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Fast and Efficient Restricted Delaunay Triangulation in Random Geometric Graphs

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Abstract. Let $G = \mathcal{G}(n, r)$ be a random geometric graph resulting from placing n nodes uniformly at random in the unit square (or the unit disk) and connecting every two nodes if and only if their Euclidean distance is at most r. Let $r_{\text{con}} = \sqrt{\frac{\log n}{\pi n}}(1 + o(1))$ be the known critical radius for connectivity when $n \to \infty$. The *restricted Delaunay graph* RDG(G) is a subgraph of G with the following properties: it is a planar graph and a spanner of G, and in particular it contains all the short edges of the Delaunay triangulation of G. While in general graphs, the construction of RDG(G) requires $\Theta(n)$ messages, we show that when $r = O(r_{\text{con}})$ and $G = \mathcal{G}(n, r)$, then with high probability, RDG(G) can be constructed locally in one round of communication with $O(\sqrt{n \log n})$ messages, and with only one-hop neighborhood information. This size of r proves that the existence of long Delaunay edges (an order larger than r_{con}) in the unit square (disk) does not significantly affect the efficiency with which good routing graphs can be maintained.

1. Introduction

A random geometric graph is a graph $\mathcal{G}(n, r)$ resulting from placing n points uniformly at random in the unit square and connecting two points if and only if their Euclidean distance is at most r. Such random graphs have long been the subject of study in relation to topics such as statistical physics and hypothesis testing [Penrose 03]. They have currently gained new relevance as a model of random wireless networks, in large part due to advances in the field of sensor networks [Estrin et al. 99, Pottie and Kaiser 00].

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Sensor networks are constructed from a large number of low-cost, low-power sensors equipped with wireless communication and limited processing capabilities. These devices are embedded densely in the environment to create a multihop network in which nodes cooperate to achieve high-level tasks. A wide range of applications of such networks has been offered in the past few years, ranging from environmental and habitat monitoring to disaster management and manufacturing process flow [Culler et al. 84].

Sensor networks carry new design challenges in that the strict energy and memory constraints of the sensors and the large scale of the network require the use of distributed, localized algorithms that minimize memory and energy use [Estrin et al. 99]. Since most of the energy required by sensor networks is consumed by radio communication, the number of messages being sent by a given algorithm is considered as the efficiency metric. Naturally, the above restrictions and the theoretical modeling of random geometric graphs have led to a variety of analytical works aimed at investigating different properties of such networks [Gupta and Kumar 98, Goel et al. 04, Muthukrishnan and Pandurangan 05, Avin and Ercal 05].

The tasks of topological control and routing in sensor networks have been studied extensively, and in particular have led to the advent of *geo-routing* [Bose et al. 99, Karp and Kung 00]. In geo-routing, an assumption is made that each node knows both its own location (i.e., its coordinates) and the location of the destination to which it wants to deliver a message (via Global Positioning System (GPS) at each node or via some other mechanism). The goal then is to find an efficient route from source to destination using only the local information (i.e., the location of its neighbors) available at each node and a limited amount of memory.

Most of the early work on this issue, beginning with the proposals [Bose et al. 99] and [Karp and Kung 00], was based on greedy forwarding combined with *face-routing* over a *planar* subgraph of the network, i.e., the message is always forwarded to the neighbor closest to the destination, and if such a neighbor does not exist, recovery from a local minimum is obtained using a route along the current face of the planar subgraph.

Although this method guarantees delivery, the efficiency of the method depends on the properties of the planar subgraph. Ideally, the subgraph should be sparse and locally constructed, but at the same time it should be a spanner, i.e., the shortest path between any pair of points is at most a constant factor longer than the shortest path in the original graph. The sparseness and locality reduce energy and memory consumption in the construction phase, while the spanner property allows efficient routing.

Several candidate graphs for geo-routing in wireless networks have recently been offered in the literature. Let G be a geometric graph (i.e., the nodes are embedded in the plane). The *relative neighborhood graph* RNG(G) [Toussaint 80], and the *Gabriel graph* GG(G) [Gabriel and Sokal 69] are both planar and can be efficiently constructed locally, but they are not good spanners, even in random geometric graphs [Bose et al. 02]. Another well-known planar graph is the Delaunay triangulation, which is known to be a spanner of the complete graph [Chew 86, Dobkin et al. 90]. Unfortunately, the Delaunay triangulation Del(G) of a geometric graph G cannot be constructed locally and may contain long edges; in other words Del(G) is not necessarily a subgraph of G.

