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BOOK REVIEW

A New Approach to Differential Geometry Using Clifford's Geometric Algebra, by John Snygg, Springer, New York, 2012, xvii + 465pp., ISBN 978-0-8176-8283-8.

The book under review is perfectly organized textbook for undergraduate students in mathematics and physics due to the large experience of the author. Based on the matrix representation of the Clifford's algebra it provides an effective and universal algebraic formalism for studying the classical differential geometry of curves and surfaces and differential geometry of Riemannian and Pseudo-Riemannian manifolds. More precisely Chapters 2,3 and 4 are devoted to the matrix interpretation of the Clifford's algebra in the Euclidean three-space, Minkowski four-space and flat n-space respectively. The key point is the construction of geometrically arising basis of three involutive and skew-commuting matrices formig a subspace in the space of the symmetric 4×4 matrices, which is isomorphic to the Euclidean three-space E^3. This basis consists of the matrices, which can be written as

$$\mathrm{e}_1 = \begin{pmatrix} \mathrm{O} & \vdots & \mathrm{L} \\ \cdots & & \cdots \\ \mathrm{L} & \vdots & \mathrm{O} \end{pmatrix}, \qquad \mathrm{e}_2 = \begin{pmatrix} \mathrm{O} & \vdots & \mathrm{M} \\ \cdots & & \cdots \\ \mathrm{M} & \vdots & \mathrm{O} \end{pmatrix}, \qquad \mathrm{e}_3 = \begin{pmatrix} \mathrm{I} & \vdots & \mathrm{O} \\ \cdots & & \cdots \\ \mathrm{O} & \vdots & -\mathrm{I} \end{pmatrix}.$$

Here $\mathrm{L} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\mathrm{M} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ are the well known matrices of reflections, I is the unit and O is the zero 2×2 matrices. Because of the above and

$$\mathrm{L}^2 = \mathrm{M}^2 = \mathrm{I}, \quad \mathrm{ML} = -\mathrm{LM} = \mathrm{J}, \quad \mathrm{J}^2 = -\mathrm{I} \quad \text{then} \quad \mathrm{e}_s^2 = \mathrm{I}, \quad \mathrm{e}_s\mathrm{e}_k = -\mathrm{e}_k\mathrm{e}_s.$$

