



## BOOK REVIEW

*Differential Geometry for Physicists and Mathematicians: Moving Frames and Differential Forms: From Euclid Past Riemann*, by Jose G. Vargas, World Scientific, Singapore 2014, xvii+292 pp., ISBN 978-9814566391.

The book under review is a non-standard book on differential geometry that also includes relevant topics in theoretical physics. Object of investigation is the theory of affine and Euclidean connections. Following Cartan's approach for studying this theory, the author uses the exterior calculus and the method of moving frames. The book is written in an appropriate style for physicists. It is also of interest to mathematicians since the basic notions of differential geometry are presented from a different point of view. A great merit of the book is that the advanced concepts like fibre bundles, connections, Lie algebras, etc., are considered after introducing them in simple structures like affine and Euclidean spaces. The book consists of Preface, thirteen Chapters structured in five Parts, three Appendices, References and Index.

**Part I - Introduction.** In Chapter 1 the author refers the readers to the selective capitalization in titles of sections according to their difficulty. The author addresses the intended readers of his book. As he says, they are "readers unsatisfied" by the style of presenting the theory of connections in other books on differential geometry. He points out that there are many misconceptions in the literature on differential geometry for physicists and gives such examples. The chapter closes with brief remarks about some basic notions like a ring, field ideal, vector space, module, etc.

**Part II - Tools.** This part has three chapters devoted to the main tools used in the book. In Chapter 2 (*Differential Forms*) the differential forms are viewed as functions of curves, surfaces, etc., but not as a covariant skew-symmetric tensors. A scalar-valued differential one-form  $\alpha$  is defined by  $\alpha = \alpha_i(x)dx^i$ ,  $i = 1, 2, \dots, n$ , where  $dx^i$  are differentials of coordinate functions  $x = (x^1, \dots, x^n)$ , the components  $\alpha_i(x)$  are smooth functions. A simple differential  $r$ -form  $\omega^1 \wedge \dots \wedge \omega^r$  is the result of the exterior product  $\wedge$  ( $\wedge$  is defined as an algebraic operation that is linear with respect to each factor and it is anticommutative with respect

to the interchange of any two factors) of  $r$  differential one-forms  $\omega^1, \dots, \omega^r$ . Any linear combination of simple  $r$ -forms is also called an  $r$ -form. By using relevant examples it is shown that the differential  $r$ -forms are integrands of  $r$ -integrals ( $r = 1, 2, 3, \dots$  for a line, surface, volume, ...) in the vector calculus. The concept of exterior products of one-forms is extended to forms of arbitrary grades and a change of basis of differential forms is discussed. A short introduction of differentiable manifolds is given. Basic facts about vector space, dual vector space, Euclidean vector space are presented in Chapter 3 (*Vector Spaces and Tensor Products*). Contrary to the usual definition of a tensor as a multilinear map of vectors, tensors are introduced as elements of the vector space which is the tensor product of vector spaces. In this chapter the author also acquaints the reader with the Clifford algebra that is important for physicists. It is the natural algebra of spaces endowed with a metric since the Clifford product is a combination of the exterior and interior product. The tangent Clifford algebras of the 3-D Euclidean vector space and of spacetime are considered. In Chapter 4 (*Exterior Differentiation*) the author presents two definitions of exterior derivative. In the first approach the generalized Stokes theorem is viewed as an implicit definition of the exterior derivative  $d\omega$  of the differential  $r$ -form  $\omega$  as follows:  $d\omega$  is the differential  $(r + 1)$ -form that satisfies the identity  $\int_{\partial S} \omega = \int_S d\omega$ , where  $S$  is a domain with a boundary  $\partial S$ . In the second definition the exterior derivative of a sum is the sum of the exterior derivatives and for a simple  $r$ -form  $\tau = f dx^{i_1} \wedge \dots \wedge dx^{i_r}$  its exterior derivative  $d\tau$  is given by  $d\tau = df \wedge dx^{i_1} \wedge \dots \wedge dx^{i_r}$ . Useful expression for the exterior product of forms of any grade is obtained. Basic operators as gradient, curl, divergence and Laplacian are defined for differential forms.