To overcome this problem, several authors have proposed the *restricted Delaunay graph* RDG(G). This is a planar subgraph of G that contains all the edges of Del(G) that are also in G and that has been proved to be a spanner of G [Gao et al. 01, Li et al. 02].

Note that by definition, RDG(G) is not unique, and different methods have been suggested for constructing such graphs [Gao et al. 01, Li et al. 02, Wang and Li 03, Araújo and Rodrigues 05].

In the context of random geometric graphs, we intend to explore the relations among the range of communication r, the number of nodes n in the graph, and some desired property P (for example, connectivity). In ad hoc and sensor networks, interference grows with increased communication radius. It is thus necessary to find a tight upper bound on the smallest radius r_P that will guarantee that P holds with high probability (w.h.p.).[1] For example, the critical radius for connectivity, r_{con}, is of special interest, and it has been shown that if $\pi r^2 \geq \pi r_{\text{con}}^2 = (\log n + \gamma_n)/n$ then $\mathcal{G}(n,r)$ is connected with probability tending to 1 as $n \to +\infty$ if and only if $\gamma_n \to +\infty$ [Penrose 97, Gupta and Kumar 98].

It is well known that the maximum edge length of the Delaunay triangulation of $\mathcal{G}(n,r)$ in the unit square, and in particular on a convex hull, is $\omega(r_{\text{con}})$ (i.e., an order larger than r_{con}). Recently, a similar result has also been proved for the unit disk [Kozma et al. 04]. Therefore, it is clear that when $r = O(r_{\text{con}})$, the Delaunay triangulation cannot be computed locally (i.e., with information obtained only from nodes that are a constant number of hops away).

In this paper, we show that if $r = O(r_{\text{con}})$, namely, of the order that guarantees connectivity, then w.h.p., we can efficiently and locally construct a *restricted Delaunay graph* RDG(G). We show that while for general graphs this construction requires $\Theta(n)$ messages, an order of $O(\sqrt{n \log n})$ messages suffices in $\mathcal{G}(n,r)$. We further present a novel algorithm that achieves this bound. Our algorithm exhibits two unique features that result in a reduced message count. First, the algorithm requires only one round of communication, and second, only "problematic" nodes are required to send messages. Our results are stated for geometric

[1] Event $\mathcal{E}_n$ occurs w.h.p. if probability $P(\mathcal{E}_n)$ is such that $\lim_{n\to\infty} P(\mathcal{E}_n) = 1$.

graphs that have some nice properties, but are not necessarily random or in a specific bounded area (i.e., square, disk). Later, we show that *random geometric graphs* in the unit square (or unit disk) have these nice properties with high probability and the results follow.

2. Preliminaries

We consider a wireless ad hoc network (or sensor network) over a set V of n nodes distributed in the unit square, where each node can communicate with all the nodes in its transmission range, i.e., a disk of radius r centered at the node. The resulting graph is a *geometric graph* $G = G(V, r)$ with V the set of nodes and $E = \{\{u, v\} \mid u, v \in V \wedge d(u, v) \leq r\}$ the set of edges. This graph is similar to the *unit disk graph* $\mathrm{UDG}(V)$ [Clark and Colbourn 91], in which the set of nodes V is in $\mathbb{R}^2$ and the radius is assumed to be one unit, but in our case we are interested in a network in a bounded area and in the relation between the number of nodes n and the transmission range r as a function of n.

Let $N(u)$ denote the neighbors of u, including u, and $N(u, v)$ the set of the common neighbors of u and v, i.e., $N(u, v) = N(v, u) = N(u) \cap N(v)$. Throughout the paper, we use three disk definitions: let $\mathrm{disk}_r(v)$ be the disk centered around v with radius r (with r omitted when the context is clear), $\mathrm{disk}(u, v)$ the disk through u, v with diameter $d(u, v)$, and $\mathrm{disk}(u, v, w)$ the unique circumcircle over u, v, and w.

