.414+5\times 10^{-4}$
$\rightleftarrows 1.414+(-5\times 10^{-4} \vee 5\times 10^{-4})$
$\rightleftarrows 1.414+(-5 \vee 5)\times 10^{-4}.$

1) We shall also use small Roman letters as variables of functions in the following.

Theorem 7. *If*

i) $$A+X \rightleftarrows A+Y$$

or

ii) $$A-X \rightleftarrows A-Y,$$

then

$$X \rightleftarrows Y.$$

Proof: Let $\bar{\alpha}$, $\bar{\xi}$ and $\bar{\eta}$ be the greatest number of A, X and Y respectively. Then, in case i)

$$\bar{\alpha}+\bar{\xi}=\bar{\alpha}+\bar{\eta}.$$

Similarly, let $\underline{\alpha}$, $\underline{\xi}$ and $\underline{\eta}$ be the least number of A, X and Y respectively. Then

$$\underline{\alpha}+\underline{\xi}=\underline{\alpha}+\underline{\eta}.$$

Therefore

$$X \rightleftarrows Y.$$

In case ii), the proof can be carried out analogously.

Theorem 8. *If A does not contain zero and if*

i) $$AX \rightleftarrows AY$$

or

ii) $$A/X \rightleftarrows A/Y,$$

then

$$X \rightleftarrows Y.$$

We can verify this theorem analogously to Theorem 7.

Example 3·4. The solution of the equation

$$(1\vee 2)+X \rightleftarrows 2\vee 5\ (=(1\vee 2)+(1\vee 3))$$

is

$$X \rightleftarrows 1\vee 3.$$

4. Multi-dimensional intervals

The intervals defined in § 2 were, strictly speaking, one-dimensional. Now, we shall study multi-dimensional intervals and their continuous mapping which also will be dealt with in the next section.

It is easy to expand the concept of one-dimensional interval to that of n-dimensional one. In the following, we shall designate a point in the n-dimensional space by $\xi=(\xi_1, \cdots, \xi_n)$.

Definition 8. n-dimensional interval: The set X of all $\boldsymbol{\xi}=(\xi_1, \cdots, \xi_n)$ satisfying the condition

$$\alpha_i \leq \xi_i \leq \beta_i$$

is called the n-dimensional interval and is denoted by $([\alpha_1, \beta_1], \cdots, [\alpha_n, \beta_n])$ or $(X_1, \cdots, X_n)$ with $X_i \rightleftarrows [\alpha_i, \beta_i]$.

We shall generally denote multi-dimensional intervals by bold Roman letters.

Example 4·1. In Fig. 5 we have the interval

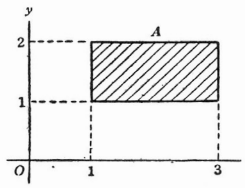

FIG. 5

which is denoted by ([1, 3], [1, 2]).

The inclusion relation, join and meet operations on n-dimensional intervals are defined analogously to those on one-dimensional intervals.

Definition 9. i) *Inclusion:* If each component X_i of interval $\mathbf{X}$ is included by the corresponding component Y_i of interval $\mathbf{Y}$, i.e., if

$$X_i \longrightarrow Y_i \quad \text{for} \quad i=1, \cdots, n,$$

then we write

$$\mathbf{X} \longrightarrow \mathbf{Y} \qquad (4\cdot 1)$$

and say that $\mathbf{X}$ is included by $\mathbf{Y}$. If $\mathbf{X} \longrightarrow \mathbf{Y}$ and $\mathbf{Y} \longrightarrow \mathbf{X}$, then we write

$$\mathbf{X} \rightleftarrows \mathbf{Y}.$$

ii) *Join:* By the join of $\mathbf{X}$ and $\mathbf{Y}$ we mean the interval Z of which the i-th component Z_i is the join of X_i and Y_i, i. e.,

$$Z \rightleftarrows (Z_1, \cdots, Z_n) \rightleftarrows (X_1 \vee Y_1, \cdots, X_n \vee Y_n)$$

and this interval is denoted by

$$X \vee Y. \qquad (4\cdot 2)$$

iii) *Meet:* By the meet of X and Y we mean the interval Z of which the i-th component Z_i is the meet of X_i and Y_i, i. e.,

$$Z \rightleftarrows (Z_1, \cdots, Z_n) \rightleftarrows (X_1 \wedge Y_1, \cdots, X_n \wedge Y_n)$$

and this interval is denoted by

$$X \wedge Y. \qquad (4\cdot 3)$$

Example 4·2. i) If

$$X \rightleftarrows ([1, 3], [1, 2])$$

and

$$Y \rightleftarrows ([0, 4], [1, 3]),$$

then, as in Fig. 6, we have the relation

$$X \longrightarrow Y.$$

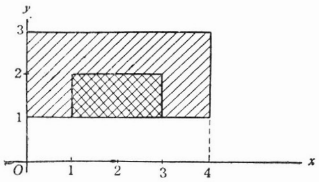

FIG. 6

ii) Using the notation of join, the interval A in Fig. 5 can be written as

$$A \rightleftarrows ([1, 3], [1, 2])$$
$$\rightleftarrows (1 \vee 3, 1 \vee 2) \rightleftarrows (1, 1) \vee (3, 2).$$

iii) If

$$X \rightleftarrows (0 \vee 4, 1 \vee 3)$$

and

$$Y \rightleftarrows (3 \vee 5, 0 \vee 2),$$

then, as in Fig. 7, we have

$$X \wedge Y \rightleftarrows \{(0 \vee 4) \wedge (3 \vee 5), (1 \vee 3) \wedge (0 \vee 2)\}$$
$$\rightleftarrows (3 \vee 4, 1 \vee 2).$$

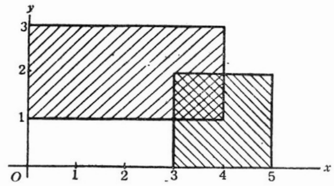

FIG. 7

Since we can consider n-dimensional intervals as interval vectors in an n-dimensional space, we shall consider some operations on interval vectors.