Part III - *Two Klein Geometries*. This part consists of Chapter 5 (*Affine Klein Geometry*) and Chapter 6 (*Euclidean Klein Geometry*). These two flat geometries (affine and Euclidean) are considered from the view point of the bundles and integrability of differential equations. In Chapter 5 an affine space  $Af(n)$  is defined as a pair of a vector space  $V^n$  and an associated set of points  $\{A, B, C, \dots\}$  satisfying some properties. The frame bundle  $F^n$  of  $Af(n)$  is introduced as the set of all frames  $(\mathbf{P}, \mathbf{e}_1, \dots, \mathbf{e}_n)$  of  $Af(n)$ , where  $\mathbf{P}$  is a point and  $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$  is a basis at  $\mathbf{P}$  of  $V^n$ . It is well known that  $F^n$  is an  $(n + n^2)$ -dimensional differentiable manifold, which is a principal fiber bundle over  $Af(n)$  (called the base space). The relation between the frames is realized by the action of the affine group  $Af(n)$  in  $F^n$ . The fibers of  $F^n$  are  $n^2$ -dimensional manifolds whose points are vector bases of  $V^n$ . The group acting in each fibre is the linear subgroup  $GL(n)$  of  $Af(n)$ . By using the basic theory of differential forms the connection equations  $d\mathbf{P} = \omega^i \mathbf{e}_i$ ,  $d\mathbf{e}_i = \omega_i^j \mathbf{e}_j$  ( $\omega_i, \omega_i^j$  are one-forms, called connection forms) and the equations of structure of affine space  $d\omega^i - \omega^j \wedge \omega_j^i = 0$ ,  $d\omega_i^j - \omega_i^k \wedge \omega_k^j = 0$

are obtained. The last two equations are also the integrability conditions of the connection equations. The author remarks that they are the basis for the Cartanian generalization of an affine space in Chapter 8. Other topics treated in this chapter are: curvilinear coordinates and bases of vector fields generated by the system of curvilinear coordinates (called holonomic or coordinate bases), non-holonomic bases, the Maurer-Cartan equations, etc. The Lie algebra of the affine group is expressed in terms of the connection forms  $(\omega^i, \omega_j^k)$ . It is shown that the connection is a differential one-form on the frame bundle  $F^n$  with values in the Lie algebra of the affine group. Chapter 6 begins with the definition of Euclidean space  $\mathcal{E}^n$  ( $\mathcal{E}^n$  is an affine space whose associated vector space is Euclidean) and Euclidean frame bundle. All frames in this bundle are Euclidean, i.e., the bases  $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$  are orthonormal. The group acting in Euclidean frame bundle is the Euclidean group, and the group in its fibers is the rotation group. The equations of structure and integrability of the connection equations of Euclidean space are discussed. The equations of structure of affine and Euclidean spaces are the same but in the Euclidean case there is an additional condition  $\omega_i^j = -\omega_j^i$  which follows from the orthonormality of the bases. This yields that the fibers are of dimension  $n(n-1)/2$ . The author also considers the extension of the Euclidean frame bundle to an affine frame bundle by the action of the linear group on the Euclidean fibers. He briefly presents: the Hodge dual of a differential  $r$ -form using Clifford algebra, the Kähler calculus of scalar-valued differential forms referring to [1] where computations in relativistic quantum mechanics with Kähler calculus are given. The Lie algebra of the Euclidean group is found.

Part IV - *Cartan Connections*. In Chapter 7 (*Generalized Geometry Made Simple*) three surfaces (Euclidean plane, two-sphere, two-torus) equipped with two Euclidean connections (the Levi-Civita connection (LCC) and the Columbus connection, which is teleparallel) are studied. These examples help the reader understand the connection on a manifold as a rule to compare tangent spaces at different points on this manifold. Chapter 8 (*Affine Connections*) is devoted to the main topic - affine connection, i.e., connection on the affine frame bundle of a differentiable manifold. This chapter begins with a presentation of the Cartan's and Kähler's interpretation of a vector field and the Lie derivative of a differential form. Next, the author simply and clearly introduces the concept of affine connection. First, an affine frame bundle of an  $n$ -dimensional differentiable manifold  $M$  is constructed in a similar way as the frame bundle  $F^n$  of an affine space. The difference is that it consists of all frames  $(\mathbf{P}, \mathbf{e}_1, \dots, \mathbf{e}_n)$ , where  $\mathbf{P}$  is a vector belonging to the tangent space  $T_x M$ ,  $x \in M$  and  $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$  is a basis of  $T_x M$ . After defining an affinely connected differentiable manifold  $M$ , its equations of structure in terms of torsion and curvature are given. A geometric interpretation

