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Minimization of the Principal Eigenvalue for an Elliptic Boundary Value Problem with Indefinite Weight, and Applications to Population Dynamics

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This paper is concerned with an indefinite weight linear eigenvalue problem which is related with biological invasions of species. We investigate the minimization of the positive principal eigenvalue under the constraint that the weight is bounded by a positive and a negative constant and the total weight is a fixed negative constant. For an arbitrary domain, it is shown that every global minimizer must be of "bang-bang" type. When the domain is an interval, it is proved that there are exactly two global minimizers, for which the weight is positive at one end of the interval and is negative in the remainder. The biological implication is that a single favorable region at one end of the habitat provides the best opportunity for the species to survive, and also that the least fragmented habitat provides the best chance for the population to maintain its genetic variability.

Key words: principal eigenvalue, global minimizer, population dynamics

1. Introduction

For more than two decades, the following linear eigenvalue problem with indefinite weight (and also more general forms)

$$\begin{cases} \Delta\varphi + \lambda m(x)\varphi = 0 & \text{in} \quad \Omega, \\ \dfrac{\partial\varphi}{\partial n} = 0 & \text{on} \quad \partial\Omega \end{cases} \tag{1.1}$$

has been extensively investigated, mainly because of its importance in the study of nonlinear mathematical models from population biology. Here, Ω is a bounded region in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, n is the outward unit normal vector on $\partial\Omega$, and $m(x) \in L^\infty(\Omega)$. Biologically, Ω refers to the habitat of a species, the zero-flux boundary condition in (1.1) means that no individuals cross the boundary of the habitat, and the function $m(x)$ represents either the selective force of the environment on alleles or the intrinsic growth rate of species at location x.

We say that λ is a *principal eigenvalue* of (1.1) if λ has a positive eigenfunction $\varphi \in H^1(\Omega)$. By standard elliptic regularity and the Sobolev embedding theorem [11], the function φ satisfies $\varphi \in W^{2,q}(\Omega) \cap C^{1,\gamma}(\overline{\Omega})$ for every $q > 1$ and every $\gamma \in (0,1)$, and $\varphi > 0$ in $\overline{\Omega}$. It is clear that $\lambda = 0$ is a principal eigenvalue of (1.1) with positive constants as its eigenfunctions. Of particular interest is the existence of positive principal eigenvalues.

Define

$$\Omega_+ = \{x \in \Omega \colon m(x) > 0\}, \qquad \Omega_- = \{x \in \Omega \colon m(x) < 0\}.$$

Before reviewing previous work on (1.1), we impose the following condition on $m(x)$:

(A1) The set Ω_+ has positive Lebesgue measure, and $\int_\Omega m < 0$.

The following result is well-known [2, 23, 13].

Theorem A. *The eigenvalue problem* (1.1) *has a positive principal eigenvalue* (*denoted by* $\lambda_1(m)$) *if and only if* (A1) *holds. Moreover,* $\lambda_1(m)$ *is the only positive principal eigenvalue, and it is also the smallest positive eigenvalue of* (1.1).

Next, we present two examples to illustrate the importance of $\lambda_1(m)$ in the study of nonlinear mathematical models from population biology. The first model concerns the evolution of gene frequencies at a single diallelic locus under the joint action of migration and selection:

$$\begin{cases} u_t = \Delta u + \lambda m(x) u(1-u) & \text{in} \quad \Omega \times \mathbb{R}^+, \\ \dfrac{\partial u}{\partial n} = 0 & \text{on} \quad \partial\Omega \times \mathbb{R}^+, \\ 0 \le u(x,0) \le 1 & \text{in} \quad \overline{\Omega}, \quad 0 < \displaystyle\int_\Omega u(x,0)\,dx < 1, \end{cases} \tag{1.2}$$

where $u(x,t)$ and $1 - u(x,t)$ represent the frequencies of two alleles A_1 and A_2 at location x and time t, respectively. The integral $\int_{\Omega_+} m$ represents the total selection force favoring A_1, whereas $\int_{\Omega_-}(-m)$ is the selection force favoring A_2. The assumption $\int_\Omega m < 0$ is equivalent to $\int_{\Omega_+} m < \int_{\Omega_-}(-m)$, i.e., allele A_2 is favored over allele A_1. The conditions on the initial data $u(x,0)$ ensure that both alleles are present initially. By the maximum principle [20], we have $0 < u(x,t) < 1$ for every $x \in \overline{\Omega}$ and every $t > 0$. Moreover,

(i) if $0 < \lambda \le \lambda_1(m)$, then $u(x,t) \to 0$ uniformly for $x \in \overline{\Omega}$ as $t \to \infty$; i.e., allele A_1 is eliminated and allele A_2 is maintained;

(ii) if $\lambda > \lambda_1(m)$, then $u(x,t) \to u^*(x)$ uniformly for $x \in \overline{\Omega}$ as $t \to \infty$, where u^* is the unique positive solution of (1.2) satisfying $u^* \in W^{2,q}(\Omega)$ for every $q > 1$ and $0 < u^*(x) < 1$ for every $x \in \overline{\Omega}$. This implies that both alleles are maintained if λ is larger than $\lambda_1(m)$.

These results were established in [12] for Hölder-continuous weight $m(x)$, and can be extended to the case of bounded measurable weight. The study of the diallelic case lays down the ground for further analysis of the much more difficult multiallelic case, and we refer to [9, 12, 18, 22, 23] and references therein for the diallelic case and [15, 16] for recent developments in the multiallelic case.

The second example is the diffusive logistic equation

$$\begin{cases} u_t = \Delta u + \lambda u[m(x) - u] & \text{in} \quad \Omega \times \mathbb{R}^+, \\ \dfrac{\partial u}{\partial n} = 0 & \text{on} \quad \partial\Omega \times \mathbb{R}^+, \\ u(x,0) \geq 0, \ u(x,0) \not\equiv 0 & \text{in} \quad \overline{\Omega}, \end{cases} \tag{1.3}$$

which first appeared in the pioneering work of Skellam [24]. Here, $u(x,t)$ represents the density of a species at location x and time t. Hence, only non-negative solutions of (1.3) are of interest. The function $m(x)$ represents the intrinsic growth rate of a species, which is positive in the favorable part of habitat (Ω_+) and negative in unfavorable one (Ω_-). The integral $\int_\Omega m$ can be viewed as a measure of the total resources in a spatially heterogeneous environment. The logistic equation (1.3) is important in understanding the effects of dispersal and spatial heterogeneity in the population growth of a single species, and we refer to [3, 5, 6] and references therein for works on (1.3). In particular, it is known that

(i) if $\lambda \leq \lambda_1(m)$, then $u(x,t) \to 0$ uniformly in $\overline{\Omega}$ as $t \to \infty$ for all non-negative and non-trivial initial data; i.e., the species goes to extinction;

(ii) if $\lambda > \lambda_1(m)$, then $u(x,t) \to u^*(x)$ uniformly in $\overline{\Omega}$ as $t \to \infty$, where u^* is the unique positive solution of (1.3) in $W^{2,q}(\Omega)$ for every $q > 1$; i.e., the species survives.

Equation (1.3) also plays a crucial role in studying the dynamics of multiple interacting species, and we refer to [6, 7, 8, 14, 17] and references therein for works on models of two competing species with diffusion and spatial heterogeneity.