Definition 10. i) *Addition:* By the sum of X and Y we mean the interval Z of which the i-th component Z_i is the sum of X_i and Y_i, i. e.,

$$Z \rightleftarrows (Z_1, \cdots, Z_n) \rightleftarrows (X_1 + Y_1, \cdots, X_n + Y_n)$$

and we write

$$Z \rightleftarrows X + Y. \qquad (4\cdot 4)$$

ii) *Subtraction:* By the difference between X and Y we mean the interval Z of which the i-th component Z_i is the difference between X_i and Y_i, i. e.,

$$Z \rightleftarrows (Z_1, \cdots, Z_n) \rightleftarrows (X_1 - Y_1, \cdots, X_n - Y_n)$$

and we write

$$Z \rightleftarrows X - Y. \qquad (4\cdot 5)$$

For the multiplication of an interval vector by an interval scalar, it is necessary to repeat such considerations as we have done on the distributive law of intervals in Theorem 5 of §3. Namely the relation

$$X(X_1, \cdots, X_n) \longrightarrow (XX_1, \cdots, XX_n) \qquad (4\cdot 6)$$

always holds, but the relation

$$X(X_1, \cdots, X_n) \rightleftarrows (XX_1, \cdots, XX_n)$$

does not.

Example 16. When

$$X \rightleftarrows 1 \vee 2$$

and

$$U \rightleftarrows 1.0 \vee 1.2,$$

we have

$$X(1,\ U) \rightleftarrows (1 \vee 2)(1,\ 1.0 \vee 1.2) \qquad (4\cdot 7)$$

and

$$(X,\ XU) \rightleftarrows (1 \vee 2,\ 1.0 \vee 2.4). \qquad (4\cdot 7')$$

The regions indicated by $X(1, U)$ and $(X, XU) \rightleftarrows (X, Y)$ are shown graphically in Fig. 8.

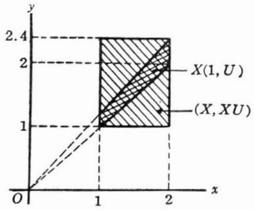

FIG. 8

Although (X, U) is a two-dimensional interval in the parameter space xu-plane, $X(1, U)$ is not an interval in the xy-plane. And $(X, XU) \rightleftarrows (X, Y)$ is an interval in the xy-plane, but it is different from the region indicated by $X(1, U)$ which is, mathematically speaking, the image of the interval (X, U) continuously mapped into the xy-plane. Since it is not sufficient to employ the interval (X, XU) for the estimation of the region or the error indicated by $X(1, U)$, we should consider not only intervals themselves, but also their continuous mapping for the interval calculus.[1]

The above consideration suggests, for example, how the coefficient errors of a system of linear equations influence the errors of the unknowns of the system and how these should be estimated.

5. Interval functions and functionals

We shall investigate here the continuous mapping of intervals into a function or a functional space. More generally, however, we should consider the continuous mapping into a topological space.[2]

Definition 11. Interval Function: By an interval function will be meant the set of all those functions of a function space which are the images of an interval continuously mapped into the function space.

For instance, if we consider a family of functions with parameters α and β

$$f(x;\ \alpha,\ \beta)$$

and if a and b are intervals respectively, then the set of all $f(x; \alpha, \beta)$ satisfying the condition

$$\alpha \epsilon a,\quad \beta \epsilon b$$

is an interval function. This will be designated by

$$f(x;\ a,\ b). \qquad (5\cdot 1)^{3)}$$

Example 5·1. The equation of motion of a single particle under gravitation is

$$s = s_0 + v_0 t + \frac{1}{2} g t^2.$$

Here, however, s_0, v_0 and g are not real numbers, but intervals and we can write

$$s = f(t;\ s_0,\ v_0,\ g),$$

using our notation.

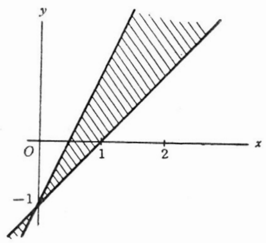

FIG. 9

1) The difference between $X(1+U)$ and $X+XU$ is similar to that which exists between $X(1, U)$ and (X, XU), but the latter difference is higher in order than the former as can be seen from the meanings of their expressions.

2) Cf. Chapter II of reference [6].

3) When we consider the mapping of an interval, we shall designate the interval by a "small" letter. Therefore $a(1+b)$ and $a+ab$ are equivalent to each other, but $A(1+B)$ and $A+AB$ are not.

Example 5·2. The interval function

$$y=-1+(1\vee 2)x$$

graphed in Fig. 9, is written in our notation, as

$$y=f(x;\ -1,\ 1\vee 2).$$

The definition of interval functionals is similar to that of interval functions as follows.

Definition 12. Interval Functional: By an interval functional will be meant the set of all those functionals of a functional space which are the images of an interval continuously mapped into the functional space.

For instance, let us consider an interval a and Dirac's function $\partial(\alpha)$ defined as

$$\partial(\alpha)f=f(\alpha)$$

where f is an arbitrary function.[1] Then the set of all $\partial(\alpha)$ satisfying the condition

$$\alpha\epsilon a$$

is an interval functional. We shall designate this interval functional by

$$\partial(a) \qquad (5\cdot 2)$$

and we have

$$\partial(a)f=f(a). \qquad (5\cdot 3)$$

The relation (5·3) is graphically explained in Fig. 10.

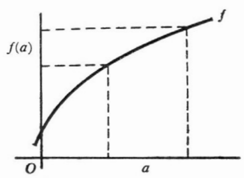

FIG. 10

The interval $\partial(a)f$ or $f(a)$ may also be regarded as the image of an interval under the mapping f and has the following properties.

Theorem 9. *If*

$$x_1 \longrightarrow x_2,\ y_1 \longrightarrow y_2,\ \cdots,$$

then

$$f(x_1,\ y_1,\ \cdots) \longrightarrow f(x_2, y_2,\ \cdots).$$

Proof: It is evident from the following graph.

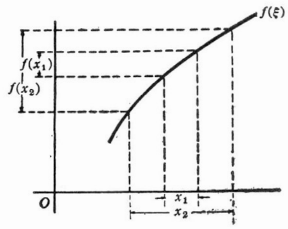

FIG. 11

Example 5·3.

$$\sin(0\vee 0.5) \longrightarrow \sin(-0.1\vee 0.6).$$

Theorem 10.

$$\left.\begin{aligned} f(x_1\vee x_2) &\longleftarrow f(x_1)\vee f(x_2), \\ f(x_1\wedge x_2) &\longrightarrow f(x_1)\wedge f(x_2). \end{aligned}\right\} \qquad (5\cdot 4)$$

Proof:

$$x_1 \longrightarrow x_1\vee x_2 \text{ and } x_2 \longrightarrow x_1\vee x_2,$$

hence, by virtue of the preceding theorem, we get

$$f(x_1) \longrightarrow f(x_1\vee x_2),$$
$$f(x_2) \longrightarrow f(x_1\vee x_2).$$

Therefore

$$f(x_1)\vee f(x_2) \longrightarrow f(x_1\vee x_2).$$

The second relation can also be verified analogously.