Next, we present additional graphs over the set of nodes V. Note that in some cases the graphs are derived directly from V, while others are a function of G (i.e., r is needed to compute them). The *Voronoi diagram* $\mathrm{Vor}(V)$ of a set of nodes (or sites) V in space is the partition of space into cells V_u, $u \in V$, such that all the points inside V_u are closer to u than to any other node in V. The *Delaunay triangulation* $\mathrm{Del}(V)$, is the dual graph of $\mathrm{Vor}(V)$: an edge $\{u, v\}$ is in $\mathrm{Del}(V)$ if and only if V_u and V_v share a common boundary.

It is well known that $\mathrm{Del}(V)$ is a spanner of the complete graph K_n [Chew 86, Dobkin et al. 90], which means that the shortest path between any two points on $\mathrm{Del}(V)$ is at most t times the shortest path on K_n, where t is a positive constant known as the *stretch factor*. In the case of $\mathrm{Del}(V)$ and the complete graph, $t \approx 5.08$.

A useful property of the Delaunay triangulation is that a triangle $\triangle uvw$ is in $\mathrm{Del}(V)$ if and only if $\mathrm{disk}(u, v, w)$ is empty, i.e., there is no other node from V in it, where for simplicity, we assume that no four points in V are cocircular [de Berg et al. 97].

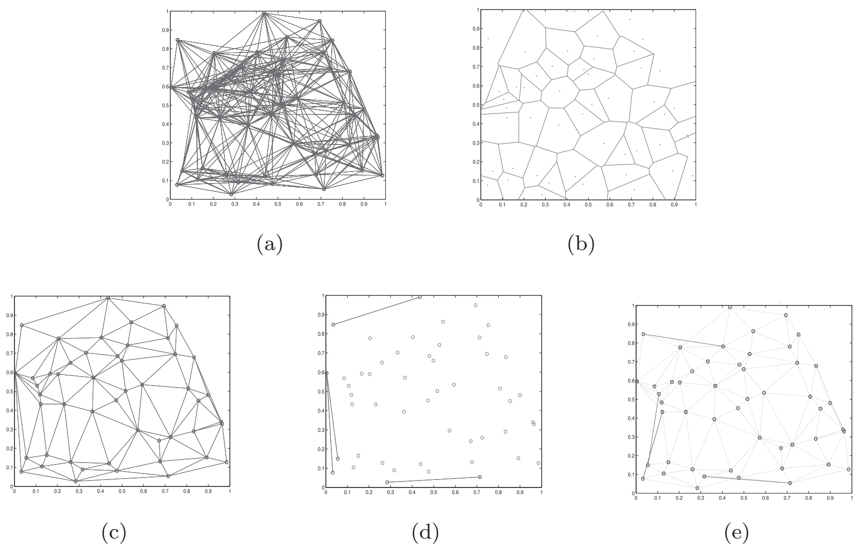

Figure 1. Various graphs over a set V of 50 random nodes in the unit square with $r = 0.3$: (a) $G(V, r)$; (b) $\mathrm{Vor}(V)$; (c) $\mathrm{Del}(V)$; (d) the edges in $\mathrm{Del}(V)$ that are longer than r; (e) $\mathrm{LocalDel}(G)$ where consistent edges are shown as dots and inconsistent edges that cross edges are shown in solid lines (these types of edges are defined formally in Definition 5.2).

Let $\mathrm{UDel}(G)$ be the subgraph of $\mathrm{Del}(V)$ that contains only the short edges of $\mathrm{Del}(V)$, that is, the edges that are shorter than r; therefore, $\mathrm{UDel}(G)$ is equal to $\mathrm{Del}(V) \cap G$ and is also a subgraph of G [Li et al. 02, Gao et al. 01].

Definition 2.1. [Gao et al. 01] A *restricted Delaunay graph* $\mathrm{RDG}(G)$ is a planar graph such that

$$\mathrm{UDel}(G) \subseteq \mathrm{RDG}(G) \subseteq G.$$

Let $T(u)$ be the set of edges in $\mathrm{Del}(N(u))$ (i.e., the Delaunay triangulation of the nodes in $N(u)$) and similarly $T(u, v) = \mathrm{Del}(N(u, v))$. Note that there may be edges in $T(u)$ and $T(u, v)$ that are not present in $\mathrm{Del}(G)$.

