A Constructive A Priori Error Estimation for Finite Element Discretizations in a Non-Convex Domain Using Singular Functions

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In solving elliptic problems by the finite element method in a bounded domain which has a re-entrant corner, the rate of convergence can be improved by adding a singular function to the usual interpolating basis. When the domain is enclosed by line segments which form a corner of $\pi/2$ or $3\pi/2$, we have obtained an explicit a priori H_0^1 error estimation of $O(h)$ and an L^2 error estimation of $O(h^2)$ for such a finite element solution of the Poisson equation. Particularly, we emphasize that all constants in our error estimates are numerically determined, which plays an essential role in the numerical verification of solutions to non-linear elliptic problems.

Key words: finite element method, a priori error estimation, Poisson equation

1. Introduction

In this paper, we consider the elliptic problem on a polygonal domain Ω which is enclosed by line segments and right angles. The domain is assumed to be connected, but not necessarily simply connected. First of all, we assume $\Omega = \Omega_0$, where Ω_0 is the L-shape domain which is shown in Fig. 1. The general case is described in Section 3.

For $f \in L^2(\Omega)$, we consider the weak solution of the following partial differential equation:

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

It is known that u has a singular function representation [9, 10],

$$u(x,y) = w(x,y) + \lambda\sigma(x,y), \tag{1.2}$$

where $w(x,y) \in H^2(\Omega) \cap H_0^1(\Omega)$, λ is a constant, $\sigma(x,y) \in H_0^1(\Omega)$ and

$$\sigma(x,y) \sim r^{2/3} \sin\left(\frac{2}{3}\theta\right)$$

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in a neighborhood of the origin. Here, (r, θ) is the polar coordinate of (x, y) where θ satisfies $0 \le \theta < 2\pi$. The choice of σ is arbitrary, but in order to simplify the calculation of H_0^1 inner product, we take

$$\sigma(x,y) = (1-x^2)(1-y^2)r^{2/3}\sin\left(\frac{2}{3}\theta\right)$$

in this paper.

In solving this problem by the finite element method, we use the square mesh with mesh size h. The mesh of $h = 1/8$ is shown in Fig. 2. We use the piecewise bilinear function,

$$\phi_{i,j}(x,y) \equiv \max\left(1 - \left|\frac{x}{h} - i\right|, 0\right) \cdot \max\left(1 - \left|\frac{y}{h} - j\right|, 0\right)$$

as the finite element basis and define Φ_h by the set of functions $\phi_{i,j}$ in $H_0^1(\Omega)$,

$$\Phi_h = \{\phi_{i,j} \mid (ih, jh) \in \Omega\}.$$

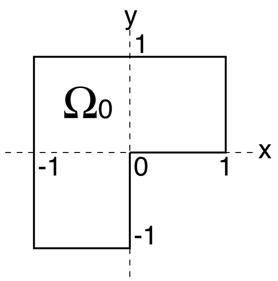

Fig. 1. The shape of Ω_0

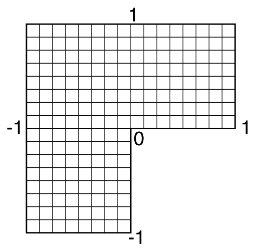

Fig. 2. The square mesh when $h = 1/8$

In general, since u does not have H^2 regularity, we can not obtain $O(h)$ error estimates with an H_0^1 norm by using this interpolating basis. Therefore, we adopt $\Phi_h \cup \{\sigma\}$ as the finite element basis. In this case, it is known that the following error estimation holds [8, 9, 10, 14]:

$$\|u_h - u\|_{H_0^1(\Omega)} \le Ch\|f\|_{L^2(\Omega)},$$

where u_h is the finite element solution. The following $O(h^2)$ estimation for the L^2-error is also obtained by the Aubin–Nitsche trick [6]:

$$\|u_h - u\|_{L^2(\Omega)} \le C^2h^2\|f\|_{L^2(\Omega)}.$$

The main purpose of this paper is to obtain this constant C.

The coefficient λ in (1.2) is often called the stress intensity factor in the context of mechanics. In our error estimation, the explicit evaluation of the coefficient λ

is essential (Lemma 4.2 and Lemma 4.6 in Section 4). For the coefficient λ, the following extraction formula holds [9, 10]:

$$\lambda = \frac{1}{\pi}\left\{\iint_{\Omega} f\eta s_- \,dx\,dy + \iint_{\Omega} u\Delta(\eta s_-)\,dx\,dy\right\},$$

where

$$s_- = r^{-2/3}\sin\left(\frac{2}{3}\theta\right)$$

and η is a smooth cut-off function which equals one at the origin and zero on $\{(x,y) \mid \max(|x|,|y|) = 1\}$. In this extraction formula, the coefficient λ does not depend on η. The Poincaré–Friedrichs inequality is needed to evaluate λ by this extraction formula. However, since the Poincaré–Friedrichs inequality is reduced to a kind of problem of eigenvalue bounds, it is not easy to obtain a good estimation, except for the case that Ω is a simple domain such as a rectangle. In this paper, instead of using a cut-off function, we use the maximum principle for the super harmonic functions to evaluate λ directly.

There are several approaches to deal with the lack of regularity at the re-entrant corner. The most simple method which is described in [8, 14] is that of adding singular functions to the finite element basis. This method is simple but is enough to obtain optimal order of H_0^1 and L^2 error bounds. The dual singular function method (DSFM) [3, 7] is presented to obtain a better approximation to the coefficient λ. DSFM consists of a system of w and λ which is derived from the extraction formula, and is often implemented as an iterative procedure. A multigrid version of this method appears in [4] and an efficient method using an improved extraction formula was presented in [5]. Another useful method is based on the local mesh refinement [2]. The advantage of using mesh refinement is that calculation of the element matrix is easy, because the information about the singular function is not needed. In this paper, we construct a priori error estimation for the finite element solution based on [8, 14]. A priori error estimation for more efficient methods such as DSFM or the method of mesh refinement is further work.

In many applications, it is useful to obtain an explicit error estimation. For example, numerical verification methods for nonlinear problems are based on explicit error bounds for linear equations [12, 17]. We emphasize that, even though there are many methods concerning with finite element solution in a non-convex domain, there is no other research which gives an *explicit* a priori H_0^1 estimation of $O(h)$ and an *explicit* a priori L^2 estimation of $O(h^2)$.

While this paper concerns a priori error estimation, a posteriori error estimation is also an important problem. See [16] for the detail of a posteriori error estimation for non-convex domains.

The present paper is organized as follows. In Section 2, we present a priori error estimation in the case that Ω is a simple L-shape domain. The general case is explained in Section 3. Section 4 contains proof of lemmas which appear in

Sections 2 and 3. We show numerical results in Section 5 and conclude this paper with Section 6.

Throughout this paper, we take the angle of polar coordinates in $[0, 2\pi)$ and 1_A denotes the function which takes value 1 if condition A holds, and takes value 0 otherwise.

2. A priori error estimation

The main purpose of this section is to prove the following theorem. Lemmas appearing in this section will be proved in Section 4.

Theorem 2.1. *When $\Omega = \Omega_0$, as to the finite element solution u_h by using $\Phi_h \cup \{\sigma\}$ as the basis, the following error estimation holds:*

$$\|u_h - u\|_{H_0^1(\Omega)} \le 1.156h\|f\|_{L^2(\Omega)}, \tag{2.1}$$

$$\|u_h - u\|_{L^2(\Omega)} \le 1.335h^2\|f\|_{L^2(\Omega)}. \tag{2.2}$$

Proof. We represent the exact solution as

$$u = w + \lambda\sigma, \tag{2.3}$$

where λ is a constant which depends on Ω and f, and w is a function which belongs to $H^2(\Omega) \cap H_0^1(\Omega)$.