Example 5·4.

$$\cos(-5°)\vee\cos 10° \longrightarrow \cos(-5°\vee 10°).$$

1) According to the theory of distributions [7], Dirac's function is a "functional". We may also treat f as an interval function.

6. Differentiation

We shall study differentiation as an application of the concept of interval functional.

The definition of differentiation is

$$f'(\xi)=\lim_{\Delta\xi\to 0}\frac{f(\xi+\Delta\xi)-f(\xi)}{\Delta\xi}, \qquad (6\cdot 1)$$

but this process seems to involve unreality. Therefore we shall try to define differentiation realistically, i. e., to define the differential coefficient on an interval.

Now, we shall introduce the derivative of Dirac's function $\delta'(\alpha)$ for which

$$\delta'(\alpha)f=f'(\alpha)$$

where f is an arbitrary differentiable function.[1] The operator $\delta'(\alpha)$ thus defined is obviously a functional and therefore for an interval a, $\delta(a)$ is an interval functional. Then we have the relation

$$\delta'(a)f=f'(a) \qquad (6\cdot 1')$$

and we shall call $\delta'(a)f$ or $f'(a)$ the differential coefficient of f on the interval a.

Next, we shall explain the meaning of the differential coefficient on an interval.

Let us consider a function $f(x)$ with the continuous derivative of the first order. By the mean value theorem we have

$$f(\xi_1+d\xi)=f(\xi_1)+f'(\xi_1+\theta d\xi)d\xi \qquad (6\cdot 2)$$

provided

$$\theta\longrightarrow 0\vee 1.$$

If x is an interval and

$$\xi_1\epsilon x \quad \text{and} \quad \xi_1+d\xi\epsilon x,$$

then

$$\xi_1+\theta d\xi\epsilon x$$

and

$$f(\xi_1+d\xi)\longrightarrow f(\xi_1)+f'(x)d\xi. \qquad (6\cdot 3)$$

Here, since we can substitute intervals x_1 and dx for ξ_1 and $d\xi$ respectively, we get the property that, so long as both x_1 and x_1+dx are included in x, then the relation

$$f(x_1+dx)\longrightarrow f(x_1)+f'(x)dx \qquad (6\cdot 4)$$

always exists.

We shall show later that relation (6·4) is very useful for applied analysis.[2]

The expansion of a function can be explained as follows.

For instance, the relation

$$f(x_{11}+dx)$$
$$\longrightarrow f(x_{11})+f'(x_1)dx+\frac{f''(x)}{2}(dx)^2 \qquad (6\cdot 5)$$

exists under the condition as in Fig. 12.

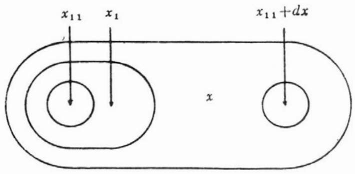

FIG. 12

7. The topological background

Since numerals are used for the analysis of continua, topology, which abstractly deals with continuity, plays an important rôle in the study of numerals. In fact, our interval calculus has a topological background.

Since numerical calculation is meaningless without error estimation, a numeral should be characterized by an error. Owing to the topological background of this paper the error concept corresponds to that of neighbourhoods which is fundamental in topology, and arithmetical operations of numerals to operations connected with topological groups which deal with neighbourhoods.

Now, we consider three elements a, b and c of a group G satisfying the relation

$$ab=c.$$

When G is a topological space, neighbourhoods U, V and W can be associated with a, b and c

---

1) It is usual for the "formal" derivative of Dirac's function $\delta(\alpha)$ that $\delta'(\alpha)$ be defined as

$$\delta'(\alpha)f=-f'(\alpha).$$

We may also treat f as an interval function.

2) See the examples of §9.

respectively as in Fig. 13.

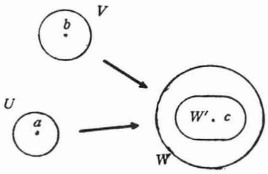

FIG. 13

If

$$UV = W'$$

and the set W' is included in the neighbourhood W, we get

$$UV \longrightarrow W.$$

This indicates the principle of interval calculus. That is, the group G corresponds to the system of the real numbers, the group operation to addition or multiplication, and neighbourhoods to intervals. In practical interval calculation it is convenient that a and b are regarded as middle points of intervals U and V respectively. This idea corresponds to the concept of systems of neighbourhoods of the identity and will be effectively used in the following chapter.

The reader who has a basic knowledge of topological groups will easily understand these circumstances.

The main concepts in the theory of topological groups which are utilized in these connexions are the definition of the topological group given as follows [6].

Definition 13. A set G of elements is called a topological group if

i) G is an abstract group,
ii) G is a topological space,
iii) the group operations in G are continuous in the topological space G.

In greater detail this condition can be formulated as follows:

i) If a and b are the elements of the set G, then for every neighbourhood W of the elements ab there exist neighbourhoods U and V of the elements a and b such that

$$UV \longrightarrow W.$$

ii) If a is an element of the set G, then for every neighbourhood V of the element a^{-1} there exists a neighbourhood U of the element a such that

$$U^{-1} \longrightarrow V.$$

CHAPTER II

APPLICATIONS

In this chapter we shall develop the practical method of the interval calculus and show its applications to numerical calculations.

8. Practical interval calculus

When the number of figures of a numeral is numerous, the above method of representation is inconvenient for practical calculation. Therefore we shall revise the method to express an interval and establish the law of arithmetical operations.

An interval A can be expressed in various ways as follows,

$$A \rightleftarrows \alpha_1 \vee \alpha_2 \rightleftarrows \frac{\alpha_1+\alpha_2}{2} + \left(\frac{\alpha_1-\alpha_2}{2} \vee \frac{\alpha_2-\alpha_1}{2}\right)$$
$$\rightleftarrows \alpha + (-\alpha_0 \vee \alpha_0), \qquad (8\cdot 1)$$

provided

$$\alpha_0 \geq 0.$$

Here α is the middle point and α_0 is the upper bound of error.

