BOOK REVIEW

Lectures on Curves, Surfaces and Projective Varieties (*A Classical View of Algebraic Geometry*), by Mauro C. Beltrametti, Ettore Carletti, Dionisio Gallarati and Giacomo M. Bragadin, European Mathematical Society Textbooks, Zürich, 2009, xv + 491pp., ISBN 978-3-03719-064-7.

The book under review represents the algebraic-geometric trends from nineteenth century and the first half of twentieth century. The motivation of the authors is not only to recall the history, but to introduce the young algebraic geometers to the natural constructions and ideas, underlying the abstract contemporary treatment.

The first chapter collects some preliminaries on projective spaces and correspondences, as well as on the duality between points and projective hyperplanes.

The second one recollects the notion of Zariski topology and Hilbert's Nullstellensatz on the correspondence between affine algebraic varieties and radical polynomial ideals. It proceeds with morphisms, rational maps and Noether Normalization Lemma. Special attention is paid to the projective algebraic varieties and their associated homogeneous radical ideals.

The third chapter is devoted to the tangent space $T_p X$ and the singularities of an algebraic variety X. The dimension of an algebraic variety is defined as the infimum of the dimensions of the tangent spaces, after showing that this infimum is attained on an open Zariski dense subset. It is proved to coincide with the transcendence degree of the field of the rational functions. Meanwhile are discussed the local parameters and the intersections with projective subspaces of sufficiently large dimension. The order (or degree) of a variety X is introduced algebraically and exhibited to coincide with the number of the intersection points with a generic projective subspace of complementary dimension. As a result, there arises the notion of a tangent cone $TC_P X$ at a singular point P of X.

The fourth chapter is on the intersection multiplicity and Bezout's Theorem for coplanar algebraic curves. It exposes Kronecker's elimination of $(r-m)$ variables as a projection with center $\mathbb{P}^{r-m-1}$ on $\mathbb{P}^m$.

The fifth chapter concerns the projective hypersurfaces. It includes a detailed discussion of the conical hypersurfaces of an arbitrary order r, characterizing their vertices as points of multiplicity r. If a hypersurface H and a projective variety X are tangent to each other at a smooth point $P \in H^{\text{smooth}} \cap X^{\text{smooth}}$, then $P \in (H \cap X)^{\text{sing}}$ is shown to be a multiple point. The union of the tangent spaces to a projective hypersurface $H \subset \mathbb{P}^n$ is treated as an algebraic envelope of hyperplanes, i.e., as a hypersurface of the dual projective space $(\mathbb{P}^n)^*$ of the hyperlanes in $\mathbb{P}^n$. Generalizing the notion of a tangent space to a hypersurface $H \subset \mathbb{P}^n$ at a smooth point P, the authors introduce the s-th polar $H_s(P)$ of P with respect to H for $1 \leq s \leq r - 2$. (For a hypersurface H of order r the last polar $H_{r-1}(P) = T_P H$ is the tangent space.) They establish the functoriality of the polars with respect to projective automorphisms $\mathbb{P}^n \to \mathbb{P}^n$. The points $P \in \mathbb{P}^n$ with $P \in \cap_{s=1}^{r-1} H_s(P)$ are shown to constitute the hypersurface H itself. The polars satisfy Permutability Theorem $(H_t(Q))_s(P) = (H_s(P))_t(Q)$, Section Theorem $\mathbb{P}^h \cap H_s(P) = (H \cap \mathbb{P}^h)_s(P)$ and Reciprocity Theorem, asserting that $Q \in H_s(P)$ implies $P \in H_{r-s}(Q)$ for all $0 < s < r$. The points $P \in H$ of multiplicity $2 \leq s \leq r-1 = \text{ord}(H)-1$ are characterized by the indeterminacy of $H_{r-s+1}(P)$ and the tangent cone $TC_P H$ to H at such P is proved to be $H_{r-s}(P)$.