Define w_h as the bilinear interpolation of w. Then, since $w \in H^2(\Omega) \cap H_0^1(\Omega)$,

$$\|w - w_h\|_{H_0^1(\Omega)} \le \frac{h}{\pi}|w|_{H^2(\Omega)}$$

holds [13]. Since Ω is a polygonal domain, the following equality holds [11]:

$$|w|_{H^2(\Omega)} = \|\Delta w\|_{L^2(\Omega)}.$$

Immediately, we have

$$\|w - w_h\|_{H_0^1(\Omega)} \le \frac{h}{\pi}\|\Delta w\|_{L^2(\Omega)}.$$

Here, let

$$\tilde{u}_h = w_h + \lambda\sigma$$

then, we have

$$\|u - \tilde{u}_h\|_{H_0^1(\Omega)} \le \frac{h}{\pi}\|\Delta w\|_{L^2(\Omega)}. \tag{2.4}$$

From (2.3), the following inequality holds:

$$\|\Delta w\|_{L^2(\Omega)} \le \|f\|_{L^2(\Omega)} + |\lambda|\, \|\Delta\sigma\|_{L^2(\Omega)}. \tag{2.5}$$

From Lemma 4.1, we have

$$\left\| \Delta\left\{(1-x^2)(1-y^2)r^{2/3}\sin\left(\frac{2}{3}\theta\right)\right\}\right\|_{L^2(\Omega)} \le \sqrt{\frac{4000}{81} - \frac{11713}{1782}\pi}. \tag{2.6}$$

For the coefficient λ, Lemma 4.2 implies

$$|\lambda| \le \frac{1}{\pi}\left\|(r^{-2/3} - 2^{-2/3}r^{2/3})\sin\left(\frac{2}{3}\theta\right)\right\|_{L^2(\Omega)} \|f(x,y)\|_{L^2(\Omega)} \tag{2.7}$$

and, from Lemma 4.3,

$$\left\|(r^{-2/3} - 2^{-2/3}r^{2/3})\sin\left(\frac{2}{3}\theta\right)\right\|_{L^2(\Omega)} \le \sqrt{\frac{3\cdot 2^{1/3}}{5}\pi}.$$

Consequently, (2.4), (2.5), (2.6) and (2.7) lead us to

$$\begin{aligned}\|u - \tilde{u}_h\|_{H_0^1(\Omega)} &\le \frac{h}{\pi}\left(1 + \sqrt{\frac{4000}{81} - \frac{11713}{1782}\pi}\cdot\frac{1}{\pi}\sqrt{\frac{3\cdot 2^{1/3}}{5}\pi}\right)\|f\|_{L^2(\Omega)} \\ &= 1.1552884253\ldots\, h\|f\|_{L^2(\Omega)} \le 1.156h\|f\|_{L^2(\Omega)}.\end{aligned}$$

Since the finite element solution u_h is the best approximation in H_0^1 space, we have

$$\|u_h - u\|_{H_0^1(\Omega)} \le \|u - \tilde{u}_h\|_{H_0^1(\Omega)} \le 1.156h\|f\|_{L^2(\Omega)}.$$

Moreover,

$$\|u_h - u\|_{L^2(\Omega)} \le 1.1553^2 h^2\|f\|_{L^2(\Omega)} \le 1.335h^2\|f\|_{L^2(\Omega)}$$

is also obtained by the Aubin–Nitsche trick. □

3. Generalization

We consider general cases in this section. We suppose that Ω is enclosed by line segments which form a corner of $\pi/2$ or $3\pi/2$. Also Ω is assumed to be connected but it is not necessarily simply connected (Fig. 3).

The following singular bases are used together with the usual interpolating basis:

$$T_k\sigma\left(\frac{x}{l_k}, \frac{y}{l_k}\right) \quad (k = 1, \ldots, n),$$

where n is a number of re-entrant corners,

$$\sigma(x,y) = \begin{cases} (1-x^2)(1-y^2)r^{2/3}\sin\left(\dfrac{2}{3}\theta\right) & ((x,y)\in\Omega_0), \\ 0 & (\text{otherwise}), \end{cases}$$

and T_k is a combination of parallel translation and rotation.

In the above, l_k denotes sizes of the singular bases. Different sizes of the singular bases are admissible. It is also admissible even if some part of the support of the singular bases are overlapped (Fig. 4).

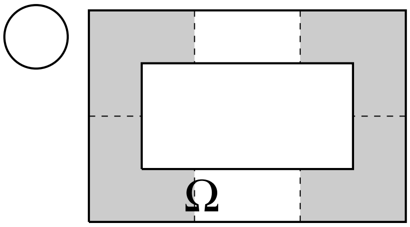

Fig. 3. Admissible pattern

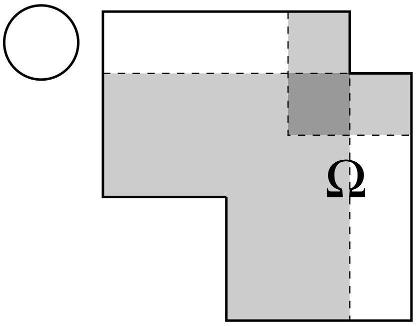

Fig. 4. Admissible pattern

There are some restrictions on defining the singular bases. Let us now suppose that Γ is the support of a singular basis, and $\partial\Gamma$ consists of line segments $\gamma_1 \sim \gamma_6$ where γ_1 and γ_6 form the re-entrant corner (Fig. 5). In this case, γ_1 and γ_6 must be contained in $\partial\Omega$ and $\gamma_2 \sim \gamma_5$ must coincide with the grid line of the mesh. Therefore, both Fig. 6 and Fig. 7 are not admissible.

In this situation, we have the following theorem.

Theorem 3.1. *For the finite element solution u_h with the basis $\Phi_h \cup \{T_k\sigma(x/l_k, y/l_k) \mid k=1,\ldots,n\}$, the following error estimation holds:*

$$\|u_h - u\|_{H_0^1(\Omega)} \le \left(0.319 + \sum_{k=1}^{n}\sqrt{0.971 + 1.469\frac{|\Omega|}{l_k^2}}\right) h\|f\|_{L^2(\Omega)}, \tag{3.1}$$

$$\|u_h - u\|_{L^2(\Omega)} \le \left(0.319 + \sum_{k=1}^{n}\sqrt{0.971 + 1.469\frac{|\Omega|}{l_k^2}}\right)^2 h^2\|f\|_{L^2(\Omega)}, \tag{3.2}$$

where $|\Omega|$ denotes the area of Ω.

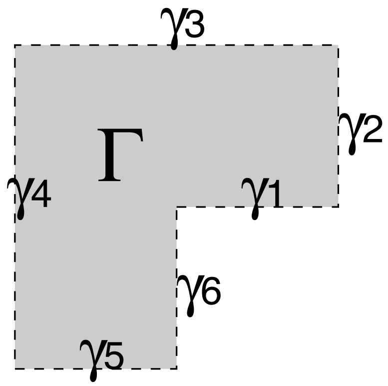

Fig. 5. The support of a singular basis

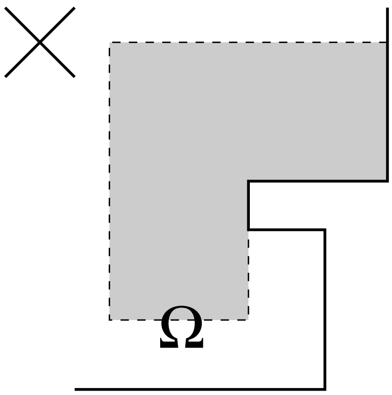

Fig. 6. Nonadmissible pattern

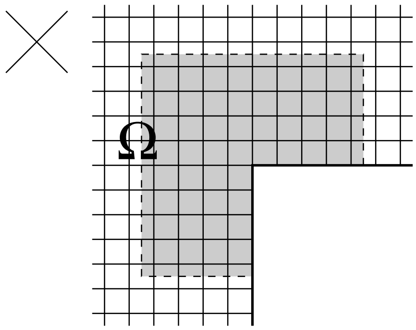

Fig. 7. Nonadmissible pattern

Proof. Let

$$u(x,y) = w(x,y) + \sum_{k=1}^{n} \lambda_k T_k \sigma\left(\frac{x}{l_k}, \frac{y}{l_k}\right), \quad w(x,y) \in H^2(\Omega) \cap H_0^1(\Omega),$$

be the exact solution. We define $\tilde{u}_h$ as follows:

$$\tilde{u}_h(x,y) = w_h(x,y) + \sum_{k=1}^{n} \lambda_k T_k \sigma\left(\frac{x}{l_k}, \frac{y}{l_k}\right),$$

where w_h denotes the bilinear interpolation of w. We also define

$$\bar{\sigma}(x,y) = \begin{cases} (1-r)^2 r^{2/3} \sin\left(\dfrac{2}{3}\theta\right) & (r < 1), \\ 0 & (r \geq 1). \end{cases}$$