Definition 14. If

$$A \rightleftarrows \alpha + (-\alpha_0 \vee \alpha_0) \quad (\alpha_0 \geq 0), \qquad (8\cdot 2)$$

then we shall write

$$A \rightleftarrows (\alpha,\ \alpha_0).$$

$$\alpha-\alpha_0 \qquad\qquad \alpha+\alpha_0$$
$$\text{———|————|————|———}$$
$$\alpha$$

FIG. 14

Example 8·1.

$$1 \pm 0.01 \rightleftarrows (1.00,\ 10^{-2}),$$
$$\langle 1.414 \rangle \rightleftarrows (1.4140,\ 5 \times 10^{-4}).$$

For practical convenience, these intervals are also written as

$$1.00;1 \quad \text{and} \quad 1.4140;5 \tag{8·3}$$

respectively.

We shall give the law of arithmetical operations applicable to the case when intervals are represented as (8·2).

If

$$\left.\begin{aligned} A &\rightleftarrows (\alpha,\ \alpha_0) \rightleftarrows \alpha+(-\alpha_0 \vee \alpha_0) \\ &\rightleftarrows \alpha-\alpha_0 \vee \alpha+\alpha_0, \\ B &\rightleftarrows (\beta,\ \beta_0) \rightleftarrows \beta+(-\beta_0 \vee \beta_0) \\ &\rightleftarrows \beta-\beta_0 \vee \beta+\beta_0, \end{aligned}\right\} \tag{8·4}$$

then in the case of addition

$$\begin{aligned} A+B &\rightleftarrows (\alpha,\ \alpha_0)+(\beta,\ \beta_0) \\ &\rightleftarrows \{\alpha+(-\alpha_0 \vee \alpha_0)\}+\{\beta+(-\beta_0 \vee \beta_0)\} \\ &\rightleftarrows \alpha+\beta+\{(-\alpha_0 \vee \alpha_0)+(-\beta_0 \vee \beta_0)\} \\ &\rightleftarrows \alpha+\beta+\{-(\alpha_0+\beta_0) \vee \alpha_0+\beta_0\} \\ &\rightleftarrows (\alpha+\beta,\ \alpha_0+\beta_0). \end{aligned} \tag{8·5}$$

Subtraction is written as follows.

$$\begin{aligned} A-B &\rightleftarrows (\alpha,\ \alpha_0)-(\beta,\ \beta_0) \\ &\rightleftarrows \{\alpha+(-\alpha_0 \vee \alpha_0)\}-\{\beta+(-\beta_0 \vee \beta_0)\} \\ &\rightleftarrows (\alpha-\beta,\ \alpha_0+\beta_0). \end{aligned} \tag{8·6}$$

In the case of multiplication, it is always possible to assume that

$$\alpha \geq \alpha_0 \geq 0, \quad \beta \geq \beta_0 \geq 0,$$

for practical calculations are performed with the absolute values and the errors are small.

Since

$$\alpha+\alpha_0 \geq \alpha-\alpha_0 \geq 0 \text{ and } \beta+\beta_0 \geq \beta-\beta_0 \geq 0,$$

we have

$$\begin{aligned} A \times B &\rightleftarrows (\alpha-\alpha_0)(\beta-\beta_0) \vee (\alpha+\alpha_0)(\beta+\beta_0) \\ &\rightleftarrows \alpha\beta-\alpha_0\beta-\alpha\beta_0+\alpha_0\beta_0 \\ &\qquad \vee \alpha\beta+\alpha_0\beta+\alpha\beta_0+\alpha_0\beta_0 \\ &\rightleftarrows \alpha\beta+\alpha_0\beta_0+(-\alpha_0\beta-\alpha\beta_0 \vee \alpha_0\beta+\alpha\beta_0) \\ &\rightleftarrows (\alpha\beta+\alpha_0\beta_0,\ \alpha_0\beta+\alpha\beta_0). \end{aligned} \tag{8·7}$$

In the case of division, it is possible to assume that

$$\alpha \geq \alpha_0 \geq 0, \quad \beta > \beta_0 \geq 0 \quad (\beta \neq \beta_0)$$

by similar reasoning to the above.

$$\begin{aligned} \frac{A}{B} &\rightleftarrows \frac{(\alpha,\ \alpha_0)}{(\beta,\ \beta_0)} \rightleftarrows \frac{\alpha-\alpha_0 \vee \alpha+\alpha_0}{\beta-\beta_0 \vee \beta+\beta_0} \\ &\rightleftarrows \frac{\alpha-\alpha_0}{\beta+\beta_0} \vee \frac{\alpha+\alpha_0}{\beta-\beta_0} \\ &\rightleftarrows \frac{1}{2}\left(\frac{\alpha-\alpha_0}{\beta+\beta_0}+\frac{\alpha+\alpha_0}{\beta-\beta_0}\right) \\ &\qquad +\frac{1}{2}\left(\frac{\alpha-\alpha_0}{\beta+\beta_0}-\frac{\alpha+\alpha_0}{\beta-\beta_0} \vee \frac{\alpha+\alpha_0}{\beta-\beta_0}-\frac{\alpha-\alpha_0}{\beta+\beta_0}\right) \\ &\rightleftarrows \frac{\alpha\beta+\alpha_0\beta_0}{\beta^2-\beta_0^2}+\left(-\frac{\alpha_0\beta+\alpha\beta_0}{\beta^2-\beta_0^2} \vee \frac{\alpha_0\beta+\alpha\beta_0}{\beta^2-\beta_0^2}\right) \\ &\rightleftarrows \left(\frac{\alpha\beta+\alpha_0\beta_0}{\beta^2-\beta_0^2},\ \frac{\alpha_0\beta+\alpha\beta_0}{\beta^2-\beta_0^2}\right). \end{aligned} \tag{8·8}$$

The above operations can be formulated as follows.

