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On the Quintic Nonlinear Schrödinger Equation Created by the Vibrations of a Square Plate on a Weakly Nonlinear Elastic Foundation and the Stability of the Uniform Solution

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Plates are common structural elements of most engineering structures, including aerospace, automotive, and civil engineering structures. The study of plates from theoretical perspective as well as experimental viewpoint is fundamental to understanding of the behavior of such structures. The dynamic characteristics of plates, such as natural vibrations, transient responses for the external forces and so on, are especially of importance in actual environments. In this paper, we conside the envelope surface created by the vibrations of a square plate on a weakly nonliner elastic foundation and analyze the stability of the uniform solution of the governing equation for the envelope surface. We derive the two-dimensional equation that governs the spatial and temporal evolution of the envelope surface on cubic nonlinear elastic foundation. The fact that the governing equation becomes the quintic nonlinear Schrödinger equation is shown. Also we obtain the stability condition of the uniform solution of the quintic nonlinear Schrödinger equation.

Key words: elastic foundation, envelope, nearly monochromatic waves, perturbation, Schrödinger equation

1. Introduction

Plates are common structural elements of most engineering structures, including aerospace, automotive, and civil engineering structures. The study of plates from theoretical perspective as well as experimental viewpoint is fundamental to understanding of the behavior of such structures. The dynamic characteristics of plates, such as natural vibrations, transient responses for the external forces and so on, are especially of importance in actual environments. In this paper, we conside the envelope surface created by the vibrations of a square plate on a weakly nonliner elastic foundation and analyze the stability of the uniform solution of the governing equation for the envelope surface. We derive the two-dimensional equation that governs the spatial and temporal evolution of the envelope surface and discuss the stability of the uniform solution.

In the course of studying the theory of plates the classical, Kirchhoff plate theory [1]–[3], in which transverse normal and shear stresses are neglected to study bending, buckling, and natural vibrations of rectangular plates, was first established. The treatment of the linear vibrations of plates is comprehensively given in the monograph [4]. The governing equations of the nonlinear vibrations of plates was also reduced [5]. The first-order shear deformation plate theory extends the kinematics of the classical, Kirchhoff plate theory by relaxing the normality restriction and allowing for arbitrary but constant rotation of transverse normals [6, 7] and finite element models are developed for the precise analysis of the plate characteristics in real problems [8].

In general, the Schrödinger equation [9] governs the spatial and temporal evolution of the amplitude of a wavepacket propagating transversely in any dispersive, lossless medium. The spatial and temporal evolution of the amplitude of a wavepacket centered around a wavenumber and a frequency is varying slowly in space and time so that it creates an envelope. In other words, the Schrödinger equation governs an envelope created by a wavepacket. The nonlinear Schrödinger equation arises in the nonlinear dispersive characteristics of propagation medium and nonlinear restoring force and so on. Many studies of a wavepacket has been carried out in water wave [10]–[12], plasma [13], fiber-optic communication systems [14], and some other area as well. Moreover, several Schrödinger type equations are derived from the wavenumber-based or directional-based spectrum of nearly monochromatice waves and their stabilities of the solutions are analyzed [15, 16]. Nearly bichromatic waves which are expanded from nearly monochromatice waves are also analyzed and the related equations govern envelopes created by nearly bichromatic waves are derived. The nonlinear dynamics and numerical simulations of its solutions are performed [17]–[20].

We can consider the force due to the elastic foundation is propotional to the second or third power of the displacement. The past studies have only treated the second power of the displacement [21]. However, when considering the nonlinearity of an elastic foundation, the third power of the displacement is naturally considered. In this paper, we derive the two-dimensional governing equation that describes the propagation of the envelope surface of a square plate on an elastic foundation using the method of multiple scales [22]. The obtained equation becomes the quintic nonlinear Schrödinger equation. We consider the stability of the uniform solution of the obtained quintic nonlinear Schrödinger equation through the modulational perturbation. The sufficient condition of the stability is shown.

The following section presents the plate equation on an elastic, weakly nonlinear foundation with cubic nonlinearity. In the third section we derive the govrning equation of the envelope surface of nearly monochromatic waves on cubic nonlinear elastic foundation using the method of multiple scales. In the fourth section we analyze the stability of the uniform solution of the obtained governing equation in the previous section.

2. Preliminary: Plate equation on an elastic, weakly nonlinear foundation [21]

We consider a square plate with side length l. The mass of the plate per unit area perpendicular to z-axis, the mass density of the plate, the area of the cross sectoin of the plate perpendicular to x-axis, the elasticity modulus, and the moment of inertia of the cross section with respect to the x-axis are denoted by μ, ρ, A_a, E, and I, respectively. Moreover, the weight W of the plate per unit area is set to be constant, that is, $W = \mu g$ (g is the gravitational acceleration) and we neglect internal damping. Then, the equation of motion for the vertical displacement of the plate $w(x, y, t)$, in which t is time, is given by

$$\begin{aligned}&\mu \frac{\partial^2 w(x,y,t)}{\partial t^2} + EI\left\{\frac{\partial^4 w(x,y,t)}{\partial x^4} + 2\frac{\partial^4 w(x,y,t)}{\partial x^2 \partial y^2} + \frac{\partial^4 w(x,y,t)}{\partial y^4}\right\}\\ &+ F(w(x,y,t)) = -\mu g,\\ &0 < x < l, \quad 0 < y < l, \quad t > 0,\end{aligned} \tag{1}$$

where F is the force in z-direction per unit area acting on the plate due to the elastic foundation. We consider the free oscillations of the plate without specific boundary conditions.

We assume that the vertical displacements of the plate are small compared to the length l. We also assume that the force $F(w)$ can be naturally written as follows:

$$F(w) = k_s w + b_s w^3, \tag{2}$$

where k_s and b_s are spring constants. Constant k_s must be positive physically. Constant b_s takes either zero or positive or negative. If b_s is zero, then the spring is a linear spring. For a linear spring, the force is proportional to the displacement. If b_s is not zero, then the spring is a nonlinear spring [23]. If b_s is positive (we call a hard spring), then the nonlinearity increases the force. If b_s is negative (we call a soft spring), then the nonlinearity decreases the force. We are interested in a nonlinear spring, that is, b_s is positive or negative. In the appendix, the governing equation with a nonlinear spring expressed by

$$F(w) = k_s w + b_s w^2 \tag{3}$$

is shown. In this case, we have the known result [21].

Then Equation (1) becomes

$$\begin{aligned}&\frac{\partial^2 w(x,y,t)}{\partial t^2} + \frac{EI}{\mu}\left\{\frac{\partial^4 w(x,y,t)}{\partial x^4} + 2\frac{\partial^4 w(x,y,t)}{\partial x^2 \partial y^2} + \frac{\partial^4 w(x,y,t)}{\partial y^4}\right\}\\ &+ \frac{k_s}{\mu} w(x,y,t) + \frac{b_s}{\mu} w(x,y,t)^3 = -g.\end{aligned} \tag{4}$$

In order to simplify equation (4) the term $-g$ will be removed by introducing the transformation

$$w(x,y,t) = \tilde{w}(x,y,t) + \frac{\mu g}{k_s} s(x,y), \tag{5}$$

where $s(x,y)$ satisfies the following time-independent equation:

$$\frac{\partial^4 s(x,y)}{\partial x^4} + 2\frac{\partial^4 s(x,y)}{\partial x^2 \partial y^2} + \frac{\partial^4 s(x,y)}{\partial y^4} + \frac{k_s}{EI}s(x,y) = -\frac{k_s}{EI}, \tag{6}$$

where $\frac{\mu g}{k_s}s(x,y)$ represents the deflection of the plate in static state due to gravity. Equation (6) is easily solved as the boundary value problem, but we are not interested in static state. $\tilde{w}(x,y,t)$ is written as follows:

$$\begin{aligned}
&\frac{\partial^2 \tilde{w}(x,y,t)}{\partial t^2} + \frac{EI}{\mu}\left\{\frac{\partial^4 \tilde{w}(x,y,t)}{\partial x^4} + 2\frac{\partial^4 \tilde{w}(x,y,t)}{\partial x^2 \partial y^2} + \frac{\partial^4 \tilde{w}(x,y,t)}{\partial y^4}\right\} + \frac{k_s}{\mu}\tilde{w}(x,y,t) \\
&+ \frac{b_s}{\mu}\left\{\tilde{w}(x,y,t)^3 + \frac{3\mu g}{k_s}\tilde{w}(x,y,t)^2 s(x,y)\right. \\
&\qquad \left. + 3\left(\frac{\mu g}{k_s}\right)^2 \tilde{w}(x,y,t)s(x,y)^2 + \left(\frac{\mu g}{k_s}\right)^3 s(x,y)^3\right\} = 0.
\end{aligned} \tag{7}$$