The matrices $\{\mathrm{L}, -\mathrm{iJ}, \mathrm{M}, \mathrm{i} = \sqrt{-1}\}$ are usually denoted as $\{\sigma_1, \sigma_2, \sigma_3\}$ which is the well known triple of the Pauli spin-matrices. Thus quaternions, octonions (closely related to the rotations of unit spheres), the set of Dirac vectors, p-vectors and q-forms take natural place in this matrix approach. All real linear combinations of the 4×4-matrices $\mathrm{I}, \mathrm{e}_s, \mathrm{e}_s\mathrm{e}_k, \mathrm{e}_s\mathrm{e}_k\mathrm{e}_t,\ \mathrm{e}_{i_1}\mathrm{e}_{i_2}\ldots\mathrm{e}_{i_p}$, (usually called p-vectors) represent the Clifford numbers and generate Clifford's algebra over the field of reals (the only considered case in this text). The last one has a natural structure of the metric space with the metric tensor $g(g_{ij} = \langle \mathrm{e}_i, \mathrm{e}_j \rangle)$ induced by the scalar product $\langle\,,\rangle$. This scalar product is defined for any vectors (i.e., matrices) $\mathbf{a} = a^{\mathrm{i}}\mathrm{e}_i$ and $\mathbf{b} = b^{\mathrm{j}}\mathrm{e}_j$ by the symmetrisation $\langle \mathbf{a}, \mathbf{b} \rangle \mathrm{I} = \frac{1}{2}(\mathbf{ab} + \mathbf{ba})$ and is extended in a natural way to the tensor spaces of type (p, q) (see Chapter 4). Chapter 4, jointly with Appendix A and Appendix B generalize above matrix representation for the Clifford algebra associated with n-dimensional Euclidean or Pseudo-Euclidean of signature $(p^{(+)}, (n-p)^{(-)})$ spaces. Concretely matrix real linear space spanned by orthonormal matrix basis $\{\mathrm{e}_1, \mathrm{e}_2, \ldots, \mathrm{e}_n\}$, $\mathrm{e}_i\mathrm{e}_k + \mathrm{e}_k\mathrm{e}_i = 2n_{ik}\mathrm{I}$, where n_{ik} generalize the Kronecker's deltas, i.e., the diagonal matrix $N(n_{ik}) = \mathrm{diag}\{\underbrace{+1, \ldots, +1}_{p \text{ copies}}, \underbrace{-1, \ldots, -1}_{(n-p) \text{ copies}}\}$ induces 2^n-dimensional matrix algebra spanned by $\{\mathrm{I}, \pm\mathrm{e}_1^{k_1}\mathrm{e}_2^{k_2}\ldots\mathrm{e}_n^{k_n}\,;\ \mathrm{e}_s^{2k} = \pm\mathrm{I}, \mathrm{e}_s^{2k+1} = \pm\mathrm{e}_s\}$ (see the Appendix A, Theorem 291). In Appendix B the author describes the metric tensor $G(g_{\alpha\beta}) = \mathcal{R}N\mathcal{R}^T$ by an orthogonal transport operator $\mathcal{R}$, mapping the matrix basis $\{\mathrm{e}_s\}$ onto the Dirac basis $\{\gamma_\alpha\}$ so that $\gamma_\alpha\gamma_\beta + \gamma_\beta\gamma_\alpha = 2g_{\alpha\beta}\mathrm{I}$. This is the matrix analogue of the first fundamental form of a smooth surface. This quite important algebraic fact leads to uniform study of differential geometry of Riemannian and Pseudo-Riemannian manifolds (curved spaces) of any dimension. All rest chapters (from 5-th till 11-th) of the book, concerning basic notions, concept and theorems of the differential geometry, can be considered as applications of the above described matrix approach. Moreover the geometric meaning of the matrix operations as rotations of the unit spheres leads to more deep understanding of the geometric meaning of the analytical objects and their properties. Thus the operator of intrinsic derivative ∇_α is defined as a linear operator over the Clifford numbers, satisfying the Leibniz rule for its finite product, so that the linear closure of the set $\{\gamma_\beta\}$ of Dirac vectors is ∇_α-invariant and the action is symmetric (with vanishing torsion) $(\nabla_\alpha\gamma_\beta = \nabla_\beta\gamma_\alpha)$. Over the components of any p-vector and scalar ∇_α acts as usual differentiation: $\nabla_\alpha \equiv \partial/\partial u^\alpha$. (See Chapter 5.3, Definition 72). Further the notion of parallel transport of a Clifford number $\mathbf{A}(\mathbf{s})$ along the curve $c(s)$ over a smooth n-dimensional surface S and the notion of geodesic curve $c(s)$ of S are defined by the condition that $\mathbf{A}(\mathbf{s})$ and the tangent vector field $\dot{c}(s)$ belong to the kernel of ∇_s (Chapter 5.4).

The components $R^\lambda_{\alpha\beta\gamma}$ of the Riemann's curvature tensor, defining the curvature operator $R_{\alpha\beta} = [\nabla_\alpha, \nabla_\beta] - \nabla_{[\alpha,\beta]}$, so that $R_{\alpha\beta}(\gamma_\sigma) = R^\lambda_{\alpha\beta\sigma}\gamma_\lambda$ and the curvature two-form are closely related with the rotations around infinitesimal loops over surface by means of Gauss-Bonnet formula for the Gaussian curvature K (see Chapters 5.5, 6.5 and 6.6). The author's view covers a large perimeter of classical geometric topics like smooth curves in n-dimensional Euclidean space described by its Frenet's frames and formulas, special surfaces (ruled, developable, minimal), curves over surface and its invariants, non-Euclidean (hyperbolic) geometry with its Poincaré model, transformations and metrics.

In each chapter the author emphasizes on the physical motivations and interpretations of the basic concepts of differential geometry from the viewpoint of the special and the general relativity of Einstein independently of Newtonian physics. In such a way all the time physicists regard and become familiar with model gravity space. More detailed descriptions of gravity models are given in Chapter 3.1: *A Small Dose of Special Relativity* and Chapter 12: *Some General Relativity*.

The book ends with four Appendices, a large list of reference sources and Index list.

From the methodical point of view this book offers the following

- a possibility for specific arrangement of different introduction courses on differential geometry for advanced undergraduate and beginning graduate math- and physics-classes
- a large number of methodically adequate problems and exercises at the end of each chapter
- methodical discussions on the use of Clifford geometric algebra.

The author provides quite interesting historical analysis including the politic situations, culture traditions and their relationships, being a background for corresponding scientific developments jointly with biographies of historically and scientifically relevant mathematicians and physicists.

This book is a natural continuation of the previous book of the author [6] and as further readings one can suggest the books for which the author himself had mentioned that he had benefited. This list includes the books by Hestenes and Sobczuk [3] and Lounesto [5] to whom the present book is dedicated. The usefulness of the Clifford ideas can be traced in the books by Jancewiez [4], Baylis [1] and Girard [2].

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