Theorem 11. *The arithmetical operations associated with interval expressions such as* (8·2) *are as follows.*

i) *Addition:*

$$(\alpha,\ \alpha_0)+(\beta,\ \beta_0) \rightleftarrows (\alpha+\beta,\ \alpha_0+\beta_0). \tag{8·5'}$$

ii) *Subtraction:*

$$(\alpha,\ \alpha_0)-(\beta,\ \beta_0) \rightleftarrows (\alpha-\beta,\ \alpha_0+\beta_0). \tag{8·6'}$$

iii) *Multiplication:*

$$(\alpha,\ \alpha_0)(\beta,\ \beta_0) \rightleftarrows (\alpha\beta+\alpha_0\beta_0,\ \alpha_0\beta+\alpha\beta_0), \tag{8·7'}$$

provided

$$\alpha \geq \alpha_0 \geq 0, \quad \beta \geq \beta_0 \geq 0.$$

iv) *Division:*

$$\frac{(\alpha,\ \alpha_0)}{(\beta,\ \beta_0)} \rightleftarrows \left(\frac{\alpha\beta+\alpha_0\beta_0}{\beta^2-\beta_0^2},\ \frac{\alpha_0\beta+\alpha\beta_0}{\beta^2-\beta_0^2}\right), \tag{8·8'}$$

provided

$$\alpha \geq \alpha_0 \geq 0, \quad \beta > \beta_0 \geq 0.$$

These operations are similar to those of complex numbers,

$$\begin{aligned} &(\xi+i\xi_0)+(\eta+i\eta_0)=(\xi+\eta)+i(\xi_0+\eta_0), \\ &(\xi+i\xi_0)-(\eta+i\eta_0)=(\xi-\eta)+i(\xi_0-\eta_0), \\ &(\xi+i\xi_0)(\eta+i\eta_0)=(\xi\eta-\xi_0\eta_0)+i(\xi_0\eta+\xi\eta_0), \\ &\frac{\xi+i\xi_0}{\eta+i\eta_0}=\frac{\xi\eta+\xi_0\eta_0}{\eta^2+\eta_0^2}+i\frac{\xi_0\eta-\xi\eta_0}{\eta^2+\eta_0^2}. \end{aligned}$$

That is, if we treat an interval A as a point on a plane in two-dimensional space as in Fig. 15, A is resolved into the two components, one of which is the middle value and the other is the upper bound of error.

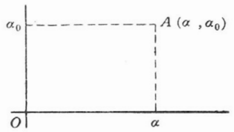

FIG. 15

Among the above operations the method of division is inconvenient practically. Therefore we need a more convenient method than (8·8) or (8·8′). We have

$$\frac{(\alpha,\ \alpha_0)}{(\beta,\ \beta_0)}\rightleftarrows\frac{\alpha-\alpha_0}{\beta+\beta_0}\vee\frac{\alpha+\alpha_0}{\beta-\beta_0}$$
$$\rightleftarrows\frac{\alpha}{\beta}+\Big(-\frac{\alpha_0\beta+\alpha\beta_0}{\beta(\beta+\beta_0)}\vee\frac{\alpha_0\beta+\alpha\beta_0}{\beta(\beta-\beta_0)}\Big).$$

Since

$$\beta+\beta_0\geq\beta-\beta_0>0,$$

we get

$$\Big(-\frac{\alpha_0\beta+\alpha\beta_0}{\beta(\beta+\beta_0)}\vee\frac{\alpha_0\beta+\alpha\beta_0}{\beta(\beta-\beta_0)}\Big)$$
$$\longrightarrow\Big(-\frac{\alpha_0\beta+\alpha\beta_0}{\beta(\beta-\beta_0)}\vee\frac{\alpha_0\beta+\alpha\beta_0}{\beta(\beta-\beta_0)}\Big)$$
$$\rightleftarrows\Big(-\frac{\alpha_0+\left(\frac{\alpha}{\beta}\right)\beta_0}{\beta-\beta_0}\vee\frac{\alpha_0+\left(\frac{\alpha}{\beta}\right)\beta_0}{\beta-\beta_0}\Big).$$

Hence we get the following theorem.

Theorem 12.

$$\frac{(\alpha,\ \alpha_0)}{(\beta,\ \beta_0)}\longrightarrow\Big(\frac{\alpha}{\beta},\ \frac{\alpha_0+\left(\frac{\alpha}{\beta}\right)\beta_0}{\beta-\beta_0}\Big)\qquad(8\cdot9)$$

provided

$$\alpha\geq\alpha_0\geq0,\quad \beta>\beta_0\geq0.$$

We first calculate $\frac{\alpha}{\beta}$ and then $\frac{\alpha_0+\left(\frac{\alpha}{\beta}\right)\beta_0}{\beta-\beta_0}$.

In practical calculations, it is often meaningless to calculate $\alpha\beta$ or α/β accurately and numerals should be rounded adequately. In such cases the following theorem is useful.

Theorem 13. *If*

$$A\rightleftarrows(\alpha,\ \alpha_0)\qquad(8\cdot10)$$

and

$$\alpha\longrightarrow(\alpha',\ \alpha_0'),$$

then

$$A\longrightarrow(\alpha',\ \alpha_0+\alpha_0').\qquad(8\cdot11)$$

Proof:

$$\alpha\longrightarrow\alpha'+(-\alpha_0'\vee\alpha_0'),$$
$$\alpha+(-\alpha_0\vee\alpha_0)$$
$$\longrightarrow\alpha'+(-\alpha_0'\vee\alpha_0')+(-\alpha_0\vee\alpha_0)$$
$$\rightleftarrows\alpha'+\{-(\alpha_0+\alpha_0')\vee\alpha_0+\alpha_0'\}$$
$$\rightleftarrows(\alpha',\ \alpha_0+\alpha_0').$$

Example 8·2. i) We have

$$1.432\longrightarrow1.43;1,$$

hence

$$1.432;50\longrightarrow1.43;6.$$

ii) For multiplication, we have

$$\alpha\beta+\alpha_0\beta_0\longrightarrow\alpha\beta+(-\alpha_0\beta_0\vee\alpha_0\beta_0)\rightleftarrows(\alpha\beta,\ \alpha_0\beta_0),$$

and hence

$$(\alpha,\ \alpha_0)(\beta,\ \beta_0)\longrightarrow(\alpha\beta,\ \alpha_0\beta_0+\alpha_0\beta+\alpha\beta_0)$$
$$\rightleftarrows(\alpha\beta,\ \alpha_0(\beta+\beta_0)+\alpha\beta_0)$$
$$\rightleftarrows\{\alpha\beta,\ \alpha_0\beta+(\alpha+\alpha_0)\beta\}.$$

This gives the method of calculation for multiplication.