Definition 2.2. The graph $\mathrm{LocalDel}(G)$ is the graph resulting from computing $T(u)$ at each node: an edge u, v is in $\mathrm{LocalDel}(G)$ if and only if there exists u such that $\{u, v\} \in T(u)$.

Figure 1 illustrates the graphs discussed above for a set V of 50 random points in the unit square.

3. Related Work

3.1. Unit Disk Graphs

The *Gabriel graph* GG(G) [Gabriel and Sokal 69] is a graph in which there is an edge $\{u, v\}$ if and only if there is no other node in disk(u, v). Bose et al. offered a distributed local algorithm to construct the Gabriel graph over a wireless network and then used face-routing to guarantee message delivery [Bose et al. 99]. Later, Bose and Morin considered different face-routing methods in triangulation and in particular in the Delaunay triangulation [Bose and Morin 99]. Karp and Kung independently proposed a greedy perimeter stateless routing (GPSR) algorithm, a memoryless routing algorithm that combines greedy forwarding and local minimum recovery and is based on face-routing over the Gabriel graph [Karp and Kung 00]. Subsequently, later work aimed at finding better planar graphs that can be constructed locally.

Gao et al. proposed the use of a restricted Delaunay graph RDG(G), a graph that contains all the short edges of the Delaunay graph and is also planar [Gao et al. 01]. They proved that RDG(G) is a Euclidean spanner of the unit disk graph G and presented an algorithm to construct it, which can in general be inefficient with $O(n^2)$ messages. Similarly, Li et al. proved that UDel(G) is a spanner of the unit disk graph G and offered a local algorithm to build a planar supergraph of UDel(G),[2] known as PLDel(G), in $\Theta(n)$ messages and $\Theta(n \log n)$ bits [Li et al. 02].

They presented yet another graph, $\mathrm{LDel}^{(k)}(G)$, a local Delaunay triangulation in which the circumcircle of u, v, w does not contain any node that is k hops away from u, v, or w. The authors proved that $\mathrm{LDel}^{(k)}(G)$, where $k \geq 1$, is a supergraph of UDel(G) and a subgraph of $\mathrm{LDel}^{(k+1)}(G)$ and therefore a spanner. In addition, they showed that for $k = 1$, $\mathrm{LDel}^{(k)}(G)$ is not planar, but for $k > 1$ it is.

Recently, Wang and Li showed how to bound the maximum degree of such graphs, since PLDel(G), or in general UDel(G), is not a graph of bounded degree [Wang and Li 03]. Arajo and Rodrigues reduced the number of steps in [Li et al. 02], but their algorithm still has the same order of messages, $\Theta(n)$ [Araújo and Rodrigues 05].

All the above algorithms are nonadaptive, i.e., in some cases they send unnecessary messages. Essentially, they require each node u to broadcast all the triangles in $T(u)$ with $\measuredangle wux \geq \pi/3$. Since the total number of such triangles (faces) in the above graphs is linear, all the algorithms require a linear number of messages.

[2] RDG(G) in the notation of [Gao et al. 01].

3.2. Random Geometric Graphs

Bose et al. proved (among other results) that the Gabriel graph is not a spanner of the unit disk graph and that in the worst case, its stretch factor is $\Theta(\sqrt{n})$ [Bose et al. 02]. Moreover, they also proved that for random geometric graphs in the unit square, the stretch factor of the Gabriel graph is w.h.p.

$$\Theta\left(\sqrt{\frac{\log n}{\log\log n}}\right),$$

which proves its inefficiency for face-routing in random networks.

Kozma et al. bounded the longest edge of $\mathrm{Del}(G)$ in a random geometric graph in the unit disk [Kozma et al. 04]. They showed that due to boundary effects, the longest edge is of order $\Theta(\sqrt[3]{\log n/n})$, an order larger than r_{con}, and they left open the question of an algorithm for the case $r = O(r_{\mathrm{con}})$.

Bern et al. proved that the Delaunay triangulation of a uniform set of points does not have bounded degree and that the maximum degree grows according to $\Theta(\log n/\log\log n)$ [Bern et al. 91]. In particular, they showed that this does not happen next to the boundary. Since the higher the degree, the greater the load imbalance, one wants a constant-degree planar graph; in our case, the algorithm we offer does not solve this problem.