In this paper we are mainly concerned with the dependence of the principal eigenvalue $\lambda_1(m)$ of (1.1) on the weight function $m(x)$. In particular, we are interested in how spatial variation in the environment of the habitat affects the maintenance of alleles or species. To be more precise, let m_0, $\underline{m}$, and $\overline{m}$ be three given positive constants with $m_0 < \underline{m}$, and assume

(A2) $-\underline{m} \leq m(x) \leq \overline{m}$ a.e. in Ω, and $\int_\Omega m \leq -m_0|\Omega|$.

We address the following mathematical question:

Question. Among all functions $m(x)$ that satisfy (A1) and (A2), which $m(x)$ will yield the smallest $\lambda_1(m)$?

This question is mainly motivated by the following biological considerations: the alleles (or species) can be maintained if and only if $\lambda > \lambda_1(m)$, and the smaller $\lambda_1(m)$ is, the more likely that both alleles can be maintained (or the species can exist, respectively). Biologically, for the genetic model, finding such a minimizing function $m(x)$ is equivalent to finding the spatial pattern in which it is easiest to maintain the less favored allele A_1 [19]; for the logistic model, it is the same as determining the optimal spatial arrangement of the favorable and unfavorable parts of the habitat for species to survive [3, 4]. This question was first addressed by Cantrell and Cosner in [3, 4] from the ecological point of view, and it remains largely open.

The question is also mathematically meaningful. Indeed, by (A1) and Theorem A, the positive principal eigenvalue $\lambda_1(m)$ is uniquely determined. Moreover, the infimum of $\lambda_1(m)$ among all those m that satisfy both (A1) and (A2) is positive. In order to see this, we recall the following theorem of Saut and Scheurer [21]:

Theorem B. *Suppose that* (A1) *holds. Then*

$$\lambda_1(m) \geq \frac{\nu_1 \left| \int_\Omega m \right|}{\int_\Omega m^2(x)\, dx + \left| \int_\Omega m \right| \sup_\Omega m},$$

where ν_1 is the smallest positive eigenvalue of the Laplace operator with homogeneous Neumann boundary condition.

As an immediate consequence of this theorem, we have

$$\lambda_1(m) \geq \frac{\nu_1 m_0}{\max(\,\overline{m}^2, \underline{m}^2) + \overline{m} \max(\,\overline{m}, \underline{m}\,)} > 0, \tag{1.4}$$

which gives a uniform positive lower bound of $\lambda_1(m)$, i.e., a bound that depends only on $\underline{m}$, $\overline{m}$, m_0, and Ω.

By the scaling $m_0 \to \mu$, $\underline{m} \to 1$, $\overline{m} \to \kappa$, $m \to m/\underline{m}$, and $\lambda \to \lambda \underline{m}$, for the sake of simplicity, we may assume that $m_0 = \mu$, $\underline{m} = 1$, and $\overline{m} = \kappa$, where $\mu < 1$ and $\kappa > 0$ are constants. With this scaling, (A2) becomes

(A2)′ $\quad -1 \leq m(x) \leq \kappa$ a.e. in Ω, and $\int_\Omega m \leq -\mu|\Omega|$.

Given $\mu < 1$ and κ, we define

$$\mathcal{M} = \big\{ m \in L^\infty(\Omega) \colon m(x) \text{ satisfies (A1) and (A2)}' \big\}, \tag{1.5}$$

and set

$$\lambda_{\inf} := \inf_{m \in \mathcal{M}} \lambda_1(m).$$

By (1.4), we see that $\lambda_{\inf} > 0$.

The existence and profile of global minimizers of $\lambda_1(m)$ in $\mathcal{M}$ with Dirichlet boundary condition was first addressed by Cantrell and Cosner in [3]. Among other things, Cantrell and Cosner showed that there exists some measurable set $E \subset \Omega$ with $|E| > 0$ such that $\lambda_1(\kappa\chi_E - \chi_{\Omega\setminus E}) = \lambda_{\inf}$. The result of Cantrell and Cosner can be viewed as saying that there exists a “bang-bang” type optimal control for minimizing $\lambda_1(m)$ in $\mathcal{M}$. For Neumann boundary condition, we show that a stronger assertion holds true:

THEOREM 1.1. *The infimum $\lambda_{\inf}$ is attained by some $m \in \mathcal{M}$. Moreover, if $\lambda_1(m) = \lambda_{\inf}$, then m can be represented as $m(x) = \kappa\chi_E - \chi_{\Omega\setminus E}$ a.e. in Ω for some measurable set $E \subset \Omega$.*

In particular, Theorem 1.1 implies that the global minimizers of $\lambda_1(m)$ in $\mathcal{M}$ must be of "bang-bang" type, and in fact must be contained in the set

$$\mathcal{M}_\alpha = \{m\colon m = \kappa\chi_E - \chi_{\Omega\setminus E} \text{ for some } E \subset \Omega \text{ with } |E| = \alpha|\Omega|\},$$

where

$$\alpha = \frac{1-\mu}{1+\kappa}. \tag{1.6}$$

It is easy to check that every $m \in \mathcal{M}_\alpha$ satisfies $\int_\Omega m = -\mu|\Omega|$, i.e., $\mathcal{M}_\alpha \subset \mathcal{M}$.

By Theorem 1.1 and the above discussion, in order to determine all of the global minimizers of $\lambda_1(m)$ in $\mathcal{M}$, it suffices to characterize $E \subset \Omega$ such that the corresponding weight function $m(x) = \kappa\chi_E - \chi_{\Omega\setminus E}$ minimizes the principal eigenvalue $\lambda_1(m)$ in $\mathcal{M}_\alpha$. The main goal of this paper is to utilize this idea to give a complete characterization of all global minimizers of $\lambda_1(m)$ in $\mathcal{M}$ when $N = 1$ and Ω is an interval. In this connection, we have

THEOREM 1.2. *Suppose that $N = 1$, $\Omega = (0,1)$, and α is as defined in (1.6). Then $\lambda_1(m) = \lambda_{\inf}$ for some function $m \in \mathcal{M}$ if and only if $m = \kappa\chi_E - \chi_{\Omega\setminus E}$ a.e. in $(0,1)$, where either $|E \cap (0,\alpha)| = \alpha$ or $|E \cap (1-\alpha,1)| = \alpha$.*

Theorem 1.2 implies that when Ω is an interval, then there are exactly two global minimizers of $\lambda_1(m)$ (up to change of a set of measure zero). This substantially improves previous work in the one-dimensional case. In [4] Cantrell and Cosner studied the case when Ω is the unit interval $(0,1)$ under three different boundary conditions (Dirichlet, Neumann, and Robin type) in a rather restricted situation. More precisely, for Neumann boundary condition, they showed that if $m(x) \equiv \kappa$ on a "single" subinterval of fixed length and $m(x) = -1$ on the remainder of the interval, then the smallest value of $\lambda_1(m)$ with $m(x)$ so restricted occurs when the subinterval where $m(x) \equiv \kappa$ is at one of the ends of the interval $(0,1)$; they also considered the situation when $m(x) \equiv -1$ in a "single" subinterval of fixed length, and $m(x) \equiv \kappa$ on the remainder of $(0,1)$. They proved that in the latter case, the smallest value for $\lambda_1(m)$ occurs when $m(x) \equiv -1$ at one of the ends of $(0,1)$. However, the method of [4] cannot be applied to a more general case where $m(x)$ has either arbitrarily many or infinitely many positive and negative intervals. Therefore, in order to study such general case, we need to develop some new ideas based on a characterization of critical points and continuous dependence of $\lambda_1(m)$ with respect to m. We note that our analysis can also be useful in handling Dirichlet, Robin, and periodic boundary conditions.