Now, we exclude the grid of the square mesh from Ω and define it as Ω_*. Then,

$$\begin{aligned}
\|u-\tilde{u}_h\|_{H_0^1(\Omega)} &= \|w-w_h\|_{H_0^1(\Omega_*)} \le \frac{h}{\pi}|w|_{H^2(\Omega_*)} \\
&= \frac{h}{\pi}\left|u-\sum_{k=1}^{n}\lambda_k T_k\sigma\left(\frac{x}{l_k},\frac{y}{l_k}\right)\right|_{H^2(\Omega_*)} \\
&\le \frac{h}{\pi}\left|u-\sum_{k=1}^{n}\lambda_k T_k\bar{\sigma}\left(\frac{x}{l_k},\frac{y}{l_k}\right)\right|_{H^2(\Omega)} \\
&\quad + \frac{h}{\pi}\left|\sum_{k=1}^{n}\lambda_k T_k\left(\sigma\left(\frac{x}{l_k},\frac{y}{l_k}\right)-\bar{\sigma}\left(\frac{x}{l_k},\frac{y}{l_k}\right)\right)\right|_{H^2(\Omega_*)} \\
&= \frac{h}{\pi}\left\|\Delta\left(u-\sum_{k=1}^{n}\lambda_k T_k\bar{\sigma}\left(\frac{x}{l_k},\frac{y}{l_k}\right)\right)\right\|_{L^2(\Omega)} \\
&\quad + \frac{h}{\pi}\left\|\sum_{k=1}^{n}\lambda_k \Delta T_k\left(\sigma\left(\frac{x}{l_k},\frac{y}{l_k}\right)-\bar{\sigma}\left(\frac{x}{l_k},\frac{y}{l_k}\right)\right)\right\|_{L^2(\Omega_*)} \\
&\le \frac{h}{\pi}\|f(x,y)\|_{L^2(\Omega)} + \frac{h}{\pi}\sum_{k=1}^{n}\frac{|\lambda_k|}{l_k}\|\Delta\bar{\sigma}(x,y)\|_{L^2(\Omega_0)} \\
&\quad + \frac{h}{\pi}\sum_{k=1}^{n}\frac{|\lambda_k|}{l_k}\|\Delta(\sigma(x,y)-\bar{\sigma}(x,y))\|_{L^2(\Omega_0)}.
\end{aligned}$$

From Lemma 4.4, we have

$$\|\Delta\bar{\sigma}(x,y)\|_{L^2(\Omega_0)} = \frac{3\sqrt{\pi}}{2}.$$

Lemma 4.5 implies that

$$\|\Delta(\sigma(x,y)-\bar{\sigma}(x,y))\|_{L^2(\Omega_0)} \le \sqrt{\frac{4000}{81}-\frac{232367}{46332}\pi}.$$

Then, we have

$$\|u-\tilde{u}_h\|_{H_0^1(\Omega)} \le \frac{h}{\pi}\|f\|_{L^2(\Omega)} + \frac{h}{\pi}\left(\frac{3\sqrt{\pi}}{2}+\sqrt{\frac{4000}{81}-\frac{232367}{46332}\pi}\right)\sum_{k=1}^{n}\frac{|\lambda_k|}{l_k}.$$

Now, from Lemma 4.6, we have the following evaluation:

$$|\lambda_k| \leq \left\{ \iint_\Omega 1_{0\leq\theta<3\pi/2} \cdot \left| G_{l_k}\left(r, \cos\left(\frac{4}{3}\theta\right), \cos\left(\frac{4}{3}\theta\right)\right)\right|^2 dx\,dy \right.$$
$$\left. + \iint_\Omega 1_{3\pi/2\leq\theta<2\pi} \cdot \left| G_{l_k}\left(r, \frac{1}{9}(8+\cos(4\theta)), 1\right)\right|^2 dx\,dy \right\}^{1/2} \|f(x,y)\|_{L^2(\Omega)},$$

where

$$G_l(r,X,Y) = \frac{l^{2/3}}{\sqrt{2}\pi}\sqrt{\sqrt{r^{-8/3}+l^{-8/3}-2r^{-4/3}l^{-4/3}X}+l^{-4/3}-r^{-4/3}Y}.$$

It follows from Lemma 4.7 that

$$|\lambda_k| \leq \frac{1}{\pi}\sqrt{\left(\frac{5}{2}-\frac{3\pi}{8}\right)l_k^2+2|\Omega|}\cdot\|f\|_{L^2(\Omega)}.$$

Then,

$$\begin{aligned}
&\|u-\tilde{u}_h\|_{H_0^1(\Omega)} \\
&\leq \frac{h}{\pi}\left(1+\left(\frac{3\sqrt{\pi}}{2}+\sqrt{\frac{4000}{81}-\frac{232367}{46332}\pi}\right)\sum_{k=1}^n \frac{1}{\pi}\sqrt{\left(\frac{5}{2}-\frac{3\pi}{8}\right)+\frac{2|\Omega|}{l_k^2}}\right)\|f\|_{L^2(\Omega)} \\
&= \left(0.31830988\ldots+\sum_{k=1}^n\sqrt{0.97070784\ldots+1.46865243\ldots\frac{|\Omega|}{l_k^2}}\right)h\|f\|_{L^2(\Omega)} \\
&\leq \left(0.319+\sum_{k=1}^n\sqrt{0.971+1.469\frac{|\Omega|}{l_k^2}}\right)h\|f\|_{L^2(\Omega)}.
\end{aligned}$$

Since the finite element solution u_h is the best approximation in H_0^1 space,

$$\begin{aligned}
\|u_h-u\|_{H_0^1(\Omega)} &\leq \|u-\tilde{u}_h\|_{H_0^1(\Omega)} \\
&\leq \left(0.319+\sum_{k=1}^n\sqrt{0.971+1.469\frac{|\Omega|}{l_k^2}}\right)h\|f\|_{L^2(\Omega)}.
\end{aligned}$$

We also obtain

$$\|u_h-u\|_{L^2(\Omega)} \leq \left(0.319+\sum_{k=1}^n\sqrt{0.971+1.469\frac{|\Omega|}{l_k^2}}\right)^2 h^2\|f\|_{L^2(\Omega)}$$

by the Aubin–Nitsche trick. □

4. Lemmas

Lemma 4.1.

$$\left\|\Delta\left\{(1-x^2)(1-y^2)r^{2/3}\sin\left(\frac{2}{3}\theta\right)\right\}\right\|_{L^2(\Omega_0)} \le \sqrt{\frac{4000}{81}-\frac{11713}{1782}\pi}.$$

Proof.

$$\begin{aligned}
&\Delta\left\{(1-x^2)(1-y^2)r^{2/3}\sin\left(\frac{2}{3}\theta\right)\right\}\\
&=\frac{2}{3}r^{2/3}\left\{(4r^2-10)\sin\left(\frac{2}{3}\theta\right)+r^2\sin\left(\frac{10}{3}\theta\right)\right\}\\
&=\frac{2}{3}r^{2/3}\sin\left(\frac{2}{3}\theta\right)\left\{9r^2-10-20r^2\sin^2\left(\frac{2}{3}\theta\right)+16r^2\sin^4\left(\frac{2}{3}\theta\right)\right\}.
\end{aligned}$$

When $1\le r\le\sqrt{2}$,

$$\begin{aligned}
&\Delta\left\{(1-x^2)(1-y^2)r^{2/3}\sin\left(\frac{2}{3}\theta\right)\right\}\\
&=\frac{2}{3}r^{2/3}\sin\left(\frac{2}{3}\theta\right)\left\{9r^2-10-4r^2\sin^2\left(\frac{2}{3}\theta\right)-16r^2\sin^2\left(\frac{2}{3}\theta\right)\left(1-\sin^2\left(\frac{2}{3}\theta\right)\right)\right\}\\
&\le\frac{2}{3}r\sin\left(\frac{2}{3}\theta\right)\left\{8-4r^2\sin^2\left(\frac{2}{3}\theta\right)\right\},
\end{aligned}$$

and

$$\begin{aligned}
&\Delta\left\{(1-x^2)(1-y^2)r^{2/3}\sin\left(\frac{2}{3}\theta\right)\right\}\\
&=-\frac{2}{3}r^{2/3}\sin\left(\frac{2}{3}\theta\right)\left\{10-4r^2\sin^2\left(\frac{2}{3}\theta\right)-r^2\left(3-4\sin^2\left(\frac{2}{3}\theta\right)\right)^2\right\}\\
&\ge-\frac{2}{3}r\sin\left(\frac{2}{3}\theta\right)\left\{10-4r^2\sin^2\left(\frac{2}{3}\theta\right)\right\},
\end{aligned}$$

which implies

$$\begin{aligned}
&\left|\Delta\left\{(1-x^2)(1-y^2)r^{2/3}\sin\left(\frac{2}{3}\theta\right)\right\}\right|\\
&\le\frac{2}{3}r\sin\left(\frac{2}{3}\theta\right)\left\{10-4r^2\sin^2\left(\frac{2}{3}\theta\right)\right\}\\
&=\frac{20\sqrt{10}}{9\sqrt{3}}-\frac{8}{3}\left(r\sin\left(\frac{2}{3}\theta\right)-\sqrt{\frac{5}{6}}\right)^2\left(r\sin\left(\frac{2}{3}\theta\right)+\sqrt{\frac{10}{3}}\right)\le\frac{20\sqrt{10}}{9\sqrt{3}}.
\end{aligned}$$