4. Computing $\mathbf{RDG}(\mathcal{G}(n,r))$

We offer an efficient algorithm to construct $\mathrm{RDG}(\mathcal{G}(n,r))$. There are several advantages to our algorithm: First, it assumes that the transmission range is of the same order as the necessary range required for connectivity; second, there is only one round of communication; and most importantly, the number of messages it sends is adaptive. Our algorithm is based on $\mathrm{LocalDel}(G)$, which is known not to be a planar graph. In the past, some proposals have been put forward to solve this problem, but all of them require a constant number of messages per node. In our algorithm, only messages that are needed to eliminate problematic edges are sent, thereby enabling us to reduce the number of messages from $\Theta(n)$ to $O(\sqrt{n\log n})$. Before describing the algorithm, we define the notion of local inconsistency.

Definition 4.1. An edge $\{u,v\}$ is *locally inconsistent* at u if $\{u,v\} \notin T(u)$ and $\{u,v\} \in T(u,v)$, and it is *locally consistent* otherwise.

The nice property of locally consistent edges is that they can be observed locally.

Algorithm 1. (Local RDG(G) construction at node u.)

1. Compute $T(u)$ and for each neighbor $v \in N(u)$ compute $T(u, v)$.
2. Keep all edges $\{u, v\} \in T(u)$.
3. If there are locally inconsistent edges, broadcast proofs for each of them to neighbors.
4. Remove edge $\{u, v\}$ if a proof of its inconsistency is received.

Lemma 4.2. *For each locally inconsistent edge $\{u, v\} \in T(u, v)$ at u there is a triangle $\triangle uwx \in T(u)$ that is the* proof *that $\{u, v\}$ is locally inconsistent, i.e., $\{w, x\}$ intersects $\{u, v\}$.*

Proof. Let w be the node such that the edge $\{u, w\} \in T(u)$ is the first edge in $T(u)$ clockwise to $\{u, v\}$. Similarly, let x be the node such that the edge $\{u, x\} \in T(u)$ is the first edge counterclockwise to $\{u, w\}$. Now, since $\{u, v\} \notin T(u)$, the edge $\{w, x\}$ does not intersect any edge in $T(u)$ (by the angle minimality of $\{u, w\}$ and $\{u, x\}$), and therefore it must be in $T(u)$. Note that $\{w, x\}$ must intersect $\{u, v\}$, and we are done. $\square$

The main results of this paper are the following theorems about the correctness and the number of messages in Algorithm 1.

Theorem 4.3. *For $r \geq \sqrt{128} r_{\mathrm{con}}$, w.h.p. Algorithm 1 computes* $\mathrm{RDG}(\mathcal{G}(n, r))$.

Theorem 4.4. *For $r \geq \sqrt{128} r_{\mathrm{con}}$, w.h.p. the number of messages in Algorithm 1 is $O(\sqrt{n \log n})$, and the number of bits is $O(\sqrt{n} (\log n)^{3/2})$.*

To prove these theorems, we next establish a few lemmas.

5. Properties of LocalDel(G)

The graph LocalDel(G) can be constructed locally without exchanging messages on the assumption that each node knows the locations of all its neighbors. Here, we assume that this information is obtained by each node using some other mechanism that is shared with other applications. We therefore do not count the messages required in this process as part of our algorithm (otherwise $\Omega(n)$

messages are necessary for any nontrivial task) and consider only algorithm-specific messages. Next, we prove some properties of LocalDel(G) that are based on the following proposition.

Proposition 5.1. *Let $V' \subseteq V$. Then $(u, v \in V' \wedge \{u, v\} \in \mathrm{Del}(V)) \Rightarrow \{u, v\} \in \mathrm{Del}(V')$.*

This result clearly follows from the fact that if two Voronoi cells V_u and V_v share a boundary in Vor(V), they must share a boundary in Vor(V'), since removing nodes cannot decrease their boundary.

Definition 5.2. An edge $\{u, v\}$ in LocalDel(G) is *consistent* if $\{u, v\} \in T(u)$ and $\{u, v\} \in T(v)$, and *inconsistent* otherwise.

Lemma 5.3. *If $\{u, v\} \in \mathrm{UDel}(G)$, then $\{u, v\}$ is a consistent edge in* LocalDel(G).