Theorem 1.2 can be viewed as filling up the gap between the result of Cantrell and Cosner and Theorem 1.1. For the genetic model, Theorem 1.2 suggests that if m is fragmented, then it is difficult to maintain the genetic variability; for the logistic model, this means that a single favorable region at one of the two ends of the whole habitat provides the best opportunity for the species to survive.

As an application of Theorem 1.2, we have

Corollary 1.3. *Suppose that* $N = 1$ *and* $\Omega = (0, 1)$. *Let* λ_α *denote the unique, positive principal eigenvalue for the weight function* $m = \kappa\chi_{(0,\alpha)} - \chi_{(\alpha,1)}$. *If* $\lambda \le \lambda_\alpha$, *then for any* $m \in \mathcal{M}$, *all solutions of* (1.2) *and* (1.3) *satisfy* $u(x, t) \to 0$ *uniformly for* $x \in \overline{\Omega}$ *as* $t \to \infty$.

Biologically, Corollary 1.3 implies that for a one-dimensional habitat, if the deleterious allele or species can not be maintained in the least fragmented step-environment, it will be eliminated in any other habitat with the same total amount of selective forces or resources.

Remark 1.4. The characterization of the optimal set E in $\mathcal{M}_\alpha$ for higher-dimensional domains is an open problem. We conjecture that if Ω is convex, then both E and Ω/E are connected (up to a set of measure zero), and $\partial E \cap \partial\Omega$ has positive measure.

This paper is organized as follows. In Section 2, we establish some fundamental properties of the eigenvalue problem (1.1) for general and bang-bang weight functions. We also show that the global minimizer must be of bang-bang type, i.e., Theorem 1.1 holds. In Section 3, we study the case where Ω is the unit interval, and give a proof of Theorem 1.2.

2. Properties of the Eigenvalue Problem

Subsection 2.1 is devoted to the proof of Theorem 1.1, which states that the global minimizers of $\lambda_1(m)$ in $\mathcal{M}$ must be of bang-bang type. In Subsection 2.2, we establish the comparison principle and continuity of positive principal eigenvalues. Finally, in Subsection 2.3 we study properties of the eigenvalue problem (1.1) with bang-bang type weight functions $m(x)$.

2.1. Bang-bang property of global minimizers

We first give the following well-known variational characterization of the positive principal eigenvalue [1, 2, 13, 23]:

Lemma 2.1. *Suppose that* (A1) *holds. Then the positive principal eigenvalue* $\lambda_1(m)$ *of* (1.1) *is given by*

$$\lambda_1(m) = \inf_{\varphi \in \mathcal{S}(m)} \frac{\int_\Omega |\nabla\varphi|^2}{\int_\Omega m(x)\varphi^2}, \tag{2.1}$$

where

$$\mathcal{S}(m) := \left\{ \varphi \in H^1(\Omega) \colon \int_\Omega m(x)\varphi^2 > 0 \right\}.$$

Moreover, $\lambda_1(m)$ *is simple, and the infimum is attained only by associated eigenfunctions that do not change sign in* $\overline{\Omega}$*.*

Next, we prove that $\lambda_{\inf}$ is attained.

LEMMA 2.2. *There exists* $m \in \mathcal{M}$ *such that* $\lambda_1(m) = \lambda_{\inf}$.

Proof. By the definition of $\lambda_{\inf}$, there exists a sequence $\{m_n\}_{n=1}^\infty$ in $\mathcal{M}$ such that $\lambda_1(m_n) \to \lambda_{\inf} > 0$ as $n \to \infty$. Let $\varphi_n > 0$ be the corresponding eigenfunction of $\lambda_1(m_n)$, uniquely determined by $\sup_\Omega \varphi_n = 1$. By standard elliptic regularity and the Sobolev embedding theorem [11], passing to a subsequence if necessary, we may assume that $\varphi_n \to \varphi_\infty$ in $W^{2,q}(\Omega) \cap C^{1,\gamma}(\overline{\Omega})$ for every $q > 1$ and every $\gamma \in (0,1)$, where $\sup_\Omega \varphi_\infty = 1$ and $\varphi_\infty \geq 0$ in Ω. Since $\|m_n\|_{L^2(\Omega)}$ is uniformly bounded, passing to a subsequence again if necessary, we may assume that $m_n(x) \to m(x)$ weakly in $L^2(\Omega)$ for some function $m \in L^2(\Omega)$, i.e., $\int_\Omega m_n\psi \to \int_\Omega m\psi$ for every $\psi \in L^2(\Omega)$ as $n \to \infty$. Choose $\psi \equiv 1$. This together with $\int_\Omega m_n \leq -\mu|\Omega|$ implies that $\int_\Omega m \leq -\mu|\Omega|$. For every non-negative L^2 function ψ, we have $-\int_\Omega \psi \leq \int_\Omega m_n\psi \leq \kappa\int_\Omega \psi$. Passing to the limit, we find that $-\int_\Omega \psi \leq \int_\Omega m\psi \leq \kappa\int_\Omega \psi$. This implies that $-1 \leq m \leq \kappa$ a.e. in Ω.

To have $m \in \mathcal{M}$, it remains to show that $m(x) > 0$ in a set of positive measure. Note that φ_∞ is a weak solution of

$$\Delta\varphi + \lambda_{\inf} m\varphi = 0 \quad \text{in} \quad \Omega, \quad \partial\varphi/\partial n = 0 \quad \text{on} \quad \partial\Omega.$$

If $m(x) \leq 0$ a.e. in Ω, then φ_∞ satisfies

$$\int_\Omega |\nabla\varphi_\infty|^2 = \lambda_{\inf}\int_\Omega m\varphi_\infty^2 \leq 0,$$

i.e., $\nabla\varphi_\infty = 0$ in Ω. Since $\sup_\Omega \varphi_\infty = 1$, we have $\varphi_\infty = 1$, which implies that $m = 0$ a.e. in Ω. However, this contradicts $\int_\Omega m \leq -\mu|\Omega| < 0$. Therefore, $m \in \mathcal{M}$. By the equation of φ_∞, we see that $\lambda_1(m) = \lambda_{\inf}$. This shows that $\lambda_{\inf}$ is attained. □

To complete the proof of Theorem 1.1, we extend some idea from [4].