The sixth chapter studies the linear systems $\Sigma = \cup_{\lambda \in \mathbb{P}^h} V(\lambda_0 f_0 + \ldots + \lambda_h f_h)$ of divisors or hypersurfaces

$$V(\lambda_0 f_0 + \lambda_1 f_1 + \ldots + \lambda_h f_h) = \{x \in \mathbb{P}^n \ ; \ \lambda_0 f_0(x) + \lambda_1 f_1(x) + \ldots + \lambda_h f_h(x) = 0\}$$

in $\mathbb{P}^n$, determined by the homogeneous polynomials $f_j(x_0, x_1, \ldots, x_n) = 0$, $0 \leq j \leq h$ of one and a same degree r. After discussing the algebraic and, in particular, the linear conditions on the coefficients of $f_j(x_0, x_1, \ldots, x_n)$ are considered the intersections of Σ with subspaces $\mathbb{P}^m \subset \mathbb{P}^n$, $m < n$. The next milestone is Bertini's First Theorem. It asserts that if the base locus $B = V(f_0) \cap V(f_0) \cap \ldots \cap V(f_h)$ of Σ does not contain a hypersurface, then the s-fold points of a generic $V(\lambda_0 f_0 + \ldots + \lambda_h f_h)$ belong to B with multiplicity $\geq (s - 1)$. Under the assumption of the aforementioned theorem, the book provides an example of a specific hypersurface, whose singularities are not contained in the base locus.

An arbitrary linear system Σ with base locus B induces a morphism

$$\varphi : \mathbb{P}^n \setminus B \to \mathbb{P}^h, \qquad \varphi(x) = [f_0(x) : f_1(x) : \ldots : f_h(x)]$$

and the projective closures $\overline{\varphi(\mathbb{P}^n \setminus B)}$ of the varieties of the form $\varphi(\mathbb{P}^n \setminus B)$ are called unirational. The book discusses Lüroth's Theorem on rationality of a unirational curve and Castelnuovo's Theorem on rationality of a unirational surface.

The projective image $V_{n,d}$ of the linear system $\Sigma_{n,d}$ of all hypersurfaces of order d in $\mathbb{P}^n$ is called Veronese variety. An arbitrary n-dimensional unirational variety is shown to be a projection of the Veronese variety $V_{n,d}$. A linear system $\Sigma = \cup_{\lambda \in \mathbb{P}^h} V(\lambda_0 f_0 + \lambda_1 f_1 + \ldots + \lambda_h f_h)$ of hypersurfaces is said to be reducible if its general member $V(\lambda_0 f_0 + \lambda_1 f_1 + \ldots + \lambda_h f_h)$ is reducible. The textbook illustrates also Bertini's Second Theorem, characterizing the reducible linear systems in $\mathbb{P}^n$, $n > 1$ as being composed with one-dimensional linear system or as having a fixed component. This chapter concludes with the description of the blow-up of $\mathbb{P}^n$ at a point or at a line.

The seventh chapter is on the projective algebraic curves. Let ν be the cardinality of the vanishing locus of the Jacobian determinant of one-dimensional linear system g_n^1 of order n on a projective curve C. Along the lines of Riemann's approach, $\nu - 2n$ is shown to be an even integer, independent of g_n^1 and $p = \frac{1}{2}\nu - n + 1$ is called genus of C. The rational curves are proved to be the ones of genus 0. Moreover, if C is an irreducible plane curve of order n, whose multiple points P_i have distinct tangents and multiplicities s_i for $1 \le i \le t$, then C is of genus

$$p = \frac{(n-1)(n-2)}{2} - \sum_{i=1}^{t} \frac{s_i(s_i - 1)}{2}.$$

By means of the two rulings on a quadric $Q \subset \mathbb{P}^3$ and the stereographic projection of Q, the authors describe the curves C of fixed order n on a non-degenerate or degenerate quadric $Q \subset \mathbb{P}^3$. Then applying Lüroth's Theorem on the rational parametrization of a rational curve, they derive that an irreducible rational curve C of order n is either a smooth rational curve $C^n \subset \mathbb{P}^n$ or a projection of a smooth rational curve $C^n \subset \mathbb{P}^n$ in $\mathbb{P}^r$, $r < n$. The smooth rational curves $C^n \subset \mathbb{P}^n$ are shown to be projectively generated, i.e., cut by n corresponding hyperplanes of n projectively referred pencils. Another result of interest is the existence of a unique smooth rational curve $C^n \subset \mathbb{P}^n$ through arbitrary generic points $P_1, \ldots, P_{n+1} \in \mathbb{P}^n$.