Using the dimensionless variables

$$\bar{w} = \frac{l}{A_a}\tilde{w}, \quad \bar{x} = \frac{\pi}{l}x, \quad \bar{y} = \frac{\pi}{l}y, \quad \bar{t} = \left(\frac{\pi}{l}\right)^2\sqrt{\frac{EI}{\mu}}t, \tag{8}$$

equation (7) becomes

$$\begin{aligned}
&\frac{\partial^2 \bar{w}}{\partial \bar{t}^2} + \frac{\partial^4 \bar{w}}{\partial \bar{x}^4} + 2\frac{\partial^4 \bar{w}}{\partial \bar{x}^2 \partial \bar{y}^2} + \frac{\partial^4 \bar{w}}{\partial \bar{y}^4} \\
&+ \frac{l^4}{\pi^4 EI}\left\{k_s \bar{w} + \frac{b_s A_a^2}{l^2}\bar{w}^3 + \frac{3b_s A_a}{l}\bar{w}^2 \frac{\mu g}{k_s}s\left(\frac{l}{\pi}\bar{x}, \frac{l}{\pi}\bar{y}\right)\right. \\
&\qquad \left. + \frac{b_s A_a^2}{l^2}\bar{w}\left(\frac{\mu g}{k_s}s\left(\frac{l}{\pi}\bar{x}, \frac{l}{\pi}\bar{y}\right)\right)^2 + \frac{b_s l}{A_a}\left(\frac{\mu g}{k_s}s\left(\frac{l}{\pi}\bar{x}, \frac{l}{\pi}\bar{y}\right)\right)^3\right\} = 0,
\end{aligned} \tag{9}$$

where we simply write $\bar{w}$, which is $\bar{w}\left(\frac{l}{\pi}\bar{x}, \frac{l}{\pi}\bar{y}, \left(\frac{l}{\pi}\right)^2\sqrt{\frac{\mu}{EI}}\,\bar{t}\right)$ exactly.

We assume that the area A_a of the cross section is small compared to the plate side length l, then we put $\tilde{\varepsilon} = \left(\frac{A_a}{l}\right)^2$ with $\tilde{\varepsilon}$ a small parameter. We also assume that the deflection of the plate in static state due to gravity, $\frac{\mu g}{k_s}s(x,y)$, is small with respect to the vertical displacement $\tilde{w}$. So, we assume that $\frac{\mu g}{k_s}s(x,y)$ is $\mathcal{O}(\tilde{\varepsilon}^n)$ with $n > 1$. Setting

$$\varepsilon = -b_s\tilde{\varepsilon}\left(\frac{l}{\pi}\right)^4\frac{1}{EI}, \quad p^2 = \left(\frac{l}{\pi}\right)^4\frac{k_s}{EI}, \tag{10}$$

equation (9) becomes

$$\frac{\partial^2 \bar{w}}{\partial \bar{t}^2} + \frac{\partial^4 \bar{w}}{\partial \bar{x}^4} + 2\frac{\partial^4 \bar{w}}{\partial \bar{x}^2 \partial \bar{y}^2} + \frac{\partial^4 \bar{w}}{\partial \bar{y}^4} + p^2\bar{w} = \varepsilon\bar{w}^3 + \mathcal{O}(\varepsilon^n), \tag{11}$$

with $n > 1$ and ε is a small parameter. We can now write the following equation governs the vertical displacement of a plate on a weakly nonlinear, elastic foundation, which describes up to $\mathcal{O}(\varepsilon^n)$, $n > 1$:

$$\begin{aligned}&\frac{\partial^2 w(x,y,t)}{\partial t^2} + \frac{\partial^4 w(x,y,t)}{\partial x^4} + 2\frac{\partial^4 w(x,y,t)}{\partial x^2 \partial y^2} + \frac{\partial^4 w(x,y,t)}{\partial y^4} + p^2 w(x,y,t)\\&= \varepsilon w(x,y,t)^3,\\&0 < x < \pi, \quad 0 < y < \pi, \quad t > 0,\end{aligned} \tag{12}$$

where we drop all bars for convenience. The first four terms in the left-hand side of equation (12) are the linear part of the plate equation and $p^2 w - \varepsilon w^3$ represents the restoring force due to the elastic foundation. Since no other external forces are considered, equation (12) describes the free osillations on an elastic, weakly nonlinear foundation.

3. Governing equation of the envelope surface of nearly monochromatic waves on an elastic, weakly nonlinear foundation

In this section we derive the governing equation for the envelope surface created by nearly monochromatic waves propagating in unidirection on an plate with weakly nonlinear foundation. Nearly monochromatic waves have the wavenumber spectrum having a peak and spreading over around a peak. So, the energy of nearly monochromatic waves is almost concentrated in a single wavenumber. The amplitude of such waves varies slowly in time and space so that creates the envelope of traveling waves.

In order to derivate the equation govern the envelope of nearly monochromatic waves centered around the wavenumber k_w and the angular frequency ω, we use the method of multiple scales [22]. We introduce the slow time scales $T_1 = \epsilon t$, and $T_2 = \epsilon^2 t$ in addition to the original time scale $T_0 = t$. Moreover we introduce the long scales $X_1 = \epsilon x$, $X_2 = \epsilon^2 x$, $Y_1 = \epsilon y$, and $Y_2 = \epsilon^2 y$ in addition to the original space scale $X_0 = x$ and $Y_0 = y$. Here ϵ is a small parameter, which is physically different from ε in equation (10). Both ϵ and ε are the same small parameter mathematically and we assume that ϵ and ε are of the same order, so we denote ϵ without distinction. Hence the time and space derivatives become

$$\begin{aligned}\frac{\partial}{\partial t} &= \frac{\partial}{\partial T_0} + \epsilon\frac{\partial}{\partial T_1} + \epsilon^2\frac{\partial}{\partial T_2},\\ \frac{\partial}{\partial x} &= \frac{\partial}{\partial X_0} + \epsilon\frac{\partial}{\partial X_1} + \epsilon^2\frac{\partial}{\partial X_2},\\ \frac{\partial}{\partial y} &= \frac{\partial}{\partial Y_0} + \epsilon\frac{\partial}{\partial Y_1} + \epsilon^2\frac{\partial}{\partial Y_2}.\end{aligned} \tag{13}$$

Then we seek a second-order solution in the form

$$w(x,y,t;\epsilon) = \sum_{n=0}^{2} \epsilon^n w_n(X_0, X_1, X_2, Y_0, Y_1, Y_2, T_0, T_1, T_2) + \mathcal{O}(\epsilon^3). \tag{14}$$

Substituting equation (14) into equation (12), using equation (13), and equating coefficients of like powers of ϵ, we obtain

$$\frac{\partial^4 w_0}{\partial X_0^4} + 2\frac{\partial^4 w_0}{\partial X_0^2 \partial Y_0^2} + \frac{\partial^4 w_0}{\partial Y_0^4} + \frac{\partial^2 w_0}{\partial T_0^2} + p^2 w_0 = 0, \tag{15}$$

$$\begin{aligned}
&\frac{\partial^4 w_1}{\partial X_0^4} + 2\frac{\partial^4 w_1}{\partial X_0^2 \partial Y_0^2} + \frac{\partial^4 w_1}{\partial Y_0^4} + \frac{\partial^2 w_1}{\partial T_0^2} + p^2 w_1 \\
&= -4\frac{\partial^4 w_0}{\partial X_0^3 \partial X_1} - 4\frac{\partial^4 w_0}{\partial X_0 \partial X_1 \partial Y_0^2} - 4\frac{\partial^4 w_0}{\partial X_0^2 \partial Y_0 \partial Y_1} \\
&\quad - 4\frac{\partial^4 w_0}{\partial Y_0^3 \partial Y_1} - 2\frac{\partial^2 w_0}{\partial T_0 \partial T_1} + w_0^3,
\end{aligned} \tag{16}$$

$$\begin{aligned}
&\frac{\partial^4 w_2}{\partial X_0^4} + 2\frac{\partial^4 w_2}{\partial X_0^2 \partial Y_0^2} + \frac{\partial^4 w_2}{\partial Y_0^4} + \frac{\partial^2 w_2}{\partial T_0^2} + p^2 w_2 \\
&= -4\frac{\partial^4 w_1}{\partial X_0^3 \partial X_1} - 4\frac{\partial^4 w_1}{\partial X_0 \partial X_1 \partial Y_0^2} - 4\frac{\partial^4 w_1}{\partial X_0^2 \partial Y_0 \partial Y_1} \\
&\quad - 4\frac{\partial^4 w_1}{\partial Y_0^3 \partial Y_1} - 2\frac{\partial^2 w_1}{\partial T_0 \partial T_1} - 6\frac{\partial^4 w_0}{\partial X_0^2 \partial X_1^2} - 4\frac{\partial^4 w_0}{\partial X_0^3 \partial X_2} - 8\frac{\partial^4 w_0}{\partial X_0 \partial X_1 \partial Y_0 \partial Y_1} \\
&\quad - 2\frac{\partial^4 w_0}{\partial X_1^2 \partial Y_0^2} - 4\frac{\partial^4 w_0}{\partial X_0 \partial X_2 \partial Y_0^2} - 2\frac{\partial^4 w_0}{\partial X_0^2 \partial Y_1^2} - 4\frac{\partial^4 w_0}{\partial X_0^2 \partial Y_0 \partial Y_2} \\
&\quad - 6\frac{\partial^4 w_0}{\partial Y_0^2 \partial Y_1^2} - 4\frac{\partial^4 w_0}{\partial Y_0^3 \partial Y_2} - \frac{\partial^2 w_0}{\partial T_1^2} - 2\frac{\partial^2 w_0}{\partial T_0 \partial T_2} + 3w_0^2 w_1.
\end{aligned} \tag{17}$$