Proof of Theorem 1.1. Suppose that $m \in \mathcal{M}$ and $\lambda_1(m) = \lambda_{\inf}$. Let $\varphi > 0$ with $\sup_\Omega \varphi = 1$ be the corresponding eigenfunction of $\lambda_1(m)$. By elliptic regularity, we have $\varphi > 0$ in $\overline{\Omega}$. For every $\eta > 0$, define

$$E_\eta = \{x \in \Omega \colon \varphi(x) > \eta\}.$$

Since $\mu < 1$, there exists some $\eta^* > 0$ such that $|E_{\eta^*}| > 0$ and

$$-\mu|\Omega| = \kappa|E_{\eta^*}| - |\Omega \setminus E_{\eta^*}|. \tag{2.2}$$

Define $m^*(x) = \kappa\chi_{E_{\eta^*}} - \chi_{\Omega\setminus E_{\eta^*}}$. Equation (2.2) ensures that $\int_\Omega m^* = -\mu|\Omega|$. Hence, we have $m^* \in \mathcal{M}$, which implies that $\lambda_{\inf} \le \lambda_1(m^*)$.

We claim that $m(x) = m^*(x)$ a.e. in Ω. To establish our assertion, we first have

$$\begin{aligned}\int_\Omega (m^* - m)\varphi^2 &= \int_{E_{\eta^*}} (\kappa - m)\varphi^2 - \int_{\Omega\setminus E_{\eta^*}} (1+m)\varphi^2 \\ &\ge (\eta^*)^2 \int_{E_{\eta^*}} (\kappa - m) - (\eta^*)^2 \int_{\Omega\setminus E_{\eta^*}} (1+m) \\ &\ge 0,\end{aligned} \tag{2.3}$$

where the last inequality follows from (2.2) and $\int_\Omega m \le -\mu|\Omega|$. Since $\int_\Omega m\varphi^2 > 0$, we have $\int_\Omega m^*\varphi^2 > 0$. Hence, $\varphi \in \mathcal{S}(m^*)$. Therefore, applying (2.1) we have

$$\lambda_1(m^*) \le \frac{\int_\Omega |\nabla\varphi|^2}{\int_\Omega m^*\varphi^2} \le \frac{\int_\Omega |\nabla\varphi|^2}{\int_\Omega m\varphi^2} = \lambda_1(m). \tag{2.4}$$

Since $\lambda_1(m) = \lambda_{\inf} \le \lambda_1(m^*)$, equalities must hold in (2.4). In particular, $\lambda_1(m^*) = \lambda_1(m)$ and

$$\lambda_1(m^*) = \frac{\int_\Omega |\nabla\varphi|^2}{\int_\Omega m^*\varphi^2}.$$

Therefore, from Lemma 2.1 we see that φ is also an eigenfunction of $\lambda_1(m^*)$, and it satisfies

$$\Delta\varphi + \lambda_1(m^*)m^*\varphi = 0 \quad \text{in} \quad \Omega, \quad \partial\varphi/\partial n = 0 \quad \text{on} \quad \partial\Omega$$

weakly in $H^1(\Omega)$ and strongly in $W^{2,q}(\Omega)$ for every $q > 1$. Since $\varphi > 0$ in $\overline{\Omega}$, we have

$$m^* = -\frac{\Delta\varphi}{\lambda_1(m^*)\varphi} = -\frac{\Delta\varphi}{\lambda_1(m)\varphi} = m$$

a.e. in Ω. This completes the proof of Theorem 1.1.

2.2. Comparison and continuity of principal eigenvalues

For later purposes, we establish two lemmas in this subsection, one of which is a comparison principle for positive principal eigenvalues of (1.1), and the other is concerned with the continuous dependence of $\lambda_1(m)$ on the weight function $m(x)$ in suitable topology. Since we can not locate their proofs of the generality needed in this paper, for the sake of completeness we include both of the proofs here.

LEMMA 2.3. *Suppose that $m_*, m^* \in L^\infty(\Omega)$, $m_* \le m^*$ a.e. in Ω, $\int_\Omega m^* < 0$, and $m_* > 0$ in a set of positive measure. Then $\lambda_1(m_*) \ge \lambda_1(m^*)$. If we further assume that $m_* < m^*$ in a set of positive measure, then $\lambda_1(m_*) > \lambda_1(m^*)$.*

Proof. Let φ_* be a positive eigenfunction associated with $\lambda_1(m_*)$. Since $\int_\Omega m_*\varphi_*^2 > 0$, we have $\int_\Omega m^*\varphi_*^2 \geq \int_\Omega m_*\varphi_*^2 > 0$. Hence $\varphi_* \in \mathcal{S}(m^*)$. Therefore, by Lemma 2.1 we have

$$\lambda_1(m^*) \leq \frac{\int_\Omega |\nabla\varphi_*^2|}{\int_\Omega m^*\varphi_*^2} \leq \frac{\int_\Omega |\nabla\varphi_*^2|}{\int_\Omega m_*\varphi_*^2} = \lambda_1(m_*). \tag{2.5}$$

If $m^* > m_*$ in a set of positive measure, then from the fact that $\varphi_* > 0$ in $\overline{\Omega}$, we have $\int_\Omega m^*\varphi_*^2 > \int_\Omega m_*\varphi_*^2$. Hence, $\lambda_1(m_*) > \lambda_1(m^*)$. □

LEMMA 2.4. *Suppose that* $\{m_k\}_{k=1}^\infty$, m *satisfy* (A1), *and* $\|m_k\|_{L^\infty(\Omega)} \leq C$ *for some constant* C *independent of* k. *If* $\|m_k - m\|_{L^1(\Omega)} \to 0$ *as* $k \to \infty$, *then* $\lambda_1(m_k) \to \lambda_1(m)$ *as* $k \to \infty$.

Proof. We first claim that there exists some $\delta > 0$ such that $\delta \leq \lambda_1(m_k) \leq 1/\delta$ for sufficiently large k. Since $\|m_k - m\|_{L^1(\Omega)} \to 0$ and $\|m_k\|_{L^\infty(\Omega)} \leq C$, we see that $\|m_k - m\|_{L^q(\Omega)} \to 0$ for every $q \geq 1$ as $k \to \infty$. To find a uniform lower bound of $\lambda_1(m_k)$, it suffices to apply Theorem B to get

$$\lambda_1(m_k) \geq \frac{\nu_1|\int_\Omega m_k|}{\int_\Omega m_k^2 + C|\int_\Omega m_k|} \to \frac{\nu_1|\int_\Omega m|}{\int_\Omega m^2 + C|\int_\Omega m|} > 0.$$

For the uniform upper bound of $\lambda_1(m_k)$, since $m > 0$ in a set of positive measure, there exists $\psi \in L^1(\Omega)$ such that $\psi \geq 0$ a.e. and $\int_\Omega m\psi > 0$. Then the proof of Theorem 3.1 in [3] (the necessity part, pp. 305–306) applies, and it yields that $\lambda_1(m_k) \leq 1/\delta$ for some $\delta > 0$ and all sufficiently large k.