Chapter 8 focuses on the linear series on algebraic curves. It shows that an arbitrary irreducible curve C has a so called ordinary model $\mathcal{C} \subset \mathbb{P}^r$, whose singularities are at worst multiple points with distinct coplanar tangents. That enables to view the algebraic curves as sets of linear branches, instead of sets of points. As a result, the intersection multiplicity of two curves splits naturally into a sum of intersection multiplicities of the first one with the linear branches of the second one. Let $X \subset \mathbb{P}^n$ be a projective curve, Σ be a linear system of hypersurfaces of order d_1 in $\mathbb{P}^n$ and $\mathcal{H} \subset \Sigma$ be the linear system of the hypersurfaces, containing X. Then the dimension r of the linear series g_d^r of order $d = d_1 \mathrm{ord}(X)$, cut out

by Σ on X is proved to be $r = \dim \Sigma - \dim \mathcal{H} - 1$. The linear equivalence class $|A|$ of an effective divisor A of degree d is referred to as a complete series of order d. An arbitrary linear series is shown to be contained in a unique complete series. Removing a divisor B from all the divisors of a linear series $|A|$, containing B, one obtains the residual series of B with respect to $|A|$. The "Restsatz" of Brill and Noether asserts that the residual divisors R of a divisor B with respect to a complete series $|A|$ constitute a complete series $|R|$, called the difference series of $|A|$ and $|B|$. Let us suppose that the projective morphism φ of a linear series g_d^r of dimension r and order d on a curve X is birational. Then g_d^r is complete if and only if $\varphi(X)$ is not a projection of a curve $X' \subset \mathbb{P}^t$, $t > r$ of the same order. After showing that under a rational morphism $\psi : X_1 \to X_2$ of curves any smooth point $P_1 \in X_1$ corresponds to a uniquely determined at worst multiple point $\psi(P_1) \in X_2$, the birational morphisms $X_1 \to X_2$ of smooth curves appear to be morphisms. Another way of defining the genus p of an algebraic curve X is by Lückensatz of Weierstrass. If $\mathcal{X}$ is the ordinary model of X then the genus p is the minimal integer, such that arbitrary $p + 1$ points of $\mathcal{X}$ constitute a divisor of a linear series g_{p+1}^r of dimension $r \geq 1$. In particular, p generic points on $\mathcal{X}$ constitute a divisor with zero-dimensional linear series. As a consequence of Weierstrass's definition of a genus, a complete linear series g_d^r on a curve $X \subset \mathbb{P}^n$ has non-negative index of speciality $i = r - d + p$. The linear series g_d^r with vanishing index of speciality $i = 0$ are called non-special. The complete linear series $|A|$ of generic divisors A with $d \geq p$ points are non-special, as well as the linear series g_d^r with $d > 2p - 2$ or $r > p - 1$. An arbitrary irreducible algebraic curve is shown to have smooth birational models in $\mathbb{P}^r$ for all $r \geq 3$ and plane models, whose singularities are at most nodes. The book argues also on the equivalence of Riemann's and Weierstrass's definitions of a genus. Let $\mathcal{C} \subset \mathbb{P}^2$ be a curve of order n, whose singularities are at most multiple points P_i with distinct tangents and multiplicities s_i, $1 \leq i \leq t$. The curves $\mathcal{C}' \subset \mathbb{P}^2$ with multiplicities $\geq s_i - 1$ at P_i for all $1 \leq i \leq t$ are said to be adjoint of $\mathcal{C}$. The adjoints of order $n - 3$ are called canonical, the linear series, cut by them on $\mathcal{C}$ is the canonical series of $\mathcal{C}$ and any divisor $K_{\mathcal{C}}$ from the canonical series is a canonical divisor of $\mathcal{C}$. Riemann-Roch Theorem asserts that if D is a divisor of degree d and index of speciality $i(D)$ on a curve $\mathcal{C} \subset \mathbb{P}^2$ of genus $p \geq 1$, then $i(D) - 1$ equals the dimension of the residual series to D with respect to the canonical series $|K_{\mathcal{C}}|$. The Clifford's Theorem that a special linear series g_d^r on a smooth curve X has $d \geq 2r$ is also discussed in the book. Further, the canonical series $|K_X|$ of a smooth curve $X \subset \mathbb{P}^n$ of genus $p \geq 2$ is shown to have no fixed points. If $|K_X|$ is not associated with a birational morphism, then it is composed with g_2^1 and X