To analyze the propagation of nearly monochromatic waves centered around the wavenumber k_w and the angular frequency ω, we take the solution of equation (15) in the form

$$\begin{aligned}
w_0 &= A(X_1, X_2, Y_1, Y_2, T_1, T_2)e^{i(k_w X_0 \cos\theta_0 + k_w Y_0 \sin\theta_0 - \omega T_0)} \\
&\quad + A^*(X_1, X_2, Y_1, Y_2, T_1, T_2)e^{-i(k_w X_0 \cos\theta_0 + k_w Y_0 \sin\theta_0 - \omega T_0)}.
\end{aligned}$$

We write this equation as follows for convenience (hereinafter, "cc" is used as the same manner):

$$w_0 = A(X_1, X_2, Y_1, Y_2, T_1, T_2)e^{i(k_w X_0 \cos\theta_0 + k_w Y_0 \sin\theta_0 - \omega T_0)} + \text{cc}. \tag{18}$$

Equation (18) describes the propagation of nearly monochromatic waves whose propagation direction is θ_0 $(0 < \theta_0 < \pi/2)$ in the (x, y) plane. In other words, A of equation (18) presents the envelope surface created by nearly monochromatic waves. We derive the governing equation for A using equations (15), (16), (17), and (18).

First, the dispersion relation for A is led by substituting equation (18) into equation (15)

$$\omega^2 = k_w^4 + p^2. \tag{19}$$

Next, substituting equation (18) into equation (16) and eliminating the terms that produce secular terms yield the following solvability condition

$$4ik_w^3\left(\cos\theta_0\frac{\partial A}{\partial X_1}+\sin\theta_0\frac{\partial A}{\partial Y_1}\right)+2i\omega\frac{\partial A}{\partial T_1}+3|A|^2A=0, \tag{20}$$

and we have the equation w_1 satisfies as follows:

$$\frac{\partial^4 w_1}{\partial X_0^4}+2\frac{\partial^4 w_1}{\partial X_0^2\partial Y_0^2}+\frac{\partial^4 w_1}{\partial Y_0^4}+\frac{\partial^2 w_1}{\partial T_0^2}+p^2w_1=A^3e^{3i(k_wX_0\cos\theta_0+k_wY_0\sin\theta_0-\omega T_0)}+\mathrm{cc}. \tag{21}$$

Then the solution of equation (21) becomes

$$w_1=\frac{A^3}{8(9k_w^4-p^2)}e^{3i(k_wX_0\cos\theta_0+k_wY_0\sin\theta_0-\omega T_0)}+\mathrm{cc}. \tag{22}$$

Similarly, we eliminate the terms that produce secular terms in the equation obtained from equations (17), (18), and (22), then we obtain the solvability condition as follows:

$$\begin{aligned}&(6k_w^2\cos^2\theta_0+2k_w^2\sin^2\theta_0)\frac{\partial^2 A}{\partial X_1^2}+4ik_w^3\cos\theta_0\frac{\partial A}{\partial X_2}+4k_w^2\sin 2\theta_0\frac{\partial^2 A}{\partial X_1\partial Y_1}\\&+(6k_w^2\sin^2\theta_0+2k_w^2\cos^2\theta_0)\frac{\partial^2 A}{\partial Y_1^2}+4ik_w^3\sin\theta_0\frac{\partial A}{\partial Y_2}+2i\omega\frac{\partial A}{\partial T_2}-\frac{\partial^2 A}{\partial T_1^2}=0.\end{aligned} \tag{23}$$

Equations (20) and (23) describe the evolution of the complex amplitude A with the slow and long scales. We obtain the governing equation for A, which is accomplished by combining equations (20) and (23). First, from equation (19), we obtain the following relations:

$$\omega\omega'=2k_w^3,\quad \omega\omega''=6k_w^2-\frac{4k_w^6}{k_w^4+p^2}, \tag{24}$$

where $\omega'=d\omega/dk_w$ and $\omega''=d^2\omega/dk_w^2$. We use these relations in the calculation below. Next, from equation (20), we have

$$\frac{\partial A}{\partial T_1}=-2\frac{k_w^3}{\omega}\left(\cos\theta_0\frac{\partial A}{\partial X_1}+\sin\theta_0\frac{\partial A}{\partial Y_1}\right)+\frac{i3}{2\omega}|A|^2A. \tag{25}$$

Differentiating both sides of equation (25) with respect to T_1 yields

$$\begin{aligned}\frac{\partial^2 A}{\partial T_1^2}&=-\omega'\left(\cos\theta_0\frac{\partial}{\partial X_1}\frac{\partial A}{\partial T_1}+\sin\theta_0\frac{\partial}{\partial Y_1}\frac{\partial A}{\partial T_1}\right)+\frac{i3}{2\omega}\left(\frac{\partial|A|^2}{\partial T_1}A+\frac{\partial A}{\partial T_1}|A|^2\right)\\&=-\omega'\left(\cos\theta_0\frac{\partial}{\partial X_1}\frac{\partial A}{\partial T_1}+\sin\theta_0\frac{\partial}{\partial Y_1}\frac{\partial A}{\partial T_1}\right)+\frac{i3}{2\omega}\left(2|A|^2\frac{\partial A}{\partial T_1}+\frac{\partial A^*}{\partial T_1}A^2\right).\end{aligned} \tag{26}$$

Substituting equation (25) to equation (26) yields

$$\begin{aligned}
\frac{\partial^2 A}{\partial T_1^2} = -\omega'\Bigg[&\cos\theta_0\frac{\partial}{\partial X_1}\left\{-\omega'\left(\cos\theta_0\frac{\partial A}{\partial X_1}+\sin\theta_0\frac{\partial A}{\partial Y_1}\right)+\frac{i3}{2\omega}|A|^2A\right\}\\
&+\sin\theta_0\frac{\partial}{\partial Y_1}\left\{-\omega'\left(\cos\theta_0\frac{\partial A}{\partial X_1}+\sin\theta_0\frac{\partial A}{\partial Y_1}\right)+\frac{i3}{2\omega}|A|^2A\right\}\Bigg]\\
&+\frac{i3}{2\omega}\Bigg[2|A|^2\left\{-\omega'\left(\cos\theta_0\frac{\partial A}{\partial X_1}+\sin\theta_0\frac{\partial A}{\partial Y_1}\right)+\frac{i3}{2\omega}|A|^2A\right\}\\
&+A^2\left\{-\omega'\left(\cos\theta_0\frac{\partial A^*}{\partial X_1}+\sin\theta_0\frac{\partial A^*}{\partial Y_1}\right)-\frac{i3}{2\omega}|A|^2A^*\right\}\Bigg]\\
=\omega'^2&\left(\cos^2\theta_0\frac{\partial^2 A}{\partial X_1^2}+2\cos\theta_0\sin\theta_0\frac{\partial^2 A}{\partial X_1\partial Y_1}+\sin^2\theta_0\frac{\partial^2 A}{\partial Y_1^2}\right)\\
-&\frac{i6\omega'}{\omega}|A|^2\left(\cos\theta_0\frac{\partial A}{\partial X_1}+\sin\theta_0\frac{\partial A}{\partial Y_1}\right)\\
-&\frac{i3\omega'}{\omega}A^2\left(\cos\theta_0\frac{\partial A^*}{\partial X_1}+\sin\theta_0\frac{\partial A^*}{\partial Y_1}\right)-\frac{9}{4\omega^2}|A|^4A. \qquad (27)
\end{aligned}$$

Substituting equation (27) into equation (23) and arranging terms leads to

$$\begin{aligned}
&i\left\{\frac{\partial A}{\partial T_2}+\omega'\left(\cos\theta_0\frac{\partial A}{\partial X_2}+\sin\theta_0\frac{\partial A}{\partial Y_2}\right)+\frac{3\omega'}{\omega^2}|A|^2\left(\cos\theta_0\frac{\partial A}{\partial X_1}+\sin\theta_0\frac{\partial A}{\partial Y_1}\right)\right.\\
&\quad\left.+\frac{3\omega'}{2\omega^2}A^2\left(\cos\theta_0\frac{\partial A^*}{\partial X_1}+\sin\theta_0\frac{\partial A^*}{\partial Y_1}\right)\right\}\\
&+\left\{\frac{1}{2}\omega''\cos^2\theta_0+\frac{1}{6}\left(\omega''+\frac{\omega'^2}{\omega}\right)\sin^2\theta_0\right\}\frac{\partial^2 A}{\partial X_1^2}\\
&+\left\{\frac{1}{2}\omega''\sin^2\theta_0+\frac{1}{6}\left(\omega''+\frac{\omega'^2}{\omega}\right)\cos^2\theta_0\right\}\frac{\partial^2 A}{\partial Y_1^2}+\frac{1}{3}\left(\omega''-\frac{\omega'^2}{2\omega}\right)\sin 2\theta_0\frac{\partial^2 A}{\partial X_1\partial Y_1}\\
&+\frac{9}{8\omega^3}|A|^4A=0. \qquad (28)
\end{aligned}$$