Then

$$\begin{aligned}
&\left\| \Delta\left\{(1-x^2)(1-y^2)r^{2/3}\sin\left(\frac{2}{3}\theta\right)\right\}\right\|^2_{L^2(\Omega_0)} \\
&\le \frac{4}{9}\int_0^1\int_0^{3\pi/2} r^{4/3}\left\{(4r^2-10)\sin\left(\frac{2}{3}\theta\right)+r^2\sin\left(\frac{10}{3}\theta\right)\right\}^2 r\,d\theta\,dr \\
&\quad + \left(|\Omega_0|-\frac{3}{4}\pi\right)\left(\frac{20\sqrt{10}}{9\sqrt{3}}\right)^2 \\
&= \frac{127}{22}\pi + \left(3-\frac{3}{4}\pi\right)\frac{4000}{243} = \frac{4000}{81} - \frac{11713}{1782}\pi.
\end{aligned}$$

□

LEMMA 4.2. *When $\Omega = \Omega_0$,*

$$|\lambda| \le \frac{1}{\pi}\left\|(r^{-2/3}-2^{-2/3}r^{2/3})\sin\left(\frac{2}{3}\theta\right)\right\|_{L^2(\Omega)} \|f(x,y)\|_{L^2(\Omega)}.$$

Proof. For any $0 < \varepsilon < 1$, let g_ε be a weak solution of the following equation:

$$\begin{cases} -\Delta g_\varepsilon = -\Delta \widetilde{g_\varepsilon} & \text{in } \Omega, \\ g_\varepsilon = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

where

$$\widetilde{g_\varepsilon}(x,y) = \begin{cases} \dfrac{1}{\pi}((2\varepsilon^{-4/3}-2^{-2/3})r^{2/3} - \varepsilon^{-8/3}r^2)\sin\left(\dfrac{2}{3}\theta\right) & (r<\varepsilon), \\ \dfrac{1}{\pi}(r^{-2/3}-2^{-2/3}r^{2/3})\sin\left(\dfrac{2}{3}\theta\right) & (\varepsilon \le r). \end{cases}$$

From the fact that $-\Delta\widetilde{g_\varepsilon} \ge 0$ in Ω, g_ε and $\widetilde{g_\varepsilon} - g_\varepsilon$ are both superharmonic functions in Ω. Since g_ε and $\widetilde{g_\varepsilon} - g_\varepsilon$ take non-negative values on $\partial\Omega$,

$$0 \le g_\varepsilon(x,y) \le \widetilde{g_\varepsilon}(x,y).$$

Consequently, we have

$$|g_\varepsilon(x,y)| \le |\widetilde{g_\varepsilon}(x,y)|.$$

From (1.1) and (4.1), taking g_ε and u as test functions,

$$\begin{aligned}
\iint_\Omega f g_\varepsilon\,dx\,dy &= \iint_\Omega \nabla u\cdot\nabla g_\varepsilon\,dx\,dy = -\iint_\Omega u\Delta\widetilde{g_\varepsilon}\,dx\,dy \\
&= \frac{32\varepsilon^{-8/3}}{9\pi}\iint_\Omega 1_{r<\varepsilon}\cdot u(x,y)\sin\left(\frac{2}{3}\theta\right)dx\,dy
\end{aligned}$$

$$
\begin{aligned}
&= \frac{32\varepsilon^{-8/3}\lambda}{9\pi}\int_0^{\varepsilon}\int_0^{3\pi/2}(1-x^2)(1-y^2)r^{2/3}\sin^2\left(\frac{2}{3}\theta\right) r\,d\theta\,dr \\
&\quad + \frac{32\varepsilon^{-8/3}}{9\pi}\int_0^{\varepsilon}\int_0^{3\pi/2} w(x,y)\sin\left(\frac{2}{3}\theta\right) r\,d\theta\,dr \\
&= \left(1-\frac{4}{7}\varepsilon^2+\frac{\varepsilon^4}{20}\right)\lambda \\
&\quad + \frac{16\varepsilon^{-8/3}}{9\pi}\int_0^{\varepsilon}\int_0^{3\pi/2}(\varepsilon^2-r^2)\frac{\partial}{\partial r}w(x,y)\sin\left(\frac{2}{3}\theta\right) d\theta\,dr.
\end{aligned}
$$

Then

$$
\begin{aligned}
&\left(1-\frac{4}{7}\varepsilon^2+\frac{\varepsilon^4}{20}\right)|\lambda| \\
&\le \|f\|_{L^2(\Omega)}\|g_\varepsilon\|_{L^2(\Omega)} \\
&\quad + \frac{16\varepsilon^{-8/3}}{9\pi}\left(\int_0^{\varepsilon}\int_0^{3\pi/2}\left|(\varepsilon^2-r^2)\sin\left(\frac{2}{3}\theta\right)\right|^{7/6} r^{-1/6}\,d\theta\,dr\right)^{6/7} \\
&\quad \times \left(\int_0^{\varepsilon}\int_0^{3\pi/2}\left|\frac{\partial}{\partial r}w(x,y)\right|^{7} r\,d\theta\,dr\right)^{1/7} \\
&\le \|f\|_{L^2(\Omega)}\|\widetilde{g_\varepsilon}\|_{L^2(\Omega)} \\
&\quad + \frac{16\varepsilon^{1/21}}{9\pi}\left(\int_0^{1}\int_0^{3\pi/2}\left|(1-r^2)\sin\left(\frac{2}{3}\theta\right)\right|^{7/6} r^{-1/6}\,d\theta\,dr\right)^{6/7} \\
&\quad \times \left(\iint_{\Omega}\left(\left|\frac{\partial}{\partial x}w(x,y)\right|+\left|\frac{\partial}{\partial y}w(x,y)\right|\right)^{7} dx\,dy\right)^{1/7}.
\end{aligned}
$$

From $w \in H^2$ and Sobolev's embedding theorem [1],

$$
\frac{\partial w}{\partial x} \in L^7(\Omega), \quad \frac{\partial w}{\partial y} \in L^7(\Omega).
$$

Thus, we have the conclusion when $\varepsilon \to 0$. $\square$

Lemma 4.3.

$$
\left\|(r^{-2/3}-2^{-2/3}r^{2/3})\sin\left(\frac{2}{3}\theta\right)\right\|_{L^2(\Omega_0)} \le \sqrt{\frac{3\cdot 2^{1/3}}{5}\pi}.
$$

Proof.

$$
\begin{aligned}
&\left\|(r^{-2/3}-2^{-2/3}r^{2/3})\sin\left(\frac{2}{3}\theta\right)\right\|^2_{L^2(\Omega_0)} \\
&\le \int_0^{\sqrt{2}}\int_0^{3\pi/2}(r^{-2/3}-2^{-2/3}r^{2/3})^2\sin^2\left(\frac{2}{3}\theta\right) r\,d\theta\,dr = \frac{3\cdot 2^{1/3}}{5}\pi.
\end{aligned}
$$

$\square$

Lemma 4.4.

$$\left\| \Delta\left\{ 1_{r<1} \cdot (1-r)^2 r^{2/3} \sin\left(\frac{2}{3}\theta\right) \right\} \right\|_{L^2(\Omega_0)} = \frac{3\sqrt{\pi}}{2}.$$