Example 8·3. i) Addition:

$$1.689;4+2.745;1\rightleftarrows4.434;5,$$
$$3.624;8+1.24;3\rightleftarrows4.864;38$$
$$\longrightarrow4.86;(0.4+3.8)$$
$$\longrightarrow4.86;5.$$

ii) Subtraction:

$$3.429;5-1.201;2\rightleftarrows2.228;7,$$
$$6.724;7-2.30;4\rightleftarrows4.424;47$$
$$\longrightarrow4.42;6.$$

iii) Multiplication: To calculate the product (0.4320;5)(0.3810;5).

$$0.4320\times0.3810=0.164592\longrightarrow0.16459;1$$
$$0.44\times50+0.39\times50<42.$$
$$\therefore\ (0.4320;5)(0.3810;5)\longrightarrow0.16459;43$$
$$\longrightarrow0.1646;5.$$

iv) Division:

$$0.1646;5 \div 0.3810;5$$

$$\longrightarrow \left(0.43202\cdots,\ \frac{5+5\times 0.44}{0.38}\times 10^{-4}\right).$$

v) In the case of multiplication and division it is convenient to proceed in the following manner.

$$\begin{array}{r} 0.4320;5 \\ \times\ 0.3810;5 \\ \hline 1.2960;15 \\ 3456;\ 4 \\ 43;\ 1 \\ ;22 \\ \hline 0.16459;42 \end{array} \qquad \begin{array}{r} 0.432\ \ ;2 \\ 0.3810;5\sqrt{0.16460;50} \\ \underline{15240;20} \\ 1220;70 \\ \underline{1143;\ 2} \\ 77;72 \\ \underline{77;\ 1} \\ ;73 \\ \underline{;76} \\ ;-3 \end{array}$$

For instance, the first operation on the left hand side is

$$4320;5\times 3 \longrightarrow 12960;15.$$

The other operation can be performed similarly.

vi) To calculate

$$\gamma=\frac{7\sqrt{2}-\pi\sqrt{3}}{\pi^2+\sqrt{3}}$$

accurately up to the error of the order of 10^{-5}.

The values rounded off sufficiently for this case are

$$\sqrt{2}\longrightarrow 1.4142140;5,$$
$$\pi\longrightarrow 3.1415930;5,$$
$$\sqrt{3}\longrightarrow 1.7320510;5.$$

Hence

$$\gamma\longrightarrow\frac{7\times 1.4142140;5-3.1415930;5\times 1.7320510;5}{(3.1415930;5)^2+1.7320510;5}$$

$$\longrightarrow\frac{9.899498;4-5.441399;3}{9.869607;4+1.732051;1}$$

$$\longrightarrow\frac{4.458099;7}{11.601658;5}$$

$$\longrightarrow 0.384264;1.$$

9. Examples of numerical calculations

We shall describe here applications of the interval calculus to simple examples of numerical calculation so that the efficiency of our method will be shown.

Example 9·1. To solve the equation

$$f(x)\equiv x^3-3x+1=0. \qquad (9\cdot 1)$$

Here, we shall calculate the root between 0 and 1. We have

$$f(0.3)=0.127,$$
$$f(0.4)=-0.136,$$

hence a root lies in

$$0.3\vee 0.4.$$

Let it be denoted by α. The function value at the point $x=0.35$ is

$$f(0.35)=-0.007125$$

We also have

$$f'(x)=3(x^2-1),$$

therefore

$$f'(0.3\vee 0.4)\longrightarrow 3\{(0.3\vee 0.4)^2-1\}$$
$$\longrightarrow -(2.52\vee 2.73).$$

Solving

$$f(0.35)+f'(0.3\vee 0.4)dx=0 \qquad (9\cdot 2)$$

we get

$$dx\longrightarrow=\frac{-0.007125}{2.52\vee 2.73}\longrightarrow-(0.0026\vee 0.0029).$$

Therefore

$$\alpha\longrightarrow 0.3471\vee 0.3474. \qquad (9\cdot 3)$$

If one needs a more accurate value, similar processes are repeated. The accuracy associated with each step will increase step after step very remarkably. Thus, we first calculate

$$f(0.3472),\ f'(0.3471\vee 0.3474)$$

and then solve

$$f(0.3472)+f'(0.3471\vee 0.3474)dx=0 \qquad (9\cdot 4)$$

with respect to dx.

Example 9·2. To solve

$$e^x=\frac{2\sin x}{x}. \qquad (9\cdot 5)^{1)}$$

1) Cf. pp. 133~136 of reference [8].

Here it is assumed that the root is known to lie in the interval

$$0.62 \vee 0.63.$$

Let

$$f(x)=xe^{x}-2\sin x, \qquad (9\cdot 6)$$

then

$$f'(x)=(1+x)e^{x}-2\cos x. \qquad (9\cdot 7)$$

We make use of the following round-off values:

$$e^{0.620} \longrightarrow \langle 1.8589280 \rangle,$$
$$e^{0.630} \longrightarrow \langle 1.8776107 \rangle,$$
$$\sin 0.620 \longrightarrow \langle 0.5810352 \rangle,$$
$$\sin 0.630 \longrightarrow \langle 0.5891447 \rangle,$$
$$\cos 0.620 \longrightarrow \langle 0.81387 \rangle,$$
$$\cos 0.630 \longrightarrow \langle 0.80815 \rangle.$$

and hence we have

$$\begin{aligned}
f(0.62) &= 0.62\, e^{0.62} - 2\sin 0.62 \\
&\longrightarrow 0.62 \times 1.858928;1 \\
&\quad -2 \times 0.581035;1 \\
&\longrightarrow 1.1525354;7 \\
&\quad -1.162070;2 \\
&\longrightarrow -0.009535;9,
\end{aligned}$$

$$\begin{aligned}
f'(0.62 \vee 0.63) &\longrightarrow \{1+(0.62 \vee 0.63)\} e^{0.62 \vee 0.63} \\
&\quad -2\cos(0.62 \vee 0.63) \\
&\longrightarrow (1.62 \vee 1.63)(1.858 \vee 1.878) \\
&\quad -2(0.808 \vee 0.814) \\
&\longrightarrow (3.009 \vee 3.062)-(1.616 \vee 1.628) \\
&\longrightarrow 1.381 \vee 1.446,
\end{aligned}$$

whence

$$dx \longrightarrow -\frac{f(0.62)}{f'(0.62 \vee 0.63)} \longrightarrow \frac{(9.526 \vee 9.544) \times 10^{-3}}{1.381 \vee 1.446}$$
$$\longrightarrow (6.58 \vee 6.91) \times 10^{-3}.$$

Let α be the required root, then

$$\alpha \longrightarrow 0.6265 \vee 0.6270. \qquad (9\cdot 8)$$

The next step is to proceed in a similar manner, calculating

$$dx \longrightarrow \frac{-f(0.6267)}{f'(0.6265 \vee 0.6270)}.$$

Example 9·3. In such a problem as the above, one needs interpolation of numerical values.