Expressing X_1, X_2, Y_1, Y_2, and T_2 in terms of the original x, y, and t variables, we obtain

$$\begin{aligned}
&i\left\{\frac{\partial A}{\partial t}+\left(\omega'+\frac{3\epsilon\omega'}{\omega^2}|A|^2\right)\left(\cos\theta_0\frac{\partial A}{\partial x}+\sin\theta_0\frac{\partial A}{\partial y}\right)\right.\\
&\quad\left.+\frac{3\epsilon\omega'}{2\omega^2}A^2\left(\cos\theta_0\frac{\partial A^*}{\partial x}+\sin\theta_0\frac{\partial A^*}{\partial y}\right)\right\}\\
&+\left\{\frac{1}{2}\omega''\cos^2\theta_0+\frac{1}{6}\left(\omega''+\frac{\omega'^2}{\omega}\right)\sin^2\theta_0\right\}\frac{\partial^2 A}{\partial x^2}\\
&+\left\{\frac{1}{2}\omega''\sin^2\theta_0+\frac{1}{6}\left(\omega''+\frac{\omega'^2}{\omega}\right)\cos^2\theta_0\right\}\frac{\partial^2 A}{\partial y^2}\\
&+\frac{1}{3}\left(\omega''-\frac{\omega'^2}{2\omega}\right)\sin 2\theta_0\frac{\partial^2 A}{\partial x\partial y}+\frac{9\epsilon^2}{8\omega^3}|A|^4A=0. \qquad (29)
\end{aligned}$$

To eliminate the cross term of $\frac{\partial^2}{\partial x \partial y}$, we transform the coordinate system by

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{t} \end{pmatrix} = \begin{pmatrix} \sqrt{\beta} & -\sqrt{\alpha} & 0 \\ \sqrt{\beta} & \sqrt{\alpha} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ t \end{pmatrix}, \tag{30}$$

where $\alpha = \frac{1}{2}\omega'' \cos^2\theta_0 + \frac{1}{6}\big(\omega'' + \frac{\omega'^2}{\omega}\big)\sin^2\theta_0$, $\beta = \frac{1}{2}\omega'' \sin^2\theta_0 + \frac{1}{6}\big(\omega'' + \frac{\omega'^2}{\omega}\big)\cos^2\theta_0$. (Note that $\alpha > 0$, $\beta > 0$ are easily found using equation (24).) We also let $c_1 = \frac{3\omega'}{\omega^2}$, $c_2 = \frac{1}{2}c_1 = \frac{3\omega'}{2\omega^2}$, $c_3 = \frac{9}{8\omega^3}$, and $\gamma = \frac{1}{3}\big(\omega'' - \frac{\omega'^2}{2\omega}\big)\sin 2\theta_0$, then equation (29) is transformed to the new coordinate system $(\tilde{x}, \tilde{y}, \tilde{t})$ as follows:

$$\begin{aligned} & i\Bigg\{\frac{\partial A}{\partial \tilde{t}} + (\omega' + \epsilon c_1|A|^2)\Bigg((\sqrt{\beta}\cos\theta_0 - \sqrt{\alpha}\sin\theta_0)\frac{\partial A}{\partial \tilde{x}} + (\sqrt{\beta}\cos\theta_0 + \sqrt{\alpha}\sin\theta_0)\frac{\partial A}{\partial \tilde{y}}\Bigg) \\ & \quad + \epsilon c_2 A^2\Bigg((\sqrt{\beta}\cos\theta_0 - \sqrt{\alpha}\sin\theta_0)\frac{\partial A^*}{\partial \tilde{x}} + (\sqrt{\beta}\cos\theta_0 + \sqrt{\alpha}\sin\theta_0)\frac{\partial A^*}{\partial \tilde{y}}\Bigg)\Bigg\} \\ & + (2\alpha\beta - \gamma\sqrt{\alpha\beta}\,)\frac{\partial^2 A}{\partial \tilde{x}^2} + (2\alpha\beta + \gamma\sqrt{\alpha\beta}\,)\frac{\partial^2 A}{\partial \tilde{y}^2} + \epsilon^2 c_3 |A|^4 A = 0. \end{aligned} \tag{31}$$

Hereinafter, we use anew x, y, and t instead of $\tilde{x}$, $\tilde{y}$, and $\tilde{t}$ in order to avoid a nuisance of symbols and we introduce the following symbols:

$$\begin{aligned} & a = \sqrt{\beta}\cos\theta_0 - \sqrt{\alpha}\sin\theta_0, \quad b = \sqrt{\beta}\cos\theta_0 + \sqrt{\alpha}\sin\theta_0, \\ & c = 2\alpha\beta - \gamma\sqrt{\alpha\beta}\,, \quad d = 2\alpha\beta + \gamma\sqrt{\alpha\beta}\,. \end{aligned} \tag{32}$$

Summarizing this section, we obtain the governing equation of the envelope surface of nearly monochromatic waves on an elastic, weakly nonlinear foundation as follows:

$$\begin{aligned} & i\Bigg\{\frac{\partial A}{\partial t} + (\omega' + \epsilon c_1|A|^2)\Bigg(a\frac{\partial A}{\partial x} + b\frac{\partial A}{\partial y}\Bigg) + \epsilon c_2 A^2\Bigg(a\frac{\partial A^*}{\partial x} + b\frac{\partial A^*}{\partial y}\Bigg)\Bigg\} \\ & + c\frac{\partial^2 A}{\partial x^2} + d\frac{\partial^2 A}{\partial y^2} + \epsilon^2 c_3|A|^4 A = 0. \end{aligned} \tag{33}$$

We note that equation (33) is a quintic nonlinear Schrödinger equation.

4. Stability analysis of the uniform solution

In this section, we analyze the stability of the uniform solution of equation (33). We obtain the following uniform solution of equation (33):

$$A(t) = A_0 e^{i(\epsilon^2 c_3 A_0^4 t + \beta_0)}, \tag{34}$$

where A_0 and β_0 are real constants. We consider a modulational perturbation [24] of equation (34) and express it in the form

$$\tilde{A}(x, y, t) = A(t)\{1 + B(x, y, t)\}, \tag{35}$$

where perturbed quantity $B(x, y, t)$ is written

$$B(x,y,t) = B_1 e^{\varOmega t + ik(x\cos\theta + y\sin\theta)} + B_2 e^{\varOmega^* t - ik(x\cos\theta + y\sin\theta)}. \tag{36}$$

Here B_i, $\varOmega$, k and θ are a complex constant, a growth rate with a complex quantity, the wavenumber and propagation direction of the modulational wave, respectively. Also superscript $*$ means complex conjugate.

THEOREM 1. *We assume that there is the innumerable perturbed quantity, presented by equation* (36), *which satisfies* $0 < \underline{c} \leq \left|\frac{B_2}{B_1}\right| \leq \bar{c}$ *as* $B_1 \to 0$. *Then the sufficient condition for the stability of the uniform solution of equation* (33) *is*

$$k^2 \geq \frac{A_0^4\{4c_3(c\cos^2\theta + d\sin^2\theta) - c_2^2(a\cos\theta + b\sin\theta)^2\}}{(c\cos^2\theta + d\sin^2\theta)^2}\epsilon^2. \tag{37}$$

Proof. Substituting equation (35) into equation (33) yields

$$\begin{aligned} &i\left[\frac{\partial A}{\partial t}(1+B) + A\frac{\partial B}{\partial t} + (\omega' + \epsilon c_1 A_0^2|1+B|^2)\left(aA\frac{\partial B}{\partial x} + bA\frac{\partial B}{\partial y}\right) + \epsilon c_2 A^2(1+B)^2 \right.\\ &\left.\times\left(aA^*\frac{\partial B^*}{\partial x} + bA^*\frac{\partial B^*}{\partial y}\right)\right] + cA\frac{\partial^2 B}{\partial x^2} + dA\frac{\partial^2 B}{\partial y^2} + \epsilon^2 c_3 A_0^4|1+B|^4 A(1+B) = 0. \end{aligned} \tag{38}$$

Using the relation

$$\frac{\partial A}{\partial t}(1+B) = i\epsilon^2 c_3 A_0^4 A(1+B), \tag{39}$$

then it follows that eqation (38) is rewritten

$$\begin{aligned} &i\left(\frac{\partial B}{\partial t} + (\omega' + \epsilon c_1 A_0^2|1+B|^2)\left(a\frac{\partial B}{\partial x} + b\frac{\partial B}{\partial y}\right) + \epsilon c_2 A_0^2(1+B)^2\left(a\frac{\partial B^*}{\partial x} + b\frac{\partial B^*}{\partial y}\right)\right)\\ &+ c\frac{\partial^2 B}{\partial x^2} + d\frac{\partial^2 B}{\partial y^2} + \epsilon^2 c_3 A_0^4(1+B)(|1+B|^4 - 1) = 0. \end{aligned} \tag{40}$$