is called hyperelliptic. The hyperelliptic curves of genus $p \geq 2$ have plane models of order $p + 2$ with a unique multiple point of multiplicity p. Two hyperelliptic curves of one and a same genus $p \geq 2$ are birational if and only if the $2p + 2$ double points of the series g_2^1 on their plane models correspond under a projective linear transformation from $\mathrm{PGL}(2, \mathbb{C})$.

The ninth chapter studies the birational maps $\mathbb{P}^2 \rightarrow \mathbb{P}^2$, called Cremona transformations. If $\psi : \mathbb{P}^2 \rightarrow \mathbb{P}^2$ is a quadratic transformation then the projective lines on the first plane are mapped onto a two-dimensional linear system Σ of conics with three base points, called homaloidal net on the second plane. Conversely, any homaloidal net of plane conics determines a quadratic transformation of $\mathbb{P}^2$. For any algebraic curve $C \subset \mathbb{P}^2$ there exist finitely many quadratic transformations, which map C onto a curve $\mathcal{C} \subset \mathbb{P}^2$, whose singularities are at worst multiple points with distinct tangents. Further, Noether-Castelnuovo's Theorem establishes that any birational transformation $\mathbb{P}^2 \rightarrow \mathbb{P}^2$ is a product of finitely many quadratic transformations and a projective linear transformation.

The next chapter is on the images $\mathcal{F}$ of $\mathbb{P}^2$ in $\mathbb{P}^r$ under the morphisms, associated with r-dimensional linear systems Σ on $\mathbb{P}^2$. These $\mathcal{F}$ are called rational surfaces. They are of the same order as Σ. There does not exist an algebraic surface $X \subset \mathbb{P}^r$ of order $< r - 1$. Any algebraic surface $X^{r-1} \subset \mathbb{P}^r$ of order $r - 1$ is rational and X^{r-1} is either a ruled surface or the Veronese surface.

The eleventh chapter studies the products of projective spaces as a special case of Segre varieties.

The twelfth chapter is on the Grassmannians. It starts with the specific example of the Klein quadric $\mathcal{Q} \subset \mathbb{P}^5$, representing the lines in $\mathbb{P}^3$ and the ruled surfaces, which are formed by the lines in $\mathbb{P}^3$, corresponding to a curve $\mathcal{C} \subset \mathcal{Q}$. Then the authors introduce the Plücker coordinates in general and show that the Plücker ideal is generated by quadrics.

Each chapter concludes with examples and exercises. Moreover, some supplementary exercises are collected in the last chapter.

The book is recommended for first acquaintance with the intuitive geometric ideas, which led historically to the appearance of the abstract algebraic geometry. Similarly to Harris [2], the exposition is built on representative examples and explicit constructions. The book under review is remarkable by its ingenuity in treating the linear series of divisors as intersections of the variety with families of hypersyrfaces. It reveals the power of the incidence relations and duality. Fair amount of the text is devoted to the rudiments of the algebraic geometry and the specific understanding of the greatest founders of the subject. The book com-

plements the famous writings on the subject like Shafarevich [4], Griffiths and Harris [1], Harris [2] and Hartshorne [3]. It represents the flavor of the classical Italian school from the viewpoint of the contemporary algebraic geometry.

References

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