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BOOK REVIEW

Dynamical Symmetry, by Carl Wulfman, World Scientific, Singapore 2011, xx + 437 pages, ISBN-13 978-981-4291-36-1.

The widely used in modern mathematics concepts of *equivalence, invariance* and *symmetry* seem very close to each other, nevertheless, each one carries its special shade of meaning. The most powerful of them seems to be *equivalence*, which is mainly used to establish the same from a definite point of view structure, carried by two sets of mathematical or physical objects. The other two concepts seem to represent the same thing but from additional to each other viewpoints: when under external action an object demonstrates definite stability properties, then *invariance* considers the situation from the viewpoint of the object's surviving strength with respect to the available external action, and *symmetry* gives accent to the nature of the external actions under which the object may change somehow, but keeps its identity. For example, all constant functions $\mathbb{R} \to \mathbb{R}$ are equivalent with respect to differentiation, the euclidean metric g in $\mathbb{R}^3$ is invariant with respect to the orthogonal group O(3). When algebraic structures are under consideration any equivalence is established by *isomorphisms*, and when topological, or smooth, structures are considered then equivalence is established by *homeomorphisms*, or *diffeomorphisms*.

Mathematics has developed powerful methods to get information about some global properties of an object through studying its local/infinitesimal properties, and the basic tool in this respect is the concept of derivative in all of its forms and generalizations. A basic moment here is to find how an object $\mathcal{A}$ changes infinitesimally with respect to other object $\mathcal{B}$, most frequently, but not necessarily, of the same nature. In this respect the central role of the concept of *vector field* defined on some manifold could hardly be disputed. A basic property of every vector field is that it generates flow, i.e., a one-parameter group φ_t of at least local diffeomorphisms of the manifold considered, and this group is parametrized by an external for the manifold parameter t. These groups of diffeomorphisms are finite transformations, and in order to get free of the parameter, mathematics makes use of the so called *Lie derivative*. In its simplest case this operator defines how a vector field X changes infinitesimally with respect to some other vector field Y taking into account the infinitesimal changes of the referent vector field Y too. The most important property of this Lie derivative is its dependence only on the differentiable structure and on *NOTHING* else. Compare for example with the covariant derivative with respect to a linear connection Γ, we immediately note that the infinitesimal change of X with respect to Γ does *NOT* take into account the infinitesimal change of Γ and this circumstance suggests that the covariant derivative is rather of kinematical nature, i.e., the effect of Γ on the local change of X can be locally neglected. We could say that the Lie derivative establishes *intrinsic* for the whole tensor algebra on a manifold local properties, therefore, its importance is out of any doubt.

We say that the vector field X is *symmetric* with respect to Y if $L_Y X = 0$. If the vector fields $\{X_1, X_2, ..., X_p\}$ define a distribution Δ on a manifold M then X is an infinitesimal symmetry of Δ if every $L_X X_i$, $i = 1, 2, ..., p$ lives in Δ, in particular, if Δ is one-dimensional it can be represented by any $Z(x) = f(x)Y(x) \neq 0, x \in M$, and then X is symmetry of this Δ if $L_X Z(x) = g(x)Z(x)$, where g is a smooth function on M. If the local symmetry X lives in Δ then it is called *characteristic* symmetry of Δ, and if X is outside Δ then it is called *shuffling* symmetry of Δ. So, the Frobenius theorem for integrability of a distribution asserts that Δ is integrable if every $X \in \Delta$ is an infinitesimal symmetry of Δ.

The Lie derivative of a differential p-form on a manifold with respect to a distribution Δ represented by the corresponding p-vector field $X_1 \wedge X_2 \wedge ... \wedge X_p$, can also be defined, moreover, the Lie derivative of a differential p-form taking values in a vector space W_1 with respect to p-vector field taking values in a vector space W_2, and a bilinear map $\varphi : W_1 \times W_2 \to W_3$, where W_3 is another vector space, can also be defined.