Passing to a subsequence if necessary, we may assume that $\lambda_1(m_k) \to \lambda_*$ for some $\lambda_* > 0$. It remains to show that $\lambda_* = \lambda_1(m)$. To this end, let $\psi_k > 0$ with $\sup_\Omega \psi_k = 1$ be the corresponding eigenfunction of $\lambda_1(m_k)$. By elliptic regularity and the Sobolev embedding theorem, passing to a subsequence if necessary, we may assume that $\psi_k \to \psi$ in $W^{2,q}(\Omega)$ for every $q > 1$, $\psi \geq 0$, $\sup_\Omega \psi = 1$, and ψ is a weak solution of $\Delta\psi + \lambda_* m\psi = 0$ in Ω and $\partial\psi/\partial n = 0$ on $\partial\Omega$. Hence, λ_* is the positive principal eigenvalue corresponding to the weight function $m(x)$. By Theorem A, we have $\lambda_* = \lambda_1(m)$. Note that λ_* is independent of the subsequence chosen. This shows that $\lambda_1(m_k) \to \lambda_1(m)$ for the whole sequence $\{m_k\}$. □

2.3. Eigenvalue problems of bang-bang type

In this subsection, we consider properties of the eigenvalue problem (1.1) in the case where m is of bang-bang type.

Let E be a measurable subset of Ω satisfying $|E| < |\Omega|/(\kappa + 1)$, and set

$$m_E(x) := \begin{cases} \kappa & \text{if} \quad x \in E, \\ -1 & \text{if} \quad x \notin E. \end{cases}$$

Then we consider the eigenvalue problem

$$\begin{cases} \Delta\varphi + \lambda m_E(x)\varphi = 0 & \text{in} \quad \Omega, \\ \dfrac{\partial\varphi}{\partial n} = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{2.6}$$

For simplicity, we denote by $\lambda_E := \lambda_1(m_E)$ the positive principal eigenvalue of (2.6), and by $\phi_E(x)$ the positive eigenfunction associated with λ_E.

As corollaries of Lemmas 2.3 and 2.4, we have the following two lemmas.

Lemma 2.5. *Let E_1 and E_2 be measurable subsets of Ω with $|E_1|, |E_2| < |\Omega|/(\kappa+1)$. If $E_1 \subset E_2$, then $\lambda_{E_1} \geq \lambda_{E_2}$. If we further assume $|E_1| < |E_2|$, then $\lambda_{E_1} > \lambda_{E_2}$.*

Lemma 2.6. *If $\{E_k\}_{k=1}^{\infty}$ and E satisfy $|E_k \setminus E| \to 0$, $|E \setminus E_k| \to 0$ as $k \to \infty$, and $|E| < |\Omega|/(\kappa+1)$, then $\lambda_{E_k} \to \lambda_E$ as $k \to \infty$.*

We say that E is *admissible* if $\phi_E(x) \geq \phi_E(y)$ for $x \in E$ and $y \in \Omega \setminus E$ a.e. We show that the minimum of λ_E can be attained only by admissible E.

Lemma 2.7. *Suppose that E is not admissible. Then there exists a measurable set $E_0 \subset \Omega$ such that $|E| = |E_0|$ and $\lambda_{E_0} < \lambda_E$.*

Proof. By assumption, there exist two measurable sets $D_1 \subset E$ and $D_2 \subset \Omega \setminus E$ with $|D_1| = |D_2|$ such that $\phi_E(x) \leq \phi_E(y)$ for $x \in D_1$ and $y \in \Omega \setminus D_2$ a.e. Define $E_0 = (E \setminus D_1) \cup D_2$. Then we have $|E| = |E_0|$. Since

$$\int_\Omega m_{E_0}\phi_E^2 - \int_\Omega m_E \phi_E^2 = (\kappa+1)\left(\int_{D_2} \phi_E^2 - \int_{D_1} \phi_E^2\right) > 0$$

and $\int_\Omega m_E \phi_E^2 > 0$, we have $\int_\Omega m_{E_0}\phi_E^2 > 0$, i.e., $\phi_E \in \mathcal{S}(m_{E_0})$. Hence, by Lemma 2.1 we obtain

$$\lambda_E = \frac{\int_\Omega |\nabla\phi_E|^2}{\int_\Omega m_E(x)\phi_E^2} > \frac{\int_\Omega |\nabla\phi_E|^2}{\int_\Omega m_{E_0}(x)\phi_E^2} \geq \lambda_{E_0}.$$

This completes the proof. □

3. One Dimensional Case

In this section we restrict our attention to the case $N = 1$. Without loss of generality, we may assume that Ω is the unit interval $(0,1)$. Thus, the eigenvalue problem (2.6) is reduced to

$$\begin{cases} \phi'' + \lambda m_E(x)\phi = 0 & \text{in} \quad (0,1), \\ \phi'(0) = 0 = \phi'(1). \end{cases} \tag{3.1}$$

Let α be a constant satisfying $0 < \alpha < 1/(\kappa+1)$, and define

$$S_\alpha = \{E \subset (0,1) \colon |E| = \alpha\}.$$

For every $E \in S_\alpha$, m_E satisfies (A1). Hence, (3.1) has a unique positive principal eigenvalue λ_E for every $E \in S_\alpha$. The goal of this section is to show that if λ_E attains the global minimum at some $E^* \in S_\alpha$, then either $|E^* \cap (0,\alpha)| = \alpha$ or $|E^* \cap (1-\alpha, 1)| = \alpha$.

3.1. Preliminaries

We define subsets of S_α by

$$S_\alpha^k := \{E \in S_\alpha \colon E \text{ consists of } k \text{ disjoint open intervals}\},$$

$k = 1, 2, \dots,$ and

$$S_\alpha^\infty := \bigcup_{k=1}^{\infty} S_\alpha^k.$$

Note that S_α^k may contain an empty interval (a,a) and two intervals tangent to each other; (a,b) and (b,c). By definition, $E \in S_\alpha^k$ is admissible if $\phi_E(x) \geq \phi_E(y)$ for any $x \in E$ and $y \in (0,1) \setminus \overline{E}$.

The following lemma is obtained immediately from definition of the admissibility.

Lemma 3.1. *The set $E \in S_\alpha^k$ is admissible if and only if $\phi_E(x)$ takes the same value at all points on $\partial\overline{E} \cap (0,1)$.*

Proof. Since ϕ_E is continuous, E is not admissible if $\phi_E(x)$ takes different values on $\partial\overline{E} \cap (0,1)$. Since $\phi_E(x)$ is convex on E and concave on $(0,1) \setminus E$, the proof is complete. $\square$

The next lemma will be needed later.

Lemma 3.2. *Let $E \in S_\alpha^k$. Suppose that $\phi_E'(c) = 0$ for some $c \in (0,1)$. Then there exists $E_0 \in S_\alpha^k$ such that*

$$\lambda_{E_0} \leq \max\{c^2, (1-c)^2\}\lambda_E.$$

Proof. We see from assumption $|E| = \alpha$ that either

$$|E \cap (0,c)| \leq c\alpha \quad \text{or} \quad |E \cap (c,1)| \leq (1-c)\alpha \tag{3.2}$$

holds. Suppose that the former is the case. Then the set

$$E_c = \{x \in (0,1) \colon cx \in E\}$$

is measurable and satisfies $|E_c| \leq \alpha$. Setting

$$\psi(x) := \phi_E(cx), \quad x \in [0,1],$$

we have

$$\begin{cases} \psi''(x) + c^2\lambda_E m_{E_c}(x)\psi(x) = 0 \quad \text{in} \quad (0,1), \\ \psi'(0) = 0 = \psi'(1). \end{cases}$$

This implies that $\lambda_{E_c} = c^2\lambda_E$ is the positive principal eigenvalue for m_{E_c}. Now we take $E_0 \in S_\alpha^k$ such that $E_0 \supset E_c$ and $|E_0| = \alpha$. Then by Lemma 2.5, we obtain $\lambda_{E_0} \le \lambda_{E_c} = c^2\lambda_E$.