Proof. By a direct calculation, we have

$$\begin{aligned}
&\left\| \Delta\left\{ 1_{r<1} \cdot (1-r)^2 r^{2/3} \sin\left(\frac{2}{3}\theta\right) \right\} \right\|^2_{L^2(\Omega_0)} \\
&= \left\| 1_{r<1} \cdot \frac{2}{3}(10r-7) r^{-1/3} \sin\left(\frac{2}{3}\theta\right) \right\|^2_{L^2(\Omega_0)} \\
&= \frac{4}{9} \int_0^1 \int_0^{3\pi/2} (10r-7)^2 r^{-2/3} \sin^2\left(\frac{2}{3}\theta\right) r \, d\theta \, dr = \frac{9}{4}\pi. \qquad \square
\end{aligned}$$

Lemma 4.5.

$$\begin{aligned}
&\left\| \Delta\left\{ ((1-x^2)(1-y^2) - 1_{r<1} \cdot (1-r)^2) r^{2/3} \sin\left(\frac{2}{3}\theta\right) \right\} \right\|_{L^2(\Omega_0)} \\
&\le \sqrt{\frac{4000}{81} - \frac{232367}{46332}\pi}.
\end{aligned}$$

Proof. Let us start with the following equation:

$$\begin{aligned}
&\Delta\left\{ ((1-x^2)(1-y^2) - (1-r)^2) r^{2/3} \sin\left(\frac{2}{3}\theta\right) \right\} \\
&= \frac{2}{3} r^{2/3} \left\{ \left(4r^2 - 20 + \frac{7}{r}\right) \sin\left(\frac{2}{3}\theta\right) + r^2 \sin\left(\frac{10}{3}\theta\right) \right\}.
\end{aligned}$$

When $r \ge 1$, in the same way as in the proof of Lemma 4.1,

$$\left| \Delta\left\{ (1-x^2)(1-y^2) r^{2/3} \sin\left(\frac{2}{3}\theta\right) \right\} \right| \le \frac{20\sqrt{10}}{9\sqrt{3}}.$$

Then

$$\begin{aligned}
&\left\| \Delta\left\{ ((1-x^2)(1-y^2) - 1_{r<1} \cdot (1-r)^2) r^{2/3} \sin\left(\frac{2}{3}\theta\right) \right\} \right\|^2_{L^2(\Omega_0)} \\
&\le \frac{4}{9} \int_0^1 \int_0^{3\pi/2} r^{4/3} \left\{ \left(4r^2 - 20 + \frac{7}{r}\right) \sin\left(\frac{2}{3}\theta\right) + r^2 \sin\left(\frac{10}{3}\theta\right) \right\}^2 r \, d\theta \, dr \\
&\quad + \left(|\Omega_0| - \frac{3}{4}\pi \right) \left(\frac{20\sqrt{10}}{9\sqrt{3}} \right)^2 \\
&= \frac{4193}{572}\pi + \left(3 - \frac{3}{4}\pi\right) \frac{4000}{243} = \frac{4000}{81} - \frac{232367}{46332}\pi. \qquad \square
\end{aligned}$$

Lemma 4.6.

$$|\lambda_k| \le \left\{ \iint_{\Omega} 1_{0\le\theta<3\pi/2} \cdot \left| G_{l_k}\left(r, \cos\left(\frac{4}{3}\theta\right), \cos\left(\frac{4}{3}\theta\right)\right)\right|^2 dx\,dy \right.$$
$$\left. + \iint_{\Omega} 1_{3\pi/2\le\theta<2\pi} \cdot \left| G_{l_k}\left(r, \frac{1}{9}(8+\cos(4\theta)), 1\right)\right|^2 dx\,dy \right\}^{1/2} \|f(x,y)\|_{L^2(\Omega)}.$$

Proof. Using parallel translation and the rotation, we move the re-entrant corner to the origin and the re-entrant angle to $[0, 3\pi/2]$.

For $\varepsilon < l_k$, let g_ε be a weak solution of following equation:

$$\begin{cases} -\Delta g_\varepsilon = -1_{0<\theta<3\pi/2} \cdot 1_{r<\varepsilon} \cdot \Delta \widetilde{g_\varepsilon} & \text{in } \Omega, \\ g_\varepsilon = 0 & \text{on } \partial\Omega, \end{cases}$$

where

$$\widetilde{g_\varepsilon}(x,y) = \begin{cases} (2\varepsilon^{-4/3}r^{4/3} - \varepsilon^{-8/3}r^{8/3}) G_{l_k}\left(r, \cos\left(\frac{4}{3}\theta\right), \cos\left(\frac{4}{3}\theta\right)\right) \\ \qquad\qquad\qquad\qquad\qquad\qquad (r < \varepsilon,\ 0 \le \theta \le 3\pi/2), \\ G_{l_k}\left(r, \cos\left(\frac{4}{3}\theta\right), \cos\left(\frac{4}{3}\theta\right)\right) \quad (r \ge \varepsilon,\ 0 \le \theta \le 3\pi/2), \\ G_{l_k}\left(r, \frac{1}{9}(8+\cos(4\theta)), 1\right) \qquad (3\pi/2 < \theta < 2\pi). \end{cases}$$

From Lemmas 4.8, 4.9 and 4.10, both g_ε and $\widetilde{g_\varepsilon} - g_\varepsilon$ are superharmonic functions in Ω. Since g_ε and $\widetilde{g_\varepsilon} - g_\varepsilon$ take non-negative values on $\partial\Omega$,

$$0 \le g_\varepsilon(x,y) \le \widetilde{g_\varepsilon}(x,y).$$

Consequently, we have

$$|g_\varepsilon(x,y)| \le |\widetilde{g_\varepsilon}(x,y)|.$$

Then, in the same way as in Lemma 4.2,

$$\begin{aligned} &\iint_{\Omega} f g_\varepsilon \, dx\,dy \\ &= \iint_{\Omega} \nabla u \cdot \nabla g_\varepsilon \, dx\,dy = -\iint_{\Omega} 1_{0<\theta<3\pi/2} \cdot 1_{r<\varepsilon} \cdot u \Delta \widetilde{g_\varepsilon} \, dx\,dy \\ &= -\iint_{\Omega} 1_{0<\theta<3\pi/2} \cdot 1_{r<\varepsilon} \cdot u(x,y) \\ &\qquad \times \Delta\left\{ (2\varepsilon^{-4/3}r^{4/3} - \varepsilon^{-8/3}r^{8/3}) G_{l_k}\left(r, \cos\left(\frac{4}{3}\theta\right), \cos\left(\frac{4}{3}\theta\right)\right)\right\} dx\,dy. \end{aligned}$$

From Lemma 4.11,

$$
\begin{aligned}
&\iint_\Omega f g_\varepsilon \, dx\, dy \\
&= \frac{32\varepsilon^{-8/3} l_k^{2/3}}{9\pi}(1+O(\varepsilon^{4/3})) \iint_\Omega 1_{r<\varepsilon} \cdot u(x,y) \sin\left(\frac{2}{3}\theta\right) dx\, dy \\
&= \frac{32\varepsilon^{-8/3} l_k^{2/3} \lambda_k}{9\pi}(1+O(\varepsilon^{4/3})) \\
&\quad \times \int_0^\varepsilon \int_0^{3\pi/2} \left(1-\frac{x^2}{l_k^2}\right)\left(1-\frac{y^2}{l_k^2}\right)\left(\frac{r}{l_k}\right)^{2/3} \sin^2\left(\frac{2}{3}\theta\right) r\, d\theta\, dr \\
&\quad + \frac{32\varepsilon^{-8/3} l_k^{2/3}}{9\pi}(1+O(\varepsilon^{4/3})) \int_0^\varepsilon \int_0^{3\pi/2} w(x,y) \sin\left(\frac{2}{3}\theta\right) r\, d\theta\, dr \\
&= (1+O(\varepsilon^{4/3}))\lambda_k \\
&\quad + \frac{32\varepsilon^{-8/3} l_k^{2/3}}{9\pi}(1+O(\varepsilon^{4/3})) \int_0^\varepsilon \int_0^{3\pi/2} w(x,y) \sin\left(\frac{2}{3}\theta\right) r\, d\theta\, dr.
\end{aligned}
$$

Again, in the same way as in Lemma 4.2,

$$
\begin{aligned}
(1+O(\varepsilon^{4/3}))|\lambda_k| &\le \|f\|_{L^2(\Omega)} \|g_\varepsilon\|_{L^2(\Omega)} + O(\varepsilon^{1/21}) \\
&\le \|f\|_{L^2(\Omega)} \|\widetilde{g_\varepsilon}\|_{L^2(\Omega)} + O(\varepsilon^{1/21}).
\end{aligned}
$$