Substituting equation (36) into equation (40) yields

$$\begin{aligned} &e^{\varOmega t + ik(x\cos\theta + y\sin\theta)}(\varTheta_{11}B_1 + \varTheta_{12}B_2^*) + e^{\varOmega^* t - ik(x\cos\theta + y\sin\theta)}(\varTheta_{21}B_1^* + \varTheta_{22}B_2)\\ &+ \sum_{\substack{m=0,1,\ldots,5\\ n=0,1,\ldots,5\\ 2\leq m+n\leq 5}} e^{m\{\varOmega t + ik(x\cos\theta + y\sin\theta)\} + n\{\varOmega^* t - ik(x\cos\theta + y\sin\theta)\}} f_{m,n}(B_1, B_2) = 0, \end{aligned} \tag{41}$$

where

$$\varTheta_{11} = i\varOmega - k(w' + \epsilon c_1 A_0^2)(a\cos\theta + b\sin\theta) - k^2(c\cos^2\theta + d\sin^2\theta) + 2\epsilon^2 c_3 A_0^4, \tag{42}$$

$$\varTheta_{12} = -\epsilon k c_2 A_0^2(a\cos\theta + b\sin\theta) + 2\epsilon^2 c_3 A_0^4, \tag{43}$$

$$\varTheta_{21} = \epsilon k c_2 A_0^2(a\cos\theta + b\sin\theta) + 2\epsilon^2 c_3 A_0^4, \tag{44}$$

$$\varTheta_{22} = i\varOmega^* + k(w' + \epsilon c_1 A_0^2)(a\cos\theta + b\sin\theta) - k^2(c\cos^2\theta + d\sin^2\theta) + 2\epsilon^2 c_3 A_0^4. \tag{45}$$

and $f_{m,n}(B_1, B_2)$ is shown in Appendix B. We find that $f_{m,n}(B_1, B_2)$ is the function of B_1 and B_2, independent of x, y, and t.

Here Substituting $(x, y, t) = (x_0, y_0, t_0)$ into equation (41) yields

$$\tilde{c}_1(\Theta_{11}B_1 + \Theta_{12}B_2^*) + \tilde{c}_1^*(\Theta_{21}B_1^* + \Theta_{22}B_2) + \sum_{\substack{m=0,1,\ldots,5 \\ n=0,1,\ldots,5 \\ 2\le m+n\le 5}} \tilde{c}_{m,n} f_{m,n}(B_1, B_2) = 0, \quad (46)$$

where $\tilde{c}_1$, $\tilde{c}_1^*$ and $\tilde{c}_{m,n}$ are complex constants as follows:

$$\begin{aligned}
\tilde{c}_1 &= e^{\Omega t_0 + ik(x_0\cos\theta + y_0\sin\theta)} \\
\tilde{c}_1^* &= e^{\Omega^* t_0 - ik(x_0\cos\theta + y_0\sin\theta)} \\
\tilde{c}_{m,n} &= e^{m\{\Omega t_0 + ik(x_0\cos\theta + y_0\sin\theta)\} + n\{\Omega^* t_0 - ik(x_0\cos\theta + y_0\sin\theta)\}}.
\end{aligned} \quad (47)$$

We also substitute $(x, y, t) = (x_1, y_1, t_1)$ into equation (41) then we have

$$\tilde{c}_1'(\Theta_{11}B_1 + \Theta_{12}B_2^*) + \tilde{c}_1^{*\prime}(\Theta_{21}B_1^* + \Theta_{22}B_2) + \sum_{\substack{m=0,1,\ldots,5 \\ n=0,1,\ldots,5 \\ 2\le m+n\le 5}} \tilde{c}_{m,n}' f_{m,n}(B_1, B_2) = 0, \quad (48)$$

where $\tilde{c}_1', \tilde{c}_1^{*\prime}$ and $\tilde{c}_{m,n}'$ are complex constants that (x_1, y_1, t_1) is subsituted into equation (47) instead of (x_0, y_0, t_0). We devide by B_1 in Equations (46) and (48) then we obtain

$$\tilde{c}_1\left(\Theta_{11} + \Theta_{12}\frac{B_2^*}{B_1}\right) + \tilde{c}_1^*\frac{B_1^*}{B_1}\left(\Theta_{21} + \Theta_{22}\frac{B_2}{B_1^*}\right) + \sum_{\substack{m=0,1,\ldots,5 \\ n=0,1,\ldots,5 \\ 2\le m+n\le 5}} \tilde{c}_{m,n}\tilde{f}_{m,n}(B_1, B_2) = 0, \quad (49)$$

$$\tilde{c}_1'\left(\Theta_{11} + \Theta_{12}\frac{B_2^*}{B_1}\right) + \tilde{c}_1^{*\prime}\frac{B_1^*}{B_1}\left(\Theta_{21} + \Theta_{22}\frac{B_2}{B_1^*}\right) + \sum_{\substack{m=0,1,\ldots,5 \\ n=0,1,\ldots,5 \\ 2\le m+n\le 5}} \tilde{c}_{m,n}'\tilde{f}_{m,n}(B_1, B_2) = 0, \quad (50)$$

where $\tilde{f}_{m,n}$, shown in Appendix C, is transformed from $f_{m,n}$ since we prepare later calculation.

From the assumption of the theorem, we can take the sequences of B_1 and B_2 such that

$$\lim_{B_1\to 0}\frac{B_2^*}{B_1} = c_b, \quad (51)$$

where $c_b \in \mathbf{C}$, since $[\underline{c}, \bar{c}]$ is compact. We immediately obtain

$$\lim_{B_1\to 0} B_2^* = 0, \quad \lim_{B_1\to 0} B_2 = 0, \quad \lim_{B_1\to 0}\frac{B_2}{B_1^*} = c_b^*, \quad \lim_{B_1\to 0}\frac{B_1^*}{B_2} = \frac{1}{c_b^*}, \quad \lim_{B_1\to 0}\frac{B_1}{B_2^*} = \frac{1}{c_b}. \quad (52)$$

Moreover, we express B_1 as $|B_1|e^{i\theta_b}$, then $\frac{B_1^*}{B_1} = e^{-2i\theta_b}$. There exsists an accumulation point of θ_{b0} in $0 \le \theta_b < 2\pi$ from the theorem of Weierstrass-Bolzano. Using the theorem of Weierstrass-Bolzano again, we can select a partially convergent sequence such as $\lim_{n\to\infty} \theta_{bn} = \theta_{b0}$ from the sequences of B_1 that satisfies equation (51). Therefore,

$$\lim_{B_1 \to 0} \frac{B_1^*}{B_1} = e^{-2i\theta_{b0}} = c_\theta. \tag{53}$$

Using equations (51), (52) and (53) and considering the state of $B_1 \to 0$, we obtain the following coupled equations from equations (49) and (50):

$$\tilde{c}_1(\Theta_{11} + c_b \Theta_{12}) + \tilde{c}_1^* c_\theta(\Theta_{21} + c_b^* \Theta_{22}) = 0, \tag{54}$$

$$\tilde{c}_1'(\Theta_{11} + c_b \Theta_{12}) + \tilde{c}_1^{*\prime} c_\theta(\Theta_{21} + c_b^* \Theta_{22}) = 0. \tag{55}$$

Combining equations (54) and (55),

$$\begin{pmatrix} \tilde{c}_1 & \tilde{c}_1^* c_\theta \\ \tilde{c}_1' & \tilde{c}_1^{*\prime} c_\theta \end{pmatrix} \begin{pmatrix} \Theta_{11} + c_b \Theta_{12} \\ \Theta_{21} + c_b^* \Theta_{22} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{56}$$

Here, we can select (x_0, y_0, t_0) and (x_1, y_1, t_1) such that a matrix $\left(\begin{smallmatrix} \tilde{c}_1 & \tilde{c}_1^* c_\theta \\ \tilde{c}_1' & \tilde{c}_1^{*\prime} c_\theta \end{smallmatrix}\right)$ has full rank. Therefore, from equation (56)

$$\begin{aligned} \Theta_{11} + c_b \Theta_{12} &= 0, \\ \Theta_{21} + c_b^* \Theta_{22} &= 0. \end{aligned} \tag{57}$$

We finally obtain the relation

$$\Theta_{11} \Theta_{22}^* - \Theta_{12} \Theta_{21}^* = 0. \tag{58}$$

We obtain Ω by solving equation (58) as follows:

$$\begin{aligned} \Omega = &-ik(w' + \epsilon c_1 A_0^2)(a\cos\theta + b\sin\theta) \\ &\pm [k^2\{\epsilon^2 A_0^4\{4c_3(c\cos^2\theta + d\sin^2\theta) - c_2^2(a\cos\theta + b\sin\theta)^2\} \\ &\qquad - (c\cos^2\theta + d\sin^2\theta)^2 k^2\}]^{\frac{1}{2}}. \end{aligned} \tag{59}$$

In order to be stable in the perturbation, the content of root of equation (59) must not be positive. This leads to the theorem. $\square$

Remark 1. We assume that $0 < \underline{c} \le \left|\frac{B_2}{B_1}\right| \le \bar{c}$ only in this theorem. Although we don't assume the convergence of $\frac{B_2^*}{B_1}$, we have the convergent point c_b determined uniquely by equation (57), which is unrelated to how to take partial sequences of B_1 and B_2.