Similarly, if the latter is the case in (3.2), then there exists $E_0 \in S_\alpha$ such that $\lambda_{E_0} \le (1-c)^2\lambda_E$. Thus, the proof is complete. $\square$

3.2. A single interval

In this subsection we consider the case where $E \in S_\alpha^1$, i.e., E is an open interval $(s, s+\alpha) \subset (0,1)$ for some $s \in [0, 1-\alpha]$.

Lemma 3.3. *The set $E \in S_\alpha^1$ is admissible if and only if E is one of* $(0,\alpha)$, $((1-\alpha)/2, (1+\alpha)/2)$, *and* $(1-\alpha, 1)$.

Proof. We first consider the case $E = (0,\alpha)$. Since $\phi_E''(x) < 0$ for $x \in (0,\alpha)$ and $\phi_E'(0) = 0$, we have $\phi_E'(x) < 0$ for $x \in (0,\alpha)$. Also, since $\phi_E''(x) > 0$ for $x \in (\alpha, 1)$ and $\phi_E'(1) = 0$, we have $\phi_E'(x) < 0$ for $x \in (\alpha,1)$. Therefore, $\phi_E(x)$ is monotone decreasing in $(0,1)$. This implies that E is admissible.

Similarly, if $E = (1-\alpha, 1)$, then $\phi_E(x)$ is monotone increasing in $(0,1)$ so that E is admissible.

Finally, assume that $E = (a,b)$ with $0 < a < b < 1$. If E is admissible, then $\phi_E(a) = \phi_E(b)$ by Lemma 3.1. This implies that $\phi_E(x)$ is symmetric with respect to $x = (a+b)/2$. Since $\phi_E''(x) > 0$ for $x \in (0,1) \setminus (a,b)$, the boundary conditions $\phi'(0) = 0 = \phi'(1)$ are satisfied only if $\phi_E'(x)$ is symmetric with respect to $x = 1/2$. Hence, $E = ((1-\alpha)/2, (1+\alpha)/2)$. Conversely, if $E = ((1-\alpha)/2, (1+\alpha)/2)$, then $\phi_E(x)$ is symmetric with respect to $x = 1/2$, monotone increasing in $(0, 1/2)$, and monotone decreasing in $(1/2, 1)$. Hence, E is admissible in this case.

Let us compute the positive principal eigenvalue for $E = (0,\alpha)$. In this case, the eigenvalue problem is written as

$$\begin{cases} \phi'' + \lambda\kappa\phi = 0 & \text{in} \quad (0,\alpha), \\ \phi'' - \lambda\phi = 0 & \text{in} \quad (\alpha, 1), \\ \phi'(0) = 0 = \phi'(1). \end{cases}$$

Assuming $\lambda > 0$ and $\phi > 0$, we may put

$$\phi(x) = \begin{cases} C_0 \cos\sqrt{\lambda\kappa}\, x, & x \in (0,\alpha), \\ C_1 \cosh\sqrt{\lambda}(1-x), & x \in (\alpha, 1), \end{cases}$$

where C_0, C_1 are positive constants and $0 < \lambda < (\pi/2\alpha)^2/\kappa$. In order to match these expressions at $x = \alpha$, the following condition must be satisfied:

$$\begin{aligned} &\det\begin{bmatrix} \phi(\alpha-0) & \phi(\alpha+0) \\ \phi'(\alpha-0) & \phi'(\alpha+0) \end{bmatrix} \\ &= \det\begin{bmatrix} \cos\sqrt{\lambda\kappa}\,\alpha & \cosh\sqrt{\lambda}(1-\alpha) \\ -\sqrt{\lambda\kappa}\sin\sqrt{\lambda\kappa}\,\alpha & -\sqrt{\lambda}\sinh\sqrt{\lambda}(1-\alpha) \end{bmatrix} \end{aligned}$$

$$= -\cos\sqrt{\lambda\kappa}\,\alpha \cdot \sqrt{\lambda}\sinh\sqrt{\lambda}(1-\alpha) + \cosh\sqrt{\lambda}(1-\alpha)\cdot\sqrt{\lambda\kappa}\sin\sqrt{\lambda\kappa}\,\alpha$$
$$= 0.$$

Thus, we obtain the characteristic equation

$$\sqrt{\kappa}\tan\sqrt{\lambda\kappa}\,\alpha = \tanh\sqrt{\lambda}(1-\alpha). \tag{3.3}$$

For every $\alpha \in (0, (\kappa+1)^{-1})$, there exists a unique solution of the characteristic equation satisfying $0 < \lambda < (\pi/2\alpha)^2/\kappa$. If we denote the unique solution by λ_α, then the positive principal eigenvalue is obtained as $\lambda_E = \lambda_\alpha$.

Similarly, we have the positive principal eigenvalue $\lambda_E = \lambda_\alpha$ for $E = (1-\alpha, 1)$.

Finally, the positive principal eigenvalue for $E = ((1-\alpha)/2, (1+\alpha)/2)$ is computed as $\lambda_E = 4\lambda_\alpha$ by solving

$$\begin{cases} \phi'' - \lambda\kappa\phi = 0 & \text{in} \quad (0, (1-\alpha)/2), \\ \phi'' + \lambda\phi = 0 & \text{in} \quad ((1-\alpha)/2, 1/2), \\ \phi'(0) = 0 = \phi'(1/2). \end{cases}$$

Thus, the following result is obtained.

Proposition 3.4. *Let λ_α be the smallest positive solution for* (3.3). *Then the minimum of λ_E in S^1_α is given by λ_α, and is attained only by $E = (0, \alpha)$ and $E = (1-\alpha, 1)$.*

Proof. Since λ_E for $E = (s, s+\alpha) \in S^1_\alpha$ is continuous in $s \in [0, 1-\alpha]$, λ_E takes the minimum in S^1_α at some $s \in [0, 1-\alpha]$. By Lemma 2.7, the minimum is attained only by admissible E. Then E is either $(0, \alpha)$ or $(1-\alpha, 0)$, and the minimal value is given by λ_α. □

3.3. A finite number of intervals

Next, let us consider the case where $E \in S^k_\alpha$ for some $k \geq 2$. Let x_i, y_i, $i = 1, 2, \ldots, k$, be variables satisfying

$$0 \leq x_1 \leq y_1 \leq x_2 \leq y_2 \leq \cdots \leq x_k \leq y_k \leq 1 \tag{3.4}$$

and

$$\sum_{i=1}^{k}(y_i - x_i) = \alpha. \tag{3.5}$$

For simplicity, we use the notation $\mathbf{x} = (x_1, y_1, \ldots, x_k, y_k) \in \mathbb{R}^{2k}$ and denote by $X^k_\alpha \subset \mathbb{R}^{2k}$ the set defined by

$$X^k_\alpha := \{\mathbf{x} \in \mathbb{R}^{2k} \colon x_1, y_1, \ldots, x_k, y_k \text{ satisfy (3.4) and (3.5)}\}.$$

For $\mathbf{x} \in X^k_\alpha$, we define $E = E(\mathbf{x}) \in S^k_\alpha$ by

$$E(\mathbf{x}) = \bigcup_{i=1}^{k}(x_i, y_i).$$

We note that (x_i, y_i) is regarded as an empty set if $x_i = y_i$.