Interval relations such as

$$e^{0.620} \longrightarrow 1.85892800;5,$$
$$e^{0.621} \longrightarrow 1.86078790;5$$

can be used to evaluate the value of

$$e^{0.620+n} \quad (n \longrightarrow 0.000 \vee 0.001) \qquad (9\cdot 9)$$

as follows.

Using the relation

$$f(x_1+dx) \longrightarrow f(x_1)+f'(x)dx$$

where

$$x_1 \longrightarrow x, \quad x_1+dx \longrightarrow x,$$

then, by means of the relation

$$e^{0.620+n} \longrightarrow e^{0.620}+e^{0.620 \vee 0.621} n, \qquad (9\cdot 10)$$

we proceed to obtain a sufficiently accurate value. For instance, when $n=0.0004$, we have

$$\begin{aligned}
e^{0.6204} &\longrightarrow 1.85892800;5 \\
&\quad +(1.8589 \vee 1.8608)0.0004 \\
&\longrightarrow 1.85892800;5+(1.8599;10)4 \times 10^{-4} \\
&\longrightarrow 1.85892800;5+7.4396;40 \times 10^{-4} \\
&\longrightarrow 1.85967196;45.
\end{aligned}$$

Thus, we have seen that making use of differential coefficients, we can increase the accuracy of interpolation.

Example 9·4. We cannot accurately calculate the value of an integral without evaluating the differential coefficients of high order. If we calculate a definite integral by Simpson's method, only a few differential coefficients of the lowest orders are enough. Here we shall study it.

Simpson's method is as follows.

$$\int_{-h}^{h} f(x)dx \longrightarrow \frac{h}{3}\{f(h)+f(-h)+4f(0)\} - \frac{h^5}{90} f^{(4)}(-h \vee h), \qquad (9\cdot 11)$$

hence it can be written

$$\begin{aligned}
\int_a^b f(x)dx &\longrightarrow \frac{h}{3}\{y_0+y_{2n}+2(y_2+y_4+\cdots+y_{2n-2}) \\
&\quad +4(y_1+y_3+\cdots+y_{2n-1})\} \\
&\quad -\frac{h^5}{90}\sum_{i=1}^{n} f^{(4)}\{x_{2i-1}+(-h \vee h)\},
\end{aligned}$$

where

$$h=\frac{b-a}{2n} \tag{9·12}$$

(cf. Fig. 16).

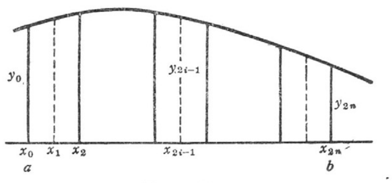

FIG. 16

For instance, to integrate

$$\int_0^1 \frac{dx}{1+x^2}\left(=\frac{\pi}{4}\right) \tag{9·13}$$

numerically one proceeds as follows.

$$f(x)=\frac{1}{1+x^2}=\frac{1}{2}\left(\frac{1}{1+ix}+\frac{1}{1-ix}\right),$$

$$f^{(4)}(x)=\frac{1}{2}\left\{\frac{(-1)^4(i)^4 4!}{(1+ix)^5}-\frac{(-1)^4(-i)^4 4!}{(1-ix)^5}\right\}$$

$$=\frac{4!}{2}\frac{\{(1-ix)^5+(1+ix)^5\}}{(1+x^2)^5}$$

$$=\frac{4!\{1-x^2(10-5x^2)\}}{(1+x^2)^5},$$

$$\frac{f^{(4)}(0\vee 0.2)}{4!}\longrightarrow\frac{1-(0\vee 0.2)^2\{10-5(0\vee 0.2)^2\}}{\{1+(0\vee 0.2)^2\}^5}$$

$$\longrightarrow\frac{1-(0\vee 0.04)(10\vee 9.8)}{(1\vee 1.04)^5}$$

$$\longrightarrow 1\vee 0.49.$$

Similarly,

$$\frac{f^{(4)}(0.2\vee 0.4)}{4!}\longrightarrow\frac{-0.57\vee 0.64}{1.2\vee 2.2}\longrightarrow(-0.48)\vee 0.54,$$

$$\frac{f^{(4)}(0.4\vee 0.6)}{4!}\longrightarrow\frac{-(0.3\vee 2.4)}{2.1\vee 4.7}\longrightarrow-(0.06\vee 1.15),$$

$$\frac{f^{(4)}(0.6\vee 0.8)}{4!}\longrightarrow\frac{-(1.4\vee 4.3)}{4.6\vee 12.0}\longrightarrow-(0.11\vee 0.94),$$

$$\frac{f^{(4)}(0.8\vee 1)}{4!}\longrightarrow\frac{-(2.2\vee 58)}{11.9\vee 32}\longrightarrow-(0.06\vee 0.49).$$

Then

$$\frac{1}{4!}\{f^{(4)}(0\vee 0.2)+f^{(4)}(0.2\vee 0.4)+f^{(4)}(0.4\vee 0.6)$$
$$+f^{(4)}(0.6\vee 0.8)+f^{(4)}(0.8\vee 1)\}$$
$$\longrightarrow -2.57\vee 1.31.$$

Hence

$$\int_0^1\frac{dx}{1+x^2}$$
$$\longrightarrow\frac{0.1}{3}\left\{1+\frac{1}{2}+2\left(\frac{1}{1.04}+\frac{1}{1.16}+\frac{1}{1.36}+\frac{1}{1.64}\right)\right.$$
$$\left.+4\left(\frac{1}{1.01}+\frac{1}{1.09}+\frac{1}{1.25}+\frac{1}{1.49}+\frac{1}{1.81}\right)\right\}$$
$$-\frac{(0.1)^5 4!}{90}(-2.6\vee 1.4).$$

Finally one gets

$$\pi\longrightarrow 3.141593;2. \tag{9·14}$$

In the above example, if statistical treatments are employed, the breadth of the interval of the final result will become narrower. Generally, it is important for the practical analysis that the interval calculus should be improved by statistical considerations. The same remark applies also to the next example.