of the affine curvature of the manifold is described. Integrability and an interpretation of the torsion are discussed. At the end of the chapter, an affine connection on manifold  $M$  is formally introduced as a differential one-form  $(\omega^i, \omega_i^j)$  on its  $n(n+1)$ -dimensional frame bundle  $B(M)$  taking values in the Lie algebra of the affine group for dimension  $n$  and satisfying some conditions. In Chapter 9 (*Euclidean Connections*) an Euclidean frame bundle of an  $n$ -dimensional differentiable manifold  $M$  is defined. It has the same structure as the affine frame bundle of  $M$  but the vector bases  $\{e_1, \dots, e_n\}$  are orthonormal or pseudo-orthonormal. An Euclidean connection on  $M$ , endowed with a metric, is introduced by a similar definition as that of an affine connection at the end of Chapter 8. In this definition the  $n(n+1)$ -dimensional frame bundle, affine group,  $GL(n)$ , torsion and the affine curvature are replaced by the  $(n+n(n-1)/2)$ -dimensional Euclidean frame bundle, Euclidean group,  $O(n)$  or Lorentz group, torsion and the Euclidean curvature, respectively. The equations of the Euclidean structure are the equations of the affine structure plus the condition that is a consequence from the orthonormality (or pseudo-orthonormality) of the metric. Formulas for an Euclidean connection in the case when the frames of the bundle are not orthonormal (or not pseudo-orthonormal) are given. This is the so-called “extension” of an Euclidean connection to an affine connection. A contorsion of Euclidean connections is defined and its components are expressed by the components of the torsion. Next, the author deals with the LCC which is an Euclidean connection with zero torsion. The components of the affine extension of the LCC with respect to the coordinate bases (the Christoffel symbols of the LCC) are expressed as usual - only by the metric and its first derivatives. The author also supplies a faster way (by inspection) to compute the components of the LCC with respect to an orthonormal basis. He gives five rules illustrated with examples. The relation between the curvature of the LCC, called Riemannian curvature, and the Euclidean curvature through the contorsion (and hence through the torsion) is presented. In the beginning of Chapter 10 (*Riemannian Pseudo-Spaces & Riemannian Spaces*) the author explains the notion of the Klein’s geometry used in the previous chapters. In the spirit of the Erlangen program, a Klein geometry is a pair  $(G, G_0)$  of a group  $G$  and its subgroup having a certain property. The transformations in  $G_0$  are the transformations from  $G$  that leave a point unchanged. Then the author comments on the so-called by Cartan “false spaces of Riemann”. Here the sense of the term “pseudo-Riemannian spaces” is not the same as usual, i.e., spaces with a non-positive definite metric. Cartan characterized the false (or the old) Riemannian geometry (until 1917 when the LCC was proposed) as a generalization of the metric part but not the affine one of the Euclidean structure. Next, the Cartan’s approach of derivation of the Riemannian curvature is reproduced. For a Riemannian space (a differentiable

manifold endowed with the LCC) the equations of structure, the Bianchi identities and the Ricci tensor are obtained. At the end of the chapter, physical applications of the Riemannian geometry towards the Einstein's equations, the Einstein's differential three-form, the Einstein's tensor and properties of its components and the Einstein's equation for the Schwarzschild's metric are presented.

Part V - *The Future?* In Chapter 11 (*Extensions of Cartan*) the author emphasizes on the fact that Cartan's theory of moving frames is an excellent way to investigate extensions of classical differential geometry of interest for physics. Moreover, this theory can be applied to all Klein geometries. Chapter 12 (*Understand the Past to Imagine the Future*) is a historical review of algebras and calculus closely related to geometry, the main ideas in the Riemann's program and Klein's Erlangen program, Cartan's approach to Finsler geometry, etc. In Chapter 13 (*A Book of Farewells*) the author explains why one should say farewell (in his opinion) to vector algebra and calculus, calculus of complex variables, Dirac's calculus, tensor calculus, auxiliary bundles. He also outlines his next book. By using the moving frame method, some facts of the geometry of curves and surfaces in 3-D Euclidean spaces are given in Appendix A. Appendix B provides biographical data about the famous mathematicians Elie Cartan (1869-1951) and Hermann Grassmann (1808-1877) concerning their work and professional life. Appendix C is a list of 51 publications by the author.

This interesting book is recommended to advanced physics and mathematics students and to scientists working in differential geometry. It has a great methodological value because of the multitude of unusual advanced concepts and applications.

## References

- [1] Vargas J. G., *The Foundations of Quantum Mechanics and the Evolution of the Cartan-Kähler Calculus*, Found. Phys. **38** (2008) 610-647, doi: 10.1007/S10701-008-9223-3.

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