Remark 2. In equation (59), Ω has one or two points on the imaginary axis when equation (37) holds. However, when equation (37) doesn't hold, Ω has one stable point and one unstable point. Therefore, Ω is structurally unstable.

Appendix A.

We consider a nonlinear spring, which characteristics is presented by

$$F(w) = k_b w + b_s w^2, \tag{A.1}$$

instead of equation (2), where $b_s \neq 0$. This characteristics is not so real in engineering. However, the result from equation (A.1) is important in mathematics. Then equation (4) becomes

$$\begin{aligned}&\frac{\partial^2 w(x,y,t)}{\partial t^2} + \frac{EI}{\mu}\left\{\frac{\partial^4 w(x,y,t)}{\partial x^4} + 2\frac{\partial^4 w(x,y,t)}{\partial x^2 \partial y^2} + \frac{\partial^4 w(x,y,t)}{\partial y^4}\right\}\\ &+ \frac{k_b}{\mu} w(x,y,t) + \frac{b_s}{\mu} w(x,y,t)^2 = -g.\end{aligned} \tag{A.2}$$

Using the same transformation (equation (5)) and dimensionless variables (equation (8)), equation (A.2) becomes

$$\begin{aligned}&\frac{\partial^2 \bar{w}}{\partial \bar{t}^2} + \frac{\partial^4 \bar{w}}{\partial \bar{x}^4} + 2\frac{\partial^4 \bar{w}}{\partial \bar{x}^2 \partial \bar{y}^2} + \frac{\partial^4 \bar{w}}{\partial \bar{y}^4}\\ &+ \frac{l^4}{\pi^4 EI}\left\{k_b \bar{w} + \frac{b_s A_a}{l}\bar{w}^2 + \frac{2b_s \mu g}{k_b} s\left(\frac{l}{\pi}\bar{x}, \frac{l}{\pi}\bar{y}\right)\bar{w} + \frac{b_s l}{A_a}\left(\frac{\mu g}{k_b} s\left(\frac{l}{\pi}\bar{x}, \frac{l}{\pi}\bar{y}\right)\right)^2\right\} = 0,\end{aligned} \tag{A.3}$$

Here we put $\tilde{\varepsilon} = \frac{A_a}{l}$ with $\tilde{\varepsilon}$ a small parameter. Using equation (10), then we have the following governing equation, which describes up to $\mathcal{O}(\varepsilon^n), n > 1$:

$$\begin{aligned}&\frac{\partial^2 w(x,y,t)}{\partial t^2} + \frac{\partial^4 w(x,y,t)}{\partial x^4} + 2\frac{\partial^4 w(x,y,t)}{\partial x^2 \partial y^2} + \frac{\partial^4 w(x,y,t)}{\partial y^4}\\ &+ p^2 w(x,y,t) = \varepsilon w(x,y,t)^2,\\ &0 < x < \pi, \quad 0 < y < \pi, \quad t > 0,\end{aligned} \tag{A.4}$$

where all bars are dropped for convenience. In this case, $p^2 w - \varepsilon w^2$ represents the restoring force due to the elastic foundation.

Substituting equation (14) into equation (A.4), we obtain

$$\frac{\partial^4 w_0}{\partial X_0^4} + 2\frac{\partial^4 w_0}{\partial X_0^2 \partial Y_0^2} + \frac{\partial^4 w_0}{\partial Y_0^4} + \frac{\partial^2 w_0}{\partial T_0^2} + p^2 w_0 = 0, \tag{A.5}$$

$$\begin{aligned}&\frac{\partial^4 w_1}{\partial X_0^4} + 2\frac{\partial^4 w_1}{\partial X_0^2 \partial Y_0^2} + \frac{\partial^4 w_1}{\partial Y_0^4} + \frac{\partial^2 w_1}{\partial T_0^2} + p^2 w_1\\ &= -4\frac{\partial^4 w_0}{\partial X_0^3 \partial X_1} - 4\frac{\partial^4 w_0}{\partial X_0 \partial X_1 \partial Y_0^2} - 4\frac{\partial^4 w_0}{\partial X_0^2 \partial Y_0 \partial Y_1}\\ &\quad - 4\frac{\partial^4 w_0}{\partial Y_0^3 \partial Y_1} - 2\frac{\partial^2 w_0}{\partial T_0 \partial T_1} + w_0^2,\end{aligned} \tag{A.6}$$

$$\begin{aligned}
&\frac{\partial^4 w_2}{\partial X_0^4} + 2\frac{\partial^4 w_2}{\partial X_0^2 \partial Y_0^2} + \frac{\partial^4 w_2}{\partial Y_0^4} + \frac{\partial^2 w_2}{\partial T_0^2} + p^2 w_2 \\
&= -4\frac{\partial^4 w_1}{\partial X_0^3 \partial X_1} - 4\frac{\partial^4 w_1}{\partial X_0 \partial X_1 \partial Y_0^2} - 4\frac{\partial^4 w_1}{\partial X_0^2 \partial Y_0 \partial Y_1} \\
&\quad - 4\frac{\partial^4 w_1}{\partial Y_0^3 \partial Y_1} - 2\frac{\partial^2 w_1}{\partial T_0 \partial T_1} - 6\frac{\partial^4 w_0}{\partial X_0^2 \partial X_1^2} - 4\frac{\partial^4 w_0}{\partial X_0^3 \partial X_2} - 8\frac{\partial^4 w_0}{\partial X_0 \partial X_1 \partial Y_0 \partial Y_1} \\
&\quad - 2\frac{\partial^4 w_0}{\partial X_1^2 \partial Y_0^2} - 4\frac{\partial^4 w_0}{\partial X_0 \partial X_2 \partial Y_0^2} - 2\frac{\partial^4 w_0}{\partial X_0^2 \partial Y_1^2} - 4\frac{\partial^4 w_0}{\partial X_0^2 \partial Y_0 \partial Y_2} \\
&\quad - 6\frac{\partial^4 w_0}{\partial Y_0^2 \partial Y_1^2} - 4\frac{\partial^4 w_0}{\partial Y_0^3 \partial Y_2} - \frac{\partial^2 w_0}{\partial T_1^2} - 2\frac{\partial^2 w_0}{\partial T_0 \partial T_2} + 2 w_0 w_1. \qquad \text{(A.7)}
\end{aligned}$$

Equation (A.5) is the same of equation (15) so that we have the same dispersion relation of equation (19). Furthermore, we have the following solvability condition from equation (A.6):

$$2k_w^3 \cos\theta_0 \frac{\partial A}{\partial X_1} + 2k_w^3 \sin\theta_0 \frac{\partial A}{\partial Y_1} + \omega \frac{\partial A}{\partial T_1} = 0, \qquad \text{(A.8)}$$

and we have the equation w_1 satisfies as follows:

$$\begin{aligned}
&\frac{\partial^4 w_1}{\partial X_0^4} + 2\frac{\partial^4 w_1}{\partial X_0^2 \partial Y_0^2} + \frac{\partial^4 w_1}{\partial Y_0^4} + \frac{\partial^2 w_1}{\partial T_0^2} + p^2 w_1 \\
&= A^2 e^{2i(k_w X_0 \cos\theta_0 + k_w Y_0 \sin\theta_0 - \omega T_0)} + |A|^2 + \text{cc}. \qquad \text{(A.9)}
\end{aligned}$$

Then the solution of equation (A.9) becomes

$$w_1 = \frac{A^2}{3(4k_w^4 - p^2)} e^{2i(k_w X_0 \cos\theta_0 + k_w Y_0 \sin\theta_0 - \omega T_0)} + \frac{1}{p^2}|A|^2 + \text{cc}. \qquad \text{(A.10)}$$