Let $\lambda_E(\mathbf{x})$ denote the positive principal eigenvalue of (3.1) for $E(\mathbf{x}) \in S_\alpha^k$. Since $\lambda_E(\mathbf{x})$ is continuous in $\mathbf{x}$ and X_α^k is compact in $\mathbb{R}^{2k}$, $\lambda_E(\mathbf{x})$ takes the minimum at some point in X_α^k. By Lemma 2.7, the minimum is attained by some admissible E. Let $\mathbf{x}_{\min}$ denote a minimal point of $\lambda_E(\mathbf{x})$. For $\mathbf{x} = \mathbf{x}_{\min}$, if $x_i = y_i$ for some i, then we may remove the interval (x_i, y_i), and if $y_i = x_{i+1}$ for some i, we can glue together two intervals (x_i, y_i) and (x_{i+1}, y_{i+1}) to create an interval (x_i, y_{i+1}). Hence, we may assume without loss of generality that $E(\mathbf{x}_{\min})$ consists of l $(\leq k)$ nonempty open intervals (x_i, y_i) with

$$0 \leq x_1 < y_1 < x_2 < y_2 < \cdots < x_l < y_l \leq 1.$$

Since $S_\alpha^l \subset S_\alpha^k$, the minimum of λ_E in S_α^l is attained by $E(\mathbf{x}_{\min})$.

Consequently, the following result is obtained.

Proposition 3.5. *The minimum of λ_E in S_α^k is attained only by $E \in S_\alpha^k$ such that $\overline{E} = [0, \alpha]$ or $\overline{E} = [1-\alpha, 1]$.*

Proof. The minimum is attained only by admissible E. Suppose that $\overline{E}$ is not connected. Then by Lemma 3.1, there exist y_i and x_{i+1} such that

$$\phi_E(y_i) = \phi_E(x_{i+1}), \quad 0 < y_i < x_{i+1} < 1,$$

and hence ϕ_E' vanishes at some $x \in (y_i, x_{i+1})$. Then by Lemma 3.2, E is not a minimal point. Hence, $\overline{E}$ must be connected. Then by Proposition 3.4, the proof is complete.

3.4. The general case

In this subsection, we consider the general case where $E \in S_\alpha$ is barely measurable and may consist of infinitely many open intervals. Our idea to deal with this case is to approximate $E \in S_\alpha$ by some $E_\varepsilon \in S_\alpha^\infty$ and use the continuity of the principal eigenvalue. If λ_{E_ε} is considerably larger than λ_α for any $E \neq (0, \alpha)$, $(1-\alpha, 1)$, then the continuity implies that λ_E is larger than λ_α.

We first approximate $E \in S_\alpha$ by some $E_\varepsilon \in S_\alpha^\infty$.

Lemma 3.6. *Let $0 < a \leq b < 1$ be fixed. Assume $E \in S_\alpha$ and $|E \cap (0, a)| = \alpha_1$, $|E \cap (a, b)| = \alpha_2$, $|E \cap (b, 1)| = \alpha_3$, where $\alpha_1, \alpha_2, \alpha_3$ are non-negative constants satisfying $\alpha_1 + \alpha_2 + \alpha_3 = \alpha$. Then for any $\varepsilon > 0$, there exists $E_\varepsilon \in S_\alpha^\infty$ such that*

$$|E \setminus E_\varepsilon| < \varepsilon, \quad |E_\varepsilon \cap (0, a)| = \alpha_1, \quad |E_\varepsilon \cap (a, b)| = \alpha_2, \quad |E_\varepsilon \cap (b, 1)| = \alpha_3.$$

Proof. It is standard (see, e.g., [10]) that for any $\varepsilon > 0$, there exists $E_0 \in S_\alpha^\infty$ such that

$$|E_0 \setminus E| < \frac{\varepsilon}{4}, \quad |E \setminus E_0| < \frac{\varepsilon}{4}.$$

Then we have

$$\alpha_1 - \frac{\varepsilon}{4} < |E_0 \cap (0, a)| < \alpha_1 + \frac{\varepsilon}{4},$$

$$\alpha_2 - \frac{\varepsilon}{4} < |E_0 \cap (a,b)| < \alpha_2 + \frac{\varepsilon}{4},$$
$$\alpha_3 - \frac{\varepsilon}{4} < |E_0 \cap (b,1)| < \alpha_3 + \frac{\varepsilon}{4}.$$

Hence, by adding and removing appropriate intervals, we can find $E_\varepsilon \in S_\alpha^\infty$ such that

$$|E_0 \setminus E_\varepsilon| < \frac{3\varepsilon}{4}, \quad |E_\varepsilon \cap (0,a)| = \alpha_1, \quad |E_\varepsilon \cap (a,b)| = \alpha_2, \quad |E_\varepsilon \cap (b,1)| = \alpha_3.$$

Since

$$|E \setminus E_\varepsilon| \leq |E \setminus E_0| + |E_0 \setminus E_\varepsilon| < \varepsilon,$$

the proof is complete. □

Let E_ε be as in the above lemma. We may assume without losing generality that $\alpha \notin E_\varepsilon$. Indeed, if there is an interval (x_i, y_i) of E_ε containing α, then we may divide it into two intervals (x_i, α) and (α, y_i). Similarly, we may assume that $1-\alpha \notin E_\varepsilon$.

Now let $0 < a \leq b < 1$ be fixed, and let x_i, y_i, where $i = 1, 2, \ldots, l+m+n$, be variables satisfying

$$\begin{cases} 0 \leq x_1 \leq y_1 \leq x_2 \leq y_2 \leq \cdots \leq x_l \leq y_l \leq a, \\ a \leq x_{l+1} \leq y_{l+1} \leq x_{l+1} \leq y_{l+2} \leq \cdots \leq x_{l+m} \leq y_{l+m} \leq b, \\ b \leq x_{l+m+1} \leq y_{l+m+1} \leq x_{l+m+1} \leq y_{l+m+2} \leq \cdots \leq x_{l+m+n} \leq y_{l+m+n} \leq 1, \\ \displaystyle\sum_{i=1}^{l} (y_i - x_i) = \alpha_1, \quad \sum_{i=l+1}^{l+m} (y_i - x_i) = \alpha_2, \quad \sum_{i=l+m+1}^{l+m+n} (y_i - x_i) = \alpha_3. \end{cases} \tag{3.6}$$

We define a compact set $X_{a,b}^{l,m,n} = X_{a,b}^{l,m,n}(\alpha_1, \alpha_2, \alpha_3) \subset \mathbb{R}^{2(l+m+n)}$ by

$$X_{a,b}^{l,m,n} = \left\{ \mathbf{x} \in \mathbb{R}^{2(l+m+n)} : \mathbf{x} \text{ satisfies (3.6)} \right\},$$

where $\mathbf{x} := (x_1, y_1, \ldots, x_{l+m+n}, y_{l+m+n})$, and put

$$E(\mathbf{x}) := \bigcup_{i=1}^{l+m+n} (x_i, y_i).$$

We denote by $\lambda_E(\mathbf{x})$ the principal eigenvalues of (3.1) for $E(\mathbf{x})$. Since $\lambda_E(\mathbf{x})$ is continuous in $\mathbf{x}$ and $X_{a,b}^{l,m,n}$ is compact, for each $(\alpha_1, \alpha_2, \alpha_3)$, $\lambda_E(\mathbf{x})$ takes the minimum at some $\mathbf{x} \in X_{a,b}^{l,m,n}$. Let $\mathbf{x}_{\min}$ be a minimal point of $\lambda_E(\mathbf{x})$ in $X_{a,b}^{l,m,n}$, and $\phi_{\min}$ be the corresponding positive eigenfunction.