Proof. The assertion of the lemma follows directly from Proposition 5.1. Since $\{u, v\} \in \mathrm{UDel}(V)$, we have $\{u, v\} \in \mathrm{Del}(V)$, $u \in N(v)$, and $v \in N(u)$. Setting $V' = N(u)$, we get $\{u, v\} \in T(u) = \mathrm{Del}(V')$, and similarly, for $V' = N(v)$, we get $\{v, u\} \in T(v) = \mathrm{Del}(V')$. □

It is clear from Lemma 5.3 that UDel(G) $\subseteq$ LocalDel(G), but it is still not RDG(G), since it may be not a planar graph. There are two types of edges in LocalDel(G), consistent and inconsistent, and both may cross other edges. First, we take care of the inconsistent edges.

Lemma 5.4. *An edge $\{u, v\} \in \mathrm{LocalDel}(G)$ is inconsistent if and only if $\{u, v\}$ is locally inconsistent at u or v.*

Proof. $\Rightarrow$: Assume that edge $\{u, v\} \in \mathrm{LocalDel}(G)$ is inconsistent. Recall that an edge $\{u, v\}$ is in LocalDel(G) if and only if it is in $T(u)$ or $T(v)$. But if it is also inconsistent, it cannot be in both. Without loss of generality, let $\{u, v\} \in T(u)$ and $\{v, u\} \notin T(v)$. By Proposition 5.1, $\{u, v\} \in T(u, v) = T(v, u)$, so $\{v, u\}$ must be locally inconsistent at v.

$\Leftarrow$: By Proposition 5.1, if an edge is consistent, it must be locally consistent at u and v. □

Next, we bound the number of proofs that each node can have for its inconsistent edges.

Lemma 5.5. *A node can have at most six proofs for all its locally inconsistent edges in* LocalDel(G).

Proof. A triangle $\triangle uwx \in T(u)$ with $\measuredangle wux \leq \pi/3$ cannot be a proof for a locally inconsistent edge $\{u, v\}$, since v must then be a neighbor of w and x, and $\mathrm{Del}(N(u))$ and $\mathrm{Del}(N(u, v))$ agree on $\{u, v\}$ and $\{w, x\}$. $\square$

5.1. Well-Distributed Geometric Graphs

We now turn to a more specific type of geometric graph. First, let us define them formally:

Definition 5.6. A geometric graph $G(V, r)$ is *well distributed* if every circle of area at least $(\pi/64)r^2$ (in the unit square) has at least one node in it.

In these graphs the nodes are distributed "nicely" across the unit square and in particular do not contain large "holes," i.e., empty circles of area larger than $(\pi/64)r^2$.

Lemma 5.7. *If $G(V, r)$ is a well-distributed geometric graph, then consistent edges do not intersect in* $\mathrm{LocalDel}(G)$.

Proof. Assume that $\{u, v\}$ and $\{w, x\}$ are two consistent edges that intersect in $\mathrm{LocalDel}(G)$. By Proposition 5.1 we can remove all nodes but u, v, w, and x from the graph and consider only the two edges that must still exist and intersect. We use [Gao et al. 01, Lemma 4.1], which states that if two edges cross, then one of the four nodes must be a neighbor of each of the other three. Without loss of generality let it be w. We now claim that both u and v are not neighbors of x.

Assume, by contradiction, that u is a neighbor of x. But in this case, since w and u see all four nodes, $T(w) = T(u)$ and either $\{w, x\}$ and $\{u, v\}$ do not intersect or at least one of them is inconsistent, which leads to a contradiction. The same is true for v. Note that since w selected $\{w, x\}$ as an edge while having information on the four nodes, $\{u, v\}$ is the non-Delaunay edge of the two edges.

Observe that x must be outside $\mathrm{disk}(u) \cup \mathrm{disk}(v)$ (otherwise, u or v sees the four nodes), so it must be the case that $d(w, x) \geq (\sqrt{3}/2)r$, since $d(u, v)$ is at most r and the edges intersect by assumption (see Figure 2). Note also that since u and v choose $\{u, v\}$ as an edge in $\mathrm{LocalDel}(G)$, the circumcircle $\mathrm{disk}(u, v, w) \cap (\mathrm{disk}(u) \cup \mathrm{disk}(v))$ must be empty.