Another solvability condition is obtained from equations (A.7) and (A.10) as follows:

$$\begin{aligned}
&(6k_w^2 \cos^2\theta_0 + 2k_w^2 \sin^2\theta_0)\frac{\partial^2 A}{\partial X_1^2} + 4ik_w^3 \cos\theta_0 \frac{\partial A}{\partial X_2} + 4k_w^2 \sin 2\theta_0 \frac{\partial^2 A}{\partial X_1 \partial Y_1} \\
&+ (6k_w^2 \sin^2\theta_0 + 2k_w^2 \cos^2\theta_0)\frac{\partial^2 A}{\partial Y_1^2} + 4ik_w^3 \sin\theta_0 \frac{\partial A}{\partial Y_2} + 2i\omega \frac{\partial A}{\partial T_2} - \frac{\partial^2 A}{\partial T_1^2} + \delta|A|^2 A \\
&= 0, \qquad \text{(A.11)}
\end{aligned}$$

where $\delta = \frac{4(6k_w^4 - p^2)}{3p^2(4k_w^4 - p^2)}$. Using equations (A.8), (A.11) and (24), we have

$$\begin{aligned}
&2i\omega \frac{\partial A}{\partial T_2} + 4ik_w^3 \left(\cos\theta_0 \frac{\partial A}{\partial X_2} + \sin\theta_0 \frac{\partial A}{\partial Y_2} \right) \\
&+ \left\{ \omega\omega'' \cos^2\theta_0 + \frac{1}{3}\left(\omega\omega'' + \omega'^2 \right) \sin^2\theta_0 \right\} \frac{\partial^2 A}{\partial X_1^2} \\
&+ \left\{ \omega\omega'' \sin^2\theta_0 + \frac{1}{3}\left(\omega\omega'' + \omega'^2 \right) \cos^2\theta_0 \right\} \frac{\partial^2 A}{\partial Y_1^2} \\
&+ \frac{1}{3}\left(2\omega\omega'' - \omega'^2 \right) \sin 2\theta_0 \frac{\partial^2 A}{\partial X_1 \partial Y_1} + \delta |A|^2 A = 0. \qquad \text{(A.12)}
\end{aligned}$$

Expressing X_1, X_2, Y_1, Y_2, and T_2 in terms of the original x, y, and t variables, we finally obtain

$$\begin{aligned}
&i\left\{ \frac{\partial A}{\partial t} + \omega' \left(\cos\theta_0 \frac{\partial A}{\partial x} + \sin\theta_0 \frac{\partial A}{\partial y} \right) \right\} \\
&+ \frac{1}{2}\left\{ \omega'' \cos^2\theta_0 + \frac{1}{3}\left(\omega'' + \frac{\omega'^2}{\omega} \right) \sin^2\theta_0 \right\} \frac{\partial^2 A}{\partial x^2} \\
&+ \frac{1}{2}\left\{ \omega'' \sin^2\theta_0 + \frac{1}{3}\left(\omega'' + \frac{\omega'^2}{\omega} \right) \cos^2\theta_0 \right\} \frac{\partial^2 A}{\partial y^2} \\
&+ \frac{1}{6}\left(2\omega'' - \frac{\omega'^2}{\omega} \right) \sin 2\theta_0 \frac{\partial^2 A}{\partial x \partial y} + \frac{\epsilon^2 \delta}{2\omega} |A|^2 A = 0. \qquad \text{(A.13)}
\end{aligned}$$

Equation (A.13) is rewritten by the transformation of equation (30) as follows:

$$\begin{aligned}
&i\left\{ \frac{\partial A}{\partial \tilde{t}} + \omega' \left((\sqrt{\beta}\cos\theta_0 - \sqrt{\alpha}\sin\theta_0) \frac{\partial A}{\partial \tilde{x}} + (\sqrt{\beta}\cos\theta_0 + \sqrt{\alpha}\sin\theta_0) \frac{\partial A}{\partial \tilde{y}} \right) \right\} \\
&+ (2\alpha\beta - \gamma\sqrt{\alpha\beta}\,) \frac{\partial^2 A}{\partial \tilde{x}^2} + (2\alpha\beta + \gamma\sqrt{\alpha\beta}\,) \frac{\partial^2 A}{\partial \tilde{y}^2} + \frac{\epsilon^2 \delta}{2\omega} |A|^2 A = 0. \qquad \text{(A.14)}
\end{aligned}$$

When using the symbols a, b, c, and d in equation (32), we obtain the governing equation of the envelope surface of nearly monochromatic waves on an elastic, weakly nonlinear foundation with the characteristics of equation (3) as follows:

$$i\left(\frac{\partial A}{\partial t} + \omega' \left(a\frac{\partial A}{\partial x} + b\frac{\partial A}{\partial y} \right) \right) + c\frac{\partial^2 A}{\partial x^2} + d\frac{\partial^2 A}{\partial y^2} + \frac{\epsilon^2 \delta}{2\omega} |A|^2 A = 0, \qquad \text{(A.15)}$$

where $\tilde{}$ is omitted for convenience. Equation (A.15) is a standard nonlinear Schrödinger equation with cubic nonlinearlity.

Appendix B.

$$\begin{aligned}
f_{2,0}(B_1, B_2) = &-\epsilon c_1 A_0^2 k (a\cos\theta + b\sin\theta)(B_1 + B_2^*) B_1 \\
&- 2\epsilon c_2 A_0^2 k (a\cos\theta + b\sin\theta) B_1 B_2^* + \epsilon^2 c_3 A_0^4 (3B_1^2 + 6B_1 B_2^* + B_2^{*2})
\end{aligned}$$

$$
\begin{aligned}
f_{0,2}(B_1,B_2) &= \epsilon c_1 A_0^2 k(a\cos\theta + b\sin\theta)(B_1^* + B_2)B_2 \\
&\quad + 2\epsilon c_2 A_0^2 k(a\cos\theta + b\sin\theta)B_1^* B_2 + \epsilon^2 c_3 A_0^4(B_1^{*2} + 6B_1^* B_2 + 3B_2^2) \\
f_{1,1}(B_1,B_2) &= \epsilon c_1 A_0^2 k(a\cos\theta + b\sin\theta)(|B_2|^2 - |B_1|^2) \\
&\quad + 2\epsilon c_2 A_0^2 k(a\cos\theta + b\sin\theta)(|B_1|^2 - |B_2|^2) \\
&\quad + \epsilon^2 c_3 A_0^4\{6(|B_1|^2 + |B_2|^2) + 6B_1B_2 + 2B_1^*B_2^*\} \\
f_{3,0}(B_1,B_2) &= -\epsilon c_1 A_0^2 k(a\cos\theta + b\sin\theta)B_1^2 B_2^* - \epsilon c_2 A_0^2 k(a\cos\theta + b\sin\theta)B_1^2 B_2^* \\
&\quad + \epsilon^2 c_3 A_0^4(B_1^3 + 6B_1^2 B_2^* + 3B_1 B_2^{*2}) \\
f_{0,3}(B_1,B_2) &= \epsilon c_1 A_0^2 k(a\cos\theta + b\sin\theta)B_1^* B_2^2 + \epsilon c_2 A_0^2 k(a\cos\theta + b\sin\theta)B_1^* B_2^2 \\
&\quad + \epsilon^2 c_3 A_0^4(B_2^3 + 6B_1^* B_2^2 + 3B_1^{*2} B_2) \\
f_{2,1}(B_1,B_2) &= -\epsilon c_1 A_0^2 k(a\cos\theta + b\sin\theta)B_1|B_1|^2 \\
&\quad + \epsilon c_2 A_0^2 k(a\cos\theta + b\sin\theta)(B_1|B_1|^2 - 2B_1|B_2|^2) \\
&\quad + \epsilon^2 c_3 A_0^4\{(|B_1|^2 + |B_2|^2)(6B_1 + 2B_2^*) \\
&\qquad + 4|B_1|^2 B_2^* + 6B_1|B_2|^2 + 3B_1^2 B_2 + B_2^*|B_2|^2\} \\
f_{1,2}(B_1,B_2) &= \epsilon c_1 A_0^2 k(a\cos\theta + b\sin\theta)B_2|B_2|^2 \\
&\quad - \epsilon c_2 A_0^2 k(a\cos\theta + b\sin\theta)(B_2|B_2|^2 - 2|B_1|^2 B_2) \\
&\quad + \epsilon^2 c_3 A_0^4\{(|B_1|^2 + |B_2|^2)(2B_1^* + 6B_2) \\
&\qquad + 4B_1^*|B_2|^2 + 6|B_1|^2 B_2 + 3B_1 B_2^2 + B_1^*|B_1|^2\} \\
f_{4,0}(B_1,B_2) &= \epsilon^2 c_3 A_0^4\{B_1^2 B_2^{*2} + 2(B_1^3 B_2^* + B_1^2 B_2^{*2})\} \\
f_{0,4}(B_1,B_2) &= \epsilon^2 c_3 A_0^4\{B_1^{*2} B_2^2 + 2(B_1^{*2} B_2^2 + B_1^* B_2^3)\} \\
f_{2,2}(B_1,B_2) &= \epsilon^2 c_3 A_0^4\{(|B_1|^2 + |B_2|^2)^2 + |B_1B_2|^2 \\
&\qquad + 2(|B_1|^2 + |B_2|^2)(2B_1B_2 + |B_1|^2 + |B_2|^2) \\
&\qquad + 2B_2(|B_1|^2 B_2^* + B_1|B_2|^2) + 2B_1(|B_1|^2 B_2 + B_1^*|B_2|^2)\} \\
f_{3,1}(B_1,B_2) &= \epsilon^2 c_3 A_0^4\{2(B_1^2 + 2B_1B_2^*)(|B_1|^2 + |B_2|^2) \\
&\qquad + 2(B_1B_2^*|B_1|^2 + B_1B_2^*|B_2|^2 + 4B_1^2|B_2|^2)\} \\
f_{1,3}(B_1,B_2) &= \epsilon^2 c_3 A_0^4\{2(B_2^2 + 2B_1^*B_2)(|B_1|^2 + |B_2|^2) \\
&\qquad + 2(B_1^*B_2|B_1|^2 + B_1^*B_2|B_2|^2 + 4|B_1|^2 B_2^2)\} \\
f_{5,0}(B_1,B_2) &= \epsilon^2 c_3 A_0^4 B_1^3 B_2^{*2} \\
f_{0,5}(B_1,B_2) &= \epsilon^2 c_3 A_0^4 B_1^{*2} B_2^3 \\
f_{4,1}(B_1,B_2) &= \epsilon^2 c_3 A_0^4\{B_1^2|B_2|^2 B_2^* + 2B_1^2 B_2^*(|B_1|^2 + |B_2|^2)\} \\
f_{1,4}(B_1,B_2) &= \epsilon^2 c_3 A_0^4\{|B_1|^2 B_1^* B_2^2 + 2B_1^* B_2^2(|B_1|^2 + |B_2|^2)\} \\
f_{3,2}(B_1,B_2) &= \epsilon^2 c_3 A_0^4\{B_1(|B_1|^2 + |B_2|^2)^2 + B_1|B_1B_2|^2 + 2B_1|B_2|^2(|B_1|^2 + |B_2|^2)\} \\
f_{2,3}(B_1,B_2) &= \epsilon^2 c_3 A_0^4\{B_2(|B_1|^2 + |B_2|^2)^2 + B_2|B_1B_2|^2 + 2|B_1|^2 B_2(|B_1|^2 + |B_2|^2)\}.
\end{aligned}
$$