Then, we have the conclusion when $\varepsilon \to 0$. □

LEMMA 4.7.

$$
\begin{aligned}
&\left\{ \iint_\Omega 1_{0\le\theta<3\pi/2} \cdot \left| G_l\left(r, \cos\left(\frac{4}{3}\theta\right), \cos\left(\frac{4}{3}\theta\right)\right)\right|^2 dx\, dy \right. \\
&\quad \left. + \iint_\Omega 1_{3\pi/2\le\theta<2\pi} \cdot \left| G_l\left(r, \frac{1}{9}(8+\cos(4\theta)), 1\right)\right|^2 dx\, dy \right\}^{1/2} \\
&\le \frac{1}{\pi}\sqrt{\left(\frac{5}{2}-\frac{3\pi}{8}\right) l^2 + 2|\Omega|}.
\end{aligned}
$$

Proof. We have the inequalities

$$
\begin{aligned}
&\left| G_l\left(r, \cos\left(\frac{4}{3}\theta\right), \cos\left(\frac{4}{3}\theta\right)\right)\right|^2 \\
&= \frac{l^{4/3}}{2\pi^2}\left\{ \sqrt{r^{-8/3}+l^{-8/3}-2r^{-4/3}l^{-4/3}\cos\left(\frac{4}{3}\theta\right)} + l^{-4/3} - r^{-4/3}\cos\left(\frac{4}{3}\theta\right) \right\} \\
&\le \frac{l^{4/3}}{2\pi^2}\left\{ |l^{-4/3}-r^{-4/3}| + \sqrt{2r^{-4/3}l^{-4/3}\left(1-\cos\left(\frac{4}{3}\theta\right)\right)} + l^{-4/3} - r^{-4/3}\cos\left(\frac{4}{3}\theta\right) \right\} \\
&= \frac{l^{4/3}}{\pi^2}\left\{ \max(l^{-4/3}-r^{-4/3}, 0) + r^{-2/3}l^{-2/3}\sin\left(\frac{2}{3}\theta\right) + r^{-4/3}\sin^2\left(\frac{2}{3}\theta\right) \right\}
\end{aligned}
$$

and

$$\begin{aligned}
&\left|G_l\left(r, \frac{1}{9}(8+\cos(4\theta)), 1\right)\right|^2 \\
&= \frac{l^{4/3}}{2\pi^2}\left\{\sqrt{r^{-8/3}+l^{-8/3}-\frac{2}{9}r^{-4/3}l^{-4/3}(8+\cos(4\theta))}+l^{-4/3}-r^{-4/3}\right\} \\
&\leq \frac{l^{4/3}}{2\pi^2}\left\{|l^{-4/3}-r^{-4/3}|+\sqrt{\frac{2}{9}r^{-4/3}l^{-4/3}(1-\cos(4\theta))}+l^{-4/3}-r^{-4/3}\right\} \\
&= \frac{l^{4/3}}{\pi^2}\left\{\max(l^{-4/3}-r^{-4/3},0)+\frac{1}{3}r^{-2/3}l^{-2/3}|\sin(2\theta)|\right\}.
\end{aligned}$$

It follows from these inequalities that

$$\begin{aligned}
&\iint_\Omega 1_{0\leq\theta<3\pi/2}\cdot\left|G_l\left(r,\cos\left(\frac{4}{3}\theta\right),\cos\left(\frac{4}{3}\theta\right)\right)\right|^2 dx\,dy \\
&+\iint_\Omega 1_{3\pi/2\leq\theta<2\pi}\cdot\left|G_l\left(r,\frac{1}{9}(8+\cos(4\theta)),1\right)\right|^2 dx\,dy \\
&\leq \frac{l^{4/3}}{\pi^2}\int_0^l\int_0^{3\pi/2}\left\{r^{-2/3}l^{-2/3}\sin\left(\frac{2}{3}\theta\right)+r^{-4/3}\sin^2\left(\frac{2}{3}\theta\right)\right\}r\,d\theta\,dr \\
&\quad+\frac{l^{4/3}}{\pi^2}\int_0^l\int_{3\pi/2}^{2\pi}\frac{1}{3}r^{-2/3}l^{-2/3}|\sin(2\theta)|r\,d\theta\,dr+\left(|\Omega|-\frac{3\pi l^2}{4}\right)\frac{l^{4/3}}{\pi^2}\cdot 2l^{-4/3} \\
&= \frac{l^2}{\pi^2}\left(\frac{5}{2}-\frac{3\pi}{8}\right)+\frac{2}{\pi^2}|\Omega|.
\end{aligned}$$

□

Lemma 4.8.

$$-\Delta G_l\left(r,\cos\left(\frac{4}{3}\theta\right),\cos\left(\frac{4}{3}\theta\right)\right)=0 \quad \textit{in } \Omega.$$

Proof.

$$G_l\left(r,\cos\left(\frac{4}{3}\theta\right),\cos\left(\frac{4}{3}\theta\right)\right)$$

is a constant times the imaginary part of

$$(z^{-4/3}-l^{-4/3})^{1/2}, \quad z=re^{i\theta}.$$

Therefore, this function is harmonic in Ω. □

Lemma 4.9. *For $r<\varepsilon<l$ and $0<\theta<3\pi/2$,*

$$-\Delta\left((2\varepsilon^{-4/3}r^{4/3}-\varepsilon^{-8/3}r^{8/3})G_l\left(r,\cos\left(\frac{4}{3}\theta\right),\cos\left(\frac{4}{3}\theta\right)\right)\right)\geq 0 \quad \textit{in } \Omega.$$

Proof. We have

$$\begin{aligned}
&-\Delta\left\{(2\varepsilon^{-4/3}r^{4/3}-\varepsilon^{-8/3}r^{8/3})G_l\left(r,\cos\left(\frac{4}{3}\theta\right),\cos\left(\frac{4}{3}\theta\right)\right)\right\}\\
&=\frac{4\sqrt{2}}{9a^2b^2cl^2\pi}\cdot\frac{(a+c-1)^2(a+c+1)^2(a-b+c)}{\left(a+c-\cos\left(\frac{4}{3}\theta\right)\right)^{3/2}},
\end{aligned}$$

where

$$a=r^{4/3}l^{-4/3},\quad b=\varepsilon^{4/3}l^{-4/3},\quad c=\sqrt{a^2+1-2a\cos\left(\frac{4}{3}\theta\right)}.$$

Since

$$a-b+c=a-b+\sqrt{a^2+1-2a\cos\left(\frac{4}{3}\theta\right)}\geq a-b+|a-1|\geq 1-b\geq 0,$$

this lemma is proved. $\square$

Lemma 4.10. *For $3\pi/2<\theta<2\pi$,*

$$-\Delta G_l\left(r,\frac{1}{9}(8+\cos(4\theta)),1\right)\geq 0\quad in\ \Omega.$$

Proof.

$$\begin{aligned}
&-\Delta G_l\left(r,\frac{1}{9}(8+\cos(4\theta)),1\right)\\
&=\frac{2\sqrt{2}\sqrt{a+d-1}}{9a^2d^3l^2\pi}(2d^3+2(a-1)^2d-4(a-1)^3+d^2),
\end{aligned}$$

where

$$a=r^{4/3}l^{-4/3},\quad d=\sqrt{a^2+1-\frac{2}{9}a(8+\cos(4t))}.$$

Since

$$d\geq\sqrt{a^2+1-2a}=|a-1|$$

holds,

$$2d^3+2(a-1)^2d-4(a-1)^3+d^2\geq 4|a-1|^3-4(a-1)^3+2(a-1)^2d+d^2\geq 0.$$

Then, this lemma is proved. $\square$

Lemma 4.11. *When $r < \varepsilon$,*

$$-\Delta\Big\{(2\varepsilon^{-4/3}r^{4/3} - \varepsilon^{-8/3}r^{8/3})G_l\Big(r, \cos\Big(\frac{4}{3}\theta\Big), \cos\Big(\frac{4}{3}\theta\Big)\Big)\Big\}$$
$$= \frac{32\varepsilon^{-8/3}l^{2/3}}{9\pi}\sin\Big(\frac{2}{3}\theta\Big)\cdot(1 + O(\varepsilon^{4/3})).$$