The concept of *dynamical symmetry* suggests that the mathematical picture considered has something to do with reality, and the parameter t is connected with the real time measured by an "appropriate device", i.e., this parameter t arises as a concequence of comparing two processes: the one that we study and the one that goes on the "appropriate device". This point of view on the appearance of time-parameter in mathematical models of physical processes clearly suggests that it must be *transversal* to the other three space coordinate parameters. This was rightly understood and realized by the well known clever men at the beginning of the last century and gave start to the relativistic viewpoint on natural processes. The corresponding mathematical structure of this view allows to associate two-dimensional distributions $\{Z, \zeta\}$ with every space-like vector field Z, carrying information about definite spatial stress in a continuous physical system, and the vector field ζ defining the referent time-process, where ζ may be time-like or null: $\zeta^2 > 0, \zeta^2 = 0$. In this way any intercommunication between any two such two-dimensional distributions can be formulated in terms of the corresponding curvature forms. So, if a time-stable physical object with dynamical structure is represented at every moment by some space-like integrable distribution $\Delta = \{X_1, ..., .X_p\}$, then in view of the time-stability of our physical object, ζ should be considered to define an infinitesimal symmetry of Δ, the ζ-extended distribution $\Delta \oplus \zeta$ will also be integrable, and the dynamical structure of $\Delta \oplus \zeta$, i.e., the various energy-momentum exchanges among any two such subsystems of $\Delta \oplus \zeta$ during its space-time development, can be described in terms of the curvature forms $\Omega_i, i = 1, ..., p$, generated by each such two-dimensional distribution $\{X_i, \zeta\}, i = 1, 2, ..., p$. The very availability of such a possibility allows to consider the concept of *symmetry* as one of great importance, and allows to understand time stability of a physical system with dynamical structure as supported and guaranteed by interaction among its subsystems.

Having in view the above remarks let's turn to the contents of the monograph.

It consists of 14 Chapters, usually ending with exercises and references, and Index.

The first impression one gets when just looking through the pages is that the author does not make use of the widely spread and used nowadays language of modern differential geometry, i.e., the manifold theory. There are no tangent and cotangent bundles, differential forms and exterior derivative, principal and jet bundles, the manifold picture of Lie groups, etc., etc. The author has chosen to represent the subject in the language of the fathers of Lie group theory and its application to symmetries of differential equations. So, the reader will have the chance to feel the spirit of time when new deeper views and creative approaches to mathematical symmetry had been worked out by subtle minds.

In the first two chapters the author introduces the basic notion of symmetry and gives many examples illustrating the main idea: an object, or relation, may have definite invariant properties with respect to some group of transformations, and the purpose is to find them and to further use them appropriately. The main examples are connected with isometries of euclidean, pseudo-euclidean and non-euclidean metrics.

The next Chapter 3 considers mainly the hamilton formulation of classical mechanics of material points, its geometrization and some symplectic aspects.

Chapter 4 introduces one-parameter transformation groups as symmetries of an ordinary differential equation from finite and infinitesimal viewpoints, gives examples, and ends with an appendix on homeomorphisms, diffeomorphisms and topology.

Chapter 5 gives more details on the introduced so far concepts and relations such as transformation of "infinitesimal displacements", contact transformations, commutator of two "operators", invariance of second order ODE.

Chapter 6 introduces many-parameter groups, Lie algebras, Cartan-Killing forms, Casimir operators and gives appropriate examples. There are two appendices on Lie groups defined by PDE, and short classification of Lie groups and algebras.

In the next Chapter 7 the author goes back to hamiltonian mechanics and the accent is on the symplectic structure of the "(P, Q) phase-space", lagrangians and hamiltonian, Poisson brackets, constants of motion, general symplectic forms and their invariance.

Chapter 8 gives more detailed symmetry study of the classical Keplerian problem.

Chapters 9,10,11 and 12 are devoted to symmetry considerations in the frame of quantum mechanics: Schrödinger equation of harmonic oscillator, Lie algebras that "generate continuous and discrete spectra", "dynamical groups" of states generated by various potentials, group symmetry considerations connected with angular momentum, symmetry groups connected with many electron atoms, approximate dynamical symmetries with examples from atomic and molecular physics.

Chapter 13 considers "Rovibronic systems" from group-theoretical viewpoint, the examples make use of $\mathrm{U}(2) \times \mathrm{U}(2), \mathrm{U}(4)$, and $\mathrm{U}(4) \times \mathrm{U}(4)$.

The final Chapter 14 gives a short symmetry consideration of Maxwell equations in non-relativistic terms: Lorentz-Poincare symmetry, conformal symmetry.

In conclusion, the author's exposition of concepts and analysis of examples follows, in my view, some kind of historically motivated view on group-symmetry aspects of some interesting from physical point of view differential equations. The examples considered could help creating careful dealing with this important branch of mathematics in every concrete group-symmetry engaged problem.

I truly hope that reading this book with "pencil in hand" and "sheet of paper" on desk will be of use to all young mathematicians and mathematically inclined young physicists. For those who decide to go further with this subject I would suggest the down cited monographs, some of which are cited also in this book.

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