Appendix C.

$$\begin{aligned}\tilde{f}_{2,0}(B_1,B_2) &= B_1\Bigg[-\epsilon c_1 A_0^2 k(a\cos\theta + b\sin\theta)\left(1+\frac{B_2^*}{B_1}\right)\\ &\quad - 2\epsilon c_2 A_0^2 k(a\cos\theta+b\sin\theta)\frac{B_2^*}{B_1} + \epsilon^2 c_3 A_0^4\left\{3+6\frac{B_2^*}{B_1}+\left(\frac{B_2^*}{B_1}\right)^2\right\}\Bigg]\\
\tilde{f}_{0,2}(B_1,B_2) &= \frac{B_1^*}{B_1}B_2\Bigg\{\epsilon c_1 A_0^2 k(a\cos\theta+b\sin\theta)\left(1+\frac{B_2}{B_1^*}\right)\\ &\quad + 2\epsilon c_2 A_0^2 k(a\cos\theta+b\sin\theta) + \epsilon^2 c_3 A_0^4\left(6+3\frac{B_2}{B_1^*}+\frac{B_1^*}{B_2}\right)\Bigg\}\\
\tilde{f}_{1,1}(B_1,B_2) &= B_1^*\Bigg\{\epsilon c_1 A_0^2 k(a\cos\theta+b\sin\theta)\left(\frac{B_2^*}{B_1}\frac{B_2}{B_1^*}-1\right)\\ &\quad + 2\epsilon c_2 A_0^2 k(a\cos\theta+b\sin\theta)\\ &\quad \times\left(1-\frac{B_2^*}{B_1}\frac{B_2}{B_1^*}\right) + \epsilon^2 c_3 A_0^4\left(6+6\frac{B_2}{B_1^*}+2\frac{B_2^*}{B_1}+6\frac{B_2^*}{B_1}\frac{B_2}{B_1^*}\right)\Bigg\}\\
\tilde{f}_{3,0}(B_1,B_2) &= B_1 B_2^*\Bigg\{-\epsilon c_1 A_0^2 k(a\cos\theta+b\sin\theta) - \epsilon c_2 A_0^2 k(a\cos\theta+b\sin\theta)\\ &\quad + \epsilon^2 c_3 A_0^4\left(6+\frac{B_1}{B_2^*}+3\frac{B_2^*}{B_1}\right)\Bigg\}\\
\tilde{f}_{0,3}(B_1,B_2) &= \frac{B_1^*}{B_1}B_2^2\Bigg\{\epsilon c_1 A_0^2 k(a\cos\theta+b\sin\theta) + \epsilon c_2 A_0^2 k(a\cos\theta+b\sin\theta)\\ &\quad + \epsilon^2 c_3 A_0^4\left(6+\frac{B_2}{B_1^*}+3\frac{B_1^*}{B_2}\right)\Bigg\}\\
\tilde{f}_{2,1}(B_1,B_2) &= |B_2|^2\Bigg\{-\epsilon c_1 A_0^2 k(a\cos\theta+b\sin\theta)\frac{B_1}{B_2^*}\frac{B_1^*}{B_2}\\ &\quad + \epsilon c_2 A_0^2 k(a\cos\theta+b\sin\theta)\left(\frac{B_1}{B_2^*}\frac{B_1^*}{B_2}-2\right)\\ &\quad + \epsilon^2 c_3 A_0^4\left(12+6\frac{B_1^*}{B_2}+3\frac{B_2^*}{B_1}+3\frac{B_1}{B_2^*}+6\frac{B_1}{B_2^*}\frac{B_1^*}{B_2}\right)\Bigg\}\\
\tilde{f}_{1,2}(B_1,B_2) &= B_1^* B_2\Bigg\{\epsilon c_1 A_0^2 k(a\cos\theta+b\sin\theta)\frac{B_2}{B_1^*}\frac{B_2^*}{B_1}\\ &\quad - \epsilon c_2 A_0^2 k(a\cos\theta+b\sin\theta)\left(\frac{B_2}{B_1^*}\frac{B_2^*}{B_1}-2\right)\\ &\quad + \epsilon^2 c_3 A_0^4\left(12+3\frac{B_1^*}{B_2}+6\frac{B_2^*}{B_1}+3\frac{B_2}{B_1^*}+6\frac{B_2}{B_1^*}\frac{B_2^*}{B_1}\right)\Bigg\}\\
\tilde{f}_{4,0}(B_1,B_2) &= B_1 B_2^{*2}\epsilon^2 c_3 A_0^4\left(3+2\frac{B_1}{B_2^*}\right)\\
\tilde{f}_{0,4}(B_1,B_2) &= B_1^*\frac{B_1^*}{B_1}B_2^2\epsilon^2 c_3 A_0^4\left(3+2\frac{B_2}{B_1^*}\right)\end{aligned}$$

$$\tilde{f}_{2,2}(B_1,B_2) = B_1^*|B_2|^2\epsilon^2 c_3 A_0^4\left(11 + 6\frac{B_1}{B_2^*} + 6\frac{B_2}{B_1^*} + 3\frac{B_1}{B_2^*}\frac{B_1^*}{B_2} + 3\frac{B_2^*}{B_1}\frac{B_2}{B_1^*}\right)$$

$$\tilde{f}_{3,1}(B_1,B_2) = |B_1|^2 B_2^*\epsilon^2 c_3 A_0^4\left(6 + 2\frac{B_1}{B_2^*} + 10\frac{B_2}{B_1^*} + 6\frac{B_2}{B_1^*}\frac{B_2^*}{B_1}\right)$$

$$\tilde{f}_{1,3}(B_1,B_2) = \frac{B_1^*}{B_1}B_2|B_2|^2\epsilon^2 c_3 A_0^4\left(6 + 2\frac{B_2}{B_1^*} + 10\frac{B_1}{B_2^*} + 6\frac{B_1}{B_2^*}\frac{B_1^*}{B_2}\right)$$

$$\tilde{f}_{5,0}(B_1,B_2) = B_1^2 B_2^{*2}\epsilon^2 c_3 A_0^4$$

$$\tilde{f}_{0,5}(B_1,B_2) = \frac{B_1^*}{B_1}B_2^4\epsilon^2 c_3 A_0^4\frac{B_1^*}{B_2}$$

$$\tilde{f}_{4,1}(B_1,B_2) = B_1|B_1|^2 B_2^*\epsilon^2 c_3 A_0^4\left(2 + 3\frac{B_2^*}{B_1}\frac{B_2}{B_1^*}\right)$$

$$\tilde{f}_{1,4}(B_1,B_2) = \frac{B_1^*}{B_1}B_2^2|B_2|^2\epsilon^2 c_3 A_0^4\left(2 + 3\frac{B_1^*}{B_2}\frac{B_1}{B_2^*}\right)$$

$$\tilde{f}_{3,2}(B_1,B_2) = |B_1|^2|B_2|^2\epsilon^2 c_3 A_0^4\left(5 + \frac{B_1}{B_2^*}\frac{B_1^*}{B_2} + 3\frac{B_2}{B_1^*}\frac{B_2^*}{B_1}\right)$$

$$\tilde{f}_{2,3}(B_1,B_2) = B_1^* B_2|B_2|^2\epsilon^2 c_3 A_0^4\left(5 + \frac{B_2^*}{B_1}\frac{B_2}{B_1^*} + 3\frac{B_1}{B_2^*}\frac{B_1^*}{B_2}\right).$$

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