Proof. Define

$$a = r^{4/3}l^{-4/3}, \quad b = \varepsilon^{4/3}l^{-4/3}, \quad c = \sqrt{a^2 + 1 - 2a\cos\Big(\frac{4}{3}\theta\Big)},$$

then we have

$$-\Delta\Big\{(2\varepsilon^{-4/3}r^{4/3} - \varepsilon^{-8/3}r^{8/3})G_l\Big(r, \cos\Big(\frac{4}{3}\theta\Big), \cos\Big(\frac{4}{3}\theta\Big)\Big)\Big\}$$
$$= \frac{4\sqrt{2}}{9a^2b^2cl^2\pi}\cdot\frac{(a+c-1)^2(a+c+1)^2(a-b+c)}{\Big(a + c - \cos\Big(\frac{4}{3}\theta\Big)\Big)^{3/2}}.$$

We can easily confirm the following expressions:

$$c = 1 + O(\varepsilon^{4/3}),$$
$$a + c - 1 = \frac{2a}{c - a + 1}\Big(1 - \cos\Big(\frac{4}{3}\theta\Big)\Big) = 2a\sin^2\Big(\frac{2}{3}\theta\Big)\cdot(1 + O(\varepsilon^{4/3})),$$
$$a + c + 1 = 2 + O(\varepsilon^{4/3}),$$
$$a - b + c = 1 + O(\varepsilon^{4/3}),$$
$$a + c - \cos\Big(\frac{4}{3}\theta\Big) = \frac{c + a + 1}{c - a + 1}\Big(1 - \cos\Big(\frac{4}{3}\theta\Big)\Big) = 2\sin^2\Big(\frac{2}{3}\theta\Big)\cdot(1 + O(\varepsilon^{4/3})).$$

Then,

$$-\Delta\Big\{(2\varepsilon^{-4/3}r^{4/3} - \varepsilon^{-8/3}r^{8/3})G_l\Big(r, \cos\Big(\frac{4}{3}\theta\Big), \cos\Big(\frac{4}{3}\theta\Big)\Big)\Big\}$$
$$= \frac{32\varepsilon^{-8/3}l^{2/3}}{9\pi}\sin\Big(\frac{2}{3}\theta\Big)\cdot(1 + O(\varepsilon^{4/3})). \qquad \square$$

5. Numerical result

In this section, numerical results are shown to confirm the validity of the error estimation. All calculations were carried out on an Intel Core 2 Duo 6700 PC at 2.66 GHz with Borland C++ compiler. There are some difficulties in calculating the H_0^1 inner product between the singular basis and the bilinear basis because the gradient of the singular basis diverges at the re-entrant corner. To deal with this difficulty, the following double exponential transformation (DE transformation) [15] is used to calculate the integral on each of the square elements:

$$\begin{aligned}
&\int_{y_l}^{y_l+h}\int_{x_k}^{x_k+h} p(x,y)\,dx\,dy \\
&\quad = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x_k+\varphi(x), y_l+\varphi(y))\varphi'(x)\varphi'(y)\,dx\,dy,
\end{aligned} \tag{5.1}$$

where $p(x,y)$ is an integrand and

$$\varphi(t) = \frac{h}{2}\left(\tanh\left(\frac{\pi}{2}\sinh t\right)+1\right).$$

We approximate (5.1) by the trapezoidal rule as follows:

$$\begin{aligned}
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x_k+\varphi(x), y_l+\varphi(y))\varphi'(x)\varphi'(y)\,dx\,dy \\
&\approx \frac{L^2}{N^2}\sum_{j=-N}^{N}\sum_{i=-N}^{N}\left(x_k+\varphi\left(\frac{kL}{N}\right), y_l+\varphi\left(\frac{kL}{N}\right)\right)\varphi'\left(\frac{kL}{N}\right)\varphi'\left(\frac{kL}{N}\right).
\end{aligned}$$

It is known that, if the integrand $p(x,y)$ is an analytic function on $(x_k, x_k+h)\times(y_l, y_l+h)$, very high accuracy is realized by this numerical integration formula. We took $L=4$ and $N=100$ which are sufficient to obtain double floating point precision. We also use this numerical integration method to compute the right-hand side vector of the finite element method and the error between the numerical solution and the exact solution.

At first, we consider the following equation on the L-shape domain which is shown in Fig. 8:

$$\begin{cases} -\Delta u = f(x,y) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.2}$$

where

$$\begin{aligned}
f(x,y) = r^{-1/3}\Bigg\{&\frac{(x-2)y(y-2)}{3}\sin\left(\frac{1}{3}\theta\right) - \frac{x(x-4)(y-1)}{3}\cos\left(\frac{1}{3}\theta\right) \\
&- \frac{(x^2-4x+y^2-2y)r}{4}\sin\left(\frac{2}{3}\theta\right)\Bigg\},
\end{aligned}$$

$$(x-1, y-1) = (r\cos\theta, r\sin\theta) \quad (0\le r,\ 0\le\theta<2\pi).$$

The exact solution of this equation is

$$u(x,y) = \frac{x(x-4)y(y-2)}{8}r^{2/3}\sin\left(\frac{2}{3}\theta\right),$$

$$(x-1, y-1) = (r\cos\theta, r\sin\theta) \quad (0\le r,\ 0\le\theta<2\pi).$$

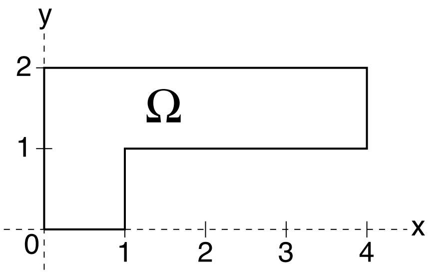

Fig. 8. L-shape domain

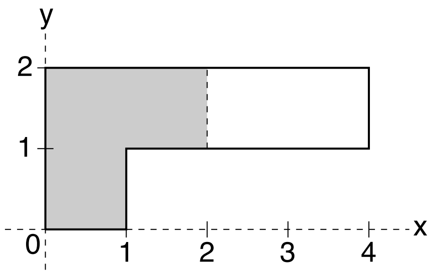

Fig. 9. An arrangement of the singular bases

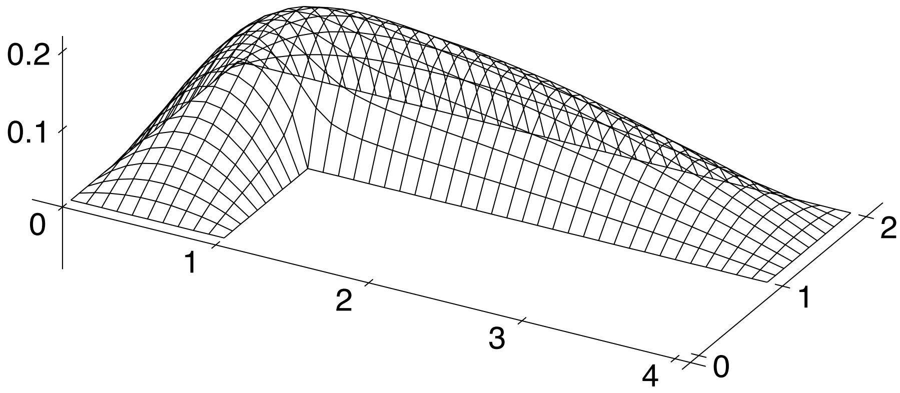

Fig. 10. The numerical solution when $h = 1/10$

The numerical results are presented in Table 1, where h is the mesh size and U_h denotes the numerical solution. Fig. 10 shows the shape of the numerical solution.

The condition numbers of the element matrix were also calculated. The definition of the condition number is the ratio of the largest to the smallest eigenvalue.

Since $\|f\|_\Omega$ is calculated to be $1.9233\ldots$, Theorem 3.1 implies that the H_0^1 and L^2 error estimates for this equation are

$$\|u_h - u\|_{H_0^1(\Omega)} \leq \left(0.319 + \sqrt{0.971 + 1.469 \cdot \frac{5}{1^2}}\right) 1.924h,$$

$$\|u_h - u\|_{L^2(\Omega)} \leq \left(0.319 + \sqrt{0.971 + 1.469 \cdot \frac{5}{1^2}}\right)^2 1.924h^2.$$

Strictly speaking, in order to obtain an explicit error estimate, it is necessary to use the verified result of $\|f\|_\Omega$. Nevertheless we use the approximate value here because the main purpose of this section is only to compare the error estimation with numerical results. The right-hand side of these inequalities are presented in Table 2. Since the a priori error estimation can be applied to arbitrary f and Ω (in other words, can be applied to the worst case), the actual error is often smaller than a priori estimate.