Lemma 3.7. *For any fixed $\beta_1, \beta_2 \in [0, \alpha)$, there exists $\lambda_{\beta_1,\beta_2} > \lambda_\alpha$ depending only on β_1 and β_2 such that $\lambda_E \geq \lambda_{\beta_1,\beta_2}$ holds for any $E \in S_\alpha^\infty$ with $|E \cap (0,\alpha)| = \beta_1$ and $|E \cap (1-\alpha, 1)| = \beta_2$.*

Proof. It suffices to consider the case where E is written as $E(\mathbf{x}_{\min})$ for some $\mathbf{x}_{\min} \in X_{a,b}^{l,m,n}$. We assume without loss of generality that the intervals of $E(\mathbf{x}_{\min})$ are separate from each other in $(0,a)$, (a,b) and $(b,1)$, respectively. We shall show that there exists $\lambda_{\beta_1,\beta_2} > \lambda_\alpha$ independent of l,m,n such that $\lambda_E(\mathbf{x}_{\min}) \geq \lambda_{\beta_1,\beta_2}$.

We note that if $\phi'_{\min}(c) = 0$ for some $c \in (0,1)$, Lemma 3.2 implies

$$\lambda_E(\mathbf{x}_{\min}) \geq \min\bigl\{(1-c)^{-2}, c^{-2}\bigr\}\lambda_{E_0}$$

for some $E_0 \in S_\alpha^\infty$. By Propositions 3.4 and 3.5, we have $\lambda_{E_0} \geq \lambda_\alpha$. Hence, if $\phi'_{\min}(c) = 0$ for some $c \in (0,1)$, we obtain

$$\lambda_E(\mathbf{x}_{\min}) \geq \min\bigl\{(1-c)^{-2}, c^{-2}\bigr\}\lambda_\alpha. \tag{3.7}$$

First we consider the case of $\alpha \leq 1/2$ and take $a = \alpha$, $b = 1-\alpha$, $\alpha_1 = \beta_1$, and $\alpha_3 = \beta_2$. Assume $l \geq 2$, i.e., $\alpha_1 > 0$ and $0 < x_l < y_l \leq a$. If $y_l < a$, then we can show in the same manner as in Lemma 3.1 that $\phi_{\min}(x_l) = \phi_{\min}(y_l)$. This implies that $\phi'_{\min}(c) = 0$ for $c = (x_l + y_l)/2$. Since $x_l > 0$ and $y_l > \alpha_1$, we have $\alpha_1/2 < c < a$. Hence, (3.7) yields

$$\lambda_E(\mathbf{x}_{\min}) \geq (1-\alpha_1/2)^{-2}\lambda_\alpha.$$

Similarly, if $y_l = a$, then $\phi_{\min}(y_{l-1}) = \phi_{\min}(x_l)$, so that $\phi'(c) = 0$ for $c = (y_{l-1} + x_l)/2$. Since $x_l > a - \alpha_1$ and $y_{l-1} > 0$, we have $(a-\alpha_1)/2 < c < a$. Hence, (3.7) yields

$$\lambda_E(\mathbf{x}_{\min}) \geq \left\{1 - \frac{a-\alpha_1}{2}\right\}^{-2}\lambda_\alpha.$$

The case $m \geq 2$ or $n \geq 2$ can be treated in the same manner. It remains to consider the case where $l,m,n \leq 1$. In this case, since $\beta_1, \beta_2 < \alpha$, we see that $E(\mathbf{x}_{\min}) \neq (0,\alpha), (1-\alpha,1)$. Hence, it follows from Propositions 3.4 and 3.5 that $\lambda_E(\mathbf{x}_{\min}) > \lambda_\alpha$ for every $l,m,n \leq 1$. Thus, we can choose $\lambda_{\beta_1,\beta_2} > \lambda_\alpha$ independent of l,m,n such that the inequality $\lambda_E \geq \lambda_{\beta_1,\beta_2}$ holds for any $E \in S_\alpha^\infty$ with $|E \cap (0,a)| = \beta_1 < \alpha$ and $|E \cap (b,1)| = \beta_2 < \alpha$.

In the case $\alpha > 1/2$, we take $a = 1-\alpha$, $b = \alpha$, $\alpha_1 + \alpha_2 = \beta_1$ and $\alpha_2 + \alpha_3 = \beta_2$. Then we easily see that $\alpha_1 = \alpha - \beta_2 > 0$ and $\alpha_2 = \alpha - \beta_1 > 0$. Once α_1, α_2, α_3 are fixed, we can use the same method as above to obtain the conclusion. □

3.5. Completion of the proof of Theorem 1.2

Now we are in a position to complete the proof of Theorem 1.2.

Proof of Theorem 1.2. Assume that $E \in S_\alpha$ satisfies $|E \cap (0,\alpha)| = \beta_1 < \alpha$ and $|E \cap (1-\alpha,1)| = \beta_2 < \alpha$. Let $\lambda_{\beta_1,\beta_2}$ be the positive constant found in Lemma 3.7. Choose

$$\delta = \frac{\lambda_{\beta_1,\beta_2} - \lambda_\alpha}{2} > 0. \tag{3.8}$$

By Lemma 3.6, for any $\varepsilon > 0$, there exists $E_\varepsilon \in S_\alpha^\infty$ such that $|E \setminus E_\varepsilon| < \varepsilon$, $|E_\varepsilon \cap (0,\alpha)| = \beta_1$, and $|E_\varepsilon \cap (1-\alpha,1)| = \beta_2$. By Lemma 3.7, we have

$$\lambda_{E_\varepsilon} \geq \lambda_{\beta_1,\beta_2}. \tag{3.9}$$

Choose $\epsilon > 0$ sufficiently small; by Lemma 2.6 we have

$$|\lambda_E - \lambda_{E_\epsilon}| \leq \frac{\delta}{2}. \tag{3.10}$$

Hence, by (3.10), (3.9), and (3.8) we have

$$\lambda_E \geq \lambda_{E_\varepsilon} - \frac{\delta}{2} \geq \lambda_{\beta_1,\beta_2} - \frac{\delta}{2} = \lambda_\alpha + \frac{\delta}{2}.$$

Thus, if $|E \cap (0,\alpha)| < \alpha$ and $|E \cap (1-\alpha,1)| < \alpha$, we obtain $\lambda_E > \lambda_\alpha$. Conversely, if $|E \cap (0,\alpha)| = \alpha$ or $|E \cap (1-\alpha,1)| = \alpha$, Proposition 3.4 implies $\lambda_E = \lambda_\alpha$. The proof is now complete. □

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