Example 9·5. To investigate the equation

$$y'=f(x,\ y). \tag{9·15}$$

We calculate the differential coefficients of high order and evaluate them on intervals.

As an example, let us solve the equation

$$y'=2-\frac{x}{y} \tag{9·16}$$

with the initial condition

$$y=1 \text{ at } x=0.$$

The x-axis is divided into

$$\mathrm{I}\rightleftarrows 0\vee 0.1,\quad \mathrm{II}\rightleftarrows 0.1\vee 0.2,\ \text{etc.}$$

Differential coefficients of high order are calculated from the relations

$$\left.\begin{aligned} yy'&=2y-x,\\ yy''+y'^2&=2y'-1,\\ &\cdots\cdots. \end{aligned}\right\} \tag{9.17}$$

y on the interval $0\vee 0.1$ is denoted by y_I and that on $0.1\vee 0.2$ by y_{II}. We then have

$$y'_I\longrightarrow 2-\frac{(0\vee 0.1)}{1}\longrightarrow 1.9\vee 2,$$

hence

$$y_I\longrightarrow 1+(1.9\vee 2)x. \tag{9.18}$$

To raise its accuracy we must first calculate the differential coefficient of the second order, i. e.,

$$y''_I \longrightarrow \frac{y'_I(2-y'_I)-1}{y_I} \longrightarrow \frac{(1.9\vee 2)(0\vee 0.1)-1}{1\vee 1.2}$$
$$\longrightarrow \frac{0\vee 0.2-1}{1\vee 1.2} \longrightarrow -\left(\frac{0.8\vee 1}{1\vee 1.2}\right)$$
$$\longrightarrow -(0.66\vee 1).$$

Since

$$y'_{x=0}=2,$$
$$y_I \longrightarrow 1+2x-\frac{0.66\vee 1}{2}x^2. \qquad (9\cdot 19)$$

Therefore

$$y_{0.1}\longrightarrow 1+0.2-(0.33\vee 0.5)\times 10^{-2}$$
$$\longrightarrow 1.1957;9,$$
$$y'_{0.1}\longrightarrow 2-(0.66\vee 1)0.1\longrightarrow 1.917;17,$$

where $y_{0.1}$ is the value of y for $x=0.1$ and $y'_{0.1}$ is analogously defined.

Next, we go to the second interval

$$0.1\vee 0.2.$$

In this interval, we have

$$y'_{II}\longrightarrow 2\vee\left(2-\frac{0.2}{1.19}\right)\longrightarrow 2\vee 1.83,$$
$$y_{II}\longrightarrow 1.19\vee 1.20+(2\vee 1.83)(x-0.1),$$
$$y_{0.2}\longrightarrow 1.19\vee 1.20+(2\vee 1.83)0.1$$
$$\longrightarrow 1.37\vee 1.40.$$

Using this value we can evaluate more accurately, i. e.,

$$y'_{II}\longrightarrow 2-\frac{0.1\vee 0.2}{1.19\vee 1.40}\longrightarrow 2-(0.07\vee 0.17)$$
$$\longrightarrow 1.83\vee 1.93.$$

Hence

$$y''_{II}\longrightarrow \frac{y'_{II}(2-y'_I)-1}{y_{II}}$$
$$\longrightarrow \frac{(1.83\vee 1.93)(0.07\vee 0.17)-1}{1.19\vee 1.40}$$
$$\longrightarrow -\left(\frac{0.67\vee 0.88}{1.19\vee 1.40}\right)\longrightarrow -(0.47\vee 0.74).$$

$$\left.\begin{aligned} y_{II}&\longrightarrow y_{0.1}+y'_{0.1}(x-0.1)+\frac{y''_{II}}{2}(x-0.1)^2\\ &\longrightarrow 1.1957;9+1.917;17(x-0.1)\\ &\quad -\frac{(0.47\vee 0.74)}{2}(x-0.1)^2.\\ y_{0.2}&\longrightarrow 1.1957;9+0.1917;17\\ &\quad -(30;7)\times 10^{-4}\longrightarrow 1.3844;33.\end{aligned}\right\} \qquad (9.20)$$

The above procedures are explained graphically in Fig. 17.

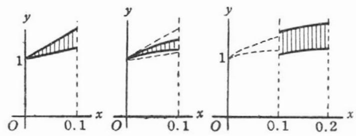

FIG. 17

As stated above, we can integrate (9·16) as accurately as we wish and carry it out over a wide region. We cut the x-axis into intervals, obtain the solution on each interval, and reconnect them again.

CONCLUSION

We have realized that numerals should be essentially considered from the topological point of view.

The chief results of this study are as follows:

1^0 Numerals are treated as intervals and the concepts of numeral and error are made clear. And an interval calculus is established algebraically from the lattice theoretical point of view so that it can be applied conveniently to the numerical calculation.

2^0 Interval functions and functionals, and their differentiation are investigated and used effectively in some examples of applied analysis.

Future problems will be:

1* To investigate problems of the numerical calculation connected with higher dimensional mathematics, for instance, matrix inversion[1), partial differential equations, etc.,

2* To investigate direct applications of the interval calculus to physical and engineering problems,

3* To revise the structure of the automatic digital computer from the standpoint of interval calculus and topology,

1) Cf. reference [1].

4* To prove the applicability to other fields, of our view that scientific laws should be stated essentially in the language of finite elements and discrete topology.

REFERENCES

[1] J. von Neumann and H. H. Goldstine, Numerical Inverting of Matrices of High Order. *Bulletin of the American Mathematical Society*, **53**, 11 (1947), 1021~1099.

[2] C. E. Shannon, *The Mathematical Theory of Communication*. The University of Illinois Press, Urbana, 1949.

[3] T. Sunaga, A Basic Theory of Communication, *Memoirs*, **2**, G-I (1958), 426~443.

[4] D. Hilbert and W. Ackermann, *Grundzüge der theoretischen Logik*. Julius Springer, Berlin, 1949.

[5] G. Birkhoff, *Lattice Theory* (Revised edition). American Mathematical Society Colloquium Publications, **25**, New York, 1940.

[6] L. Pontrjagin, *Topological Groups*. Princeton University Press, Princeton, 1946.

[7] L. Schwartz, *Théorie des distributions, I*. Hermann, Paris, 1950.

[8] W. Sibagaki, *Foundation of Numerical Calculation* (in Japanese). Kawade-shobo, Tokyo, 1951.