Table 1. Numerical results for the L-shape domain

h	$\|U_h - u\|_{H_0^1(\Omega)}$	$\|U_h - u\|_{L^2(\Omega)}$	Degree of freedom	Condition number
1/20	3.9052×10^{-2}	1.0684×10^{-3}	1882	4.9653×10^{4}
1/40	1.9576×10^{-2}	2.6905×10^{-4}	7762	6.3104×10^{5}
1/60	1.3066×10^{-2}	1.2008×10^{-4}	17642	2.7351×10^{6}
1/80	9.8064×10^{-3}	6.7750×10^{-5}	31522	7.6650×10^{6}
1/100	7.8494×10^{-3}	4.3464×10^{-5}	49402	1.6951×10^{7}

Table 2. A priori error estimation for the L-shape domain

h	$\|u_h - u\|_{H_0^1(\Omega)}$	$\|u_h - u\|_{L^2(\Omega)}$
1/20	$3.081043\ldots \times 10^{-1}$	$4.933901\ldots \times 10^{-2}$
1/40	$1.540521\ldots \times 10^{-1}$	$1.233475\ldots \times 10^{-2}$
1/60	$1.027014\ldots \times 10^{-1}$	$5.482113\ldots \times 10^{-3}$
1/80	$7.702607\ldots \times 10^{-2}$	$3.083688\ldots \times 10^{-3}$
1/100	$6.162086\ldots \times 10^{-2}$	$1.973560\ldots \times 10^{-3}$

The second result is a case when Ω is the H-shape domain shown in Fig. 11. Fig. 12 shows an arrangement of the singular bases. In this situation, we consider the following equation:

$$\begin{cases} -\Delta u = f(x-1, y-1) + f(3-x, y-1) & \\ \qquad\quad + f(3-x, 2-y) + f(x-1, 2-y) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.3}$$

where

$$\begin{aligned} f(x,y) = {} & 1_{-1<x<2} \cdot 1_{-1<y<1} \cdot r^{-1/3} \\ & \times \left\{ \frac{x(x-2)(y+1)(y-1)^2}{3} \sin\left(\frac{1}{3}\theta\right) \right. \\ & \quad - \frac{(x+1)(x-2)^2(3y+1)(y-1)}{9} \cos\left(\frac{1}{3}\theta\right) \\ & \quad \left. - \left(\frac{(x-1)(y+1)(y-1)^2}{2} + \frac{(x+1)(x-2)^2(3y-1)}{6} \right) r \sin\left(\frac{2}{3}\theta\right) \right\}, \\ (x,y) = {} & (r\cos\theta, r\sin\theta) \quad (0 \le r,\ 0 \le \theta < 2\pi). \end{aligned}$$

The exact solution of this equation is

$$u = g(x-1, y-1) + g(3-x, y-1) + g(3-x, 2-y) + g(x-1, 2-y),$$

where

$$g(x,y) = 1_{-1<x<2} \cdot 1_{-1<y<1} \cdot \frac{(x+1)(x-2)^2(y+1)(y-1)^2}{12} r^{2/3} \sin\left(\frac{2}{3}\theta\right),$$

$$(x,y) = (r\cos\theta, r\sin\theta) \quad (0 \le r,\ 0 \le \theta < 2\pi).$$

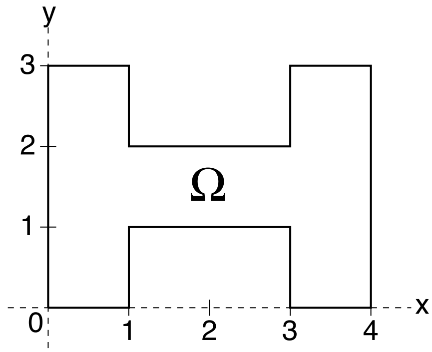

Fig. 11. H-shape domain

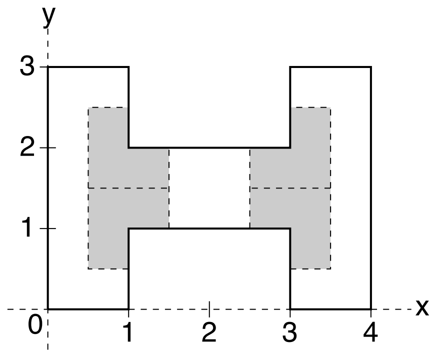

Fig. 12. An arrangement of the singular bases

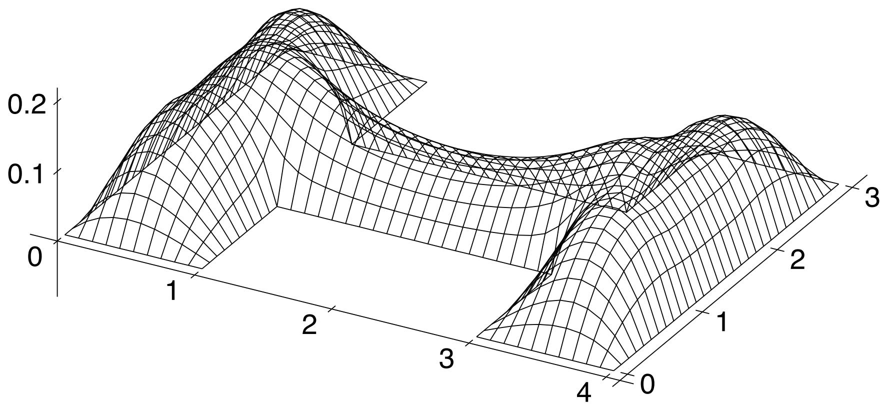

Fig. 13. The numerical solution when $h = 1/10$

The numerical results are presented in Table 3 and Fig. 13 shows the shape of the numerical solution.

The L^2-norm of the right-hand side of (5.3) is calculated to be $2.8228\ldots$ Then, from Theorem 3.1, the H_0^1 and L^2 error estimates for this equation are obtained as follows:

$$\|u_h - u\|_{H_0^1(\Omega)} \le \left(0.319 + 4\sqrt{0.971 + 1.469 \cdot \frac{8}{0.5^2}}\right) 2.823h,$$

$$\|u_h - u\|_{L^2(\Omega)} \le \left(0.319 + 4\sqrt{0.971 + 1.469 \cdot \frac{8}{0.5^2}}\right)^2 2.823h^2.$$

The right-hand side of these inequalities are presented in Table 4.

As we can see in Table 3 and Table 4, the value of the a priori error estimation becomes larger as the number of the re-entrant corners increases and when the support of the singular function becomes smaller.

Table 3. Numerical results for the H-shape domain

h	$\|U_h - U_{h/2}\|_{H_0^1(\Omega)}$	$\|U_h - U_{h/2}\|_{L^2(\Omega)}$	Degree of freedom	Condition number
1/20	3.8100×10^{-2}	6.3213×10^{-4}	3025	3.9510×10^{3}
1/40	1.9789×10^{-2}	1.7619×10^{-4}	12445	5.0101×10^{4}
1/60	1.3490×10^{-2}	8.3725×10^{-5}	28265	2.2167×10^{5}
1/80	1.0274×10^{-2}	4.9357×10^{-5}	50485	6.3251×10^{5}
1/100	8.3140×10^{-3}	3.2724×10^{-5}	79105	1.4194×10^{6}

Table 4. A priori error estimation for the H-shape domain

h	$\|u_h - u\|_{H_0^1(\Omega)}$	$\|u_h - u\|_{L^2(\Omega)}$
1/20	$3.955834\ldots$	$5.543261\ldots$
1/40	$1.977917\ldots$	$1.385815\ldots$
1/60	$1.318611\ldots$	$0.615917\ldots$
1/80	$0.988958\ldots$	$0.346453\ldots$
1/100	$0.791166\ldots$	$0.221730\ldots$

6. Concluding remark

We presented a constructive a priori error estimation for a finite element solution in a polygonal domain by using singular functions. The results are only valid when the domain is bounded by line segments which form a right angle. However, it seems possible to extend this method to the general polygonal domain with triangular mesh. Numerical calculations are carried out to confirm the validity of the error estimation. A priori error estimation for another more efficient method such as DSFM or method of mesh refinement remains as future work. But from the viewpoint of application to the numerical verification of non-linear problems, the most important thing is obtaining an a priori error estimation by any method.

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