In particular, this implies that the disk D of diameter $(\sqrt{3}/2)r$, which is tangent to the midpoint between u and v, is empty (see the gray disk in Figure 3).

Since w, x, u, v are all in the unit square, it must be the case that at least half of D is also inside the unit square. This half of D contains a circle D' of radius

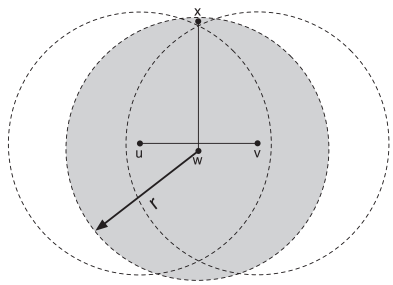

Figure 2. A case in which edges $\{w, x\}$ and $\{u, v\}$ are consistent and intersect in LocalDel(G).

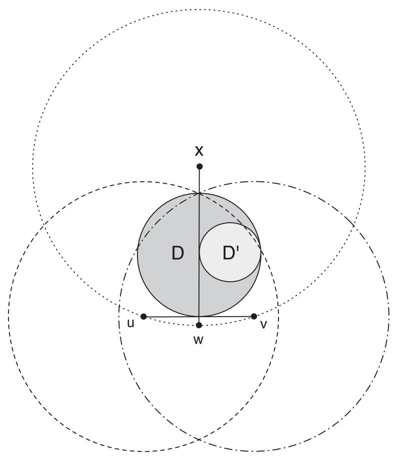

Figure 3. A disk D' that must be included in the area of disk$(u, v, w) \cap ($disk$(u) \cup$ disk$(v))$.

$(\sqrt{3}/8)r$ and has area $(3\pi/64)r^2$. Since G is well distributed, there is at least one node in D', contradicting the consistency of $\{u, v\}$ in LocalDel(G). $\square$

This lemma is at the core of our algorithm. For a well-distributed graph G, all one needs to do to compute RDG(G) is to remove all inconsistent edges. Note, however, that even for well-distributed G, there may be inconsistent edges in LocalDel(G). As Figure 4 illustrates, an edge $\{u, v\}$ can be inconsistent at v, since the area of disk$(v) \cap$ disk(u, v, w) (the gray area in the figure) can become arbitrarily small next to the boundaries of the unit square. (For a similar reason, it can be shown that long Delaunay edges can exist in Del(G), and in particular on the convex hull of V.)

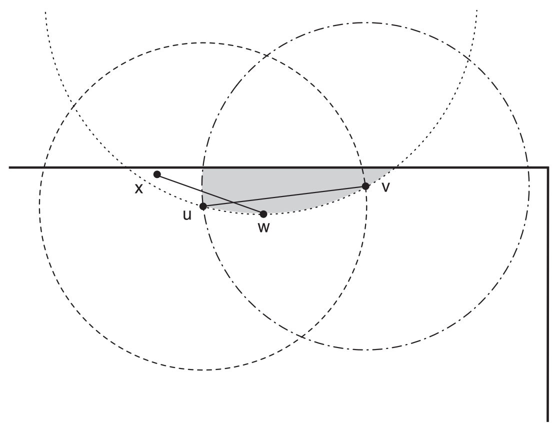

Figure 4. An example in which an inconsistent edge $\{u, v\}$ exists next to the border of the unit square.

Before formally proving the correctness of Algorithm 1, we need to show that random geometric graphs are well distributed. We will do so by utilizing a coupon-collector argument.

Lemma 5.8. *If $r \geq \sqrt{128} r_{\text{con}}$, then w.h.p. $\mathcal{G}(n, r)$ is well distributed.*

Proof. We first claim that if we partition the unit square into bins of size $(\pi/128)r^2$, every bin will have w.h.p. at least one node. We then show that every circle of area $(\pi/64)r^2$ contains at least one such bin. Recall that $r_{\text{con}}^2 = (\log n + \gamma_n)/\pi n$ and $\gamma_n \to \infty$. Partition the unit square into square bins of size $(\pi/128)r^2$, so that the number of bins is at most B, where

$$B = \frac{128}{\pi r^2} = \frac{128}{\pi} \frac{\pi n}{128(\log n + \gamma_n)} = \frac{n}{\log n + \gamma_n}.$$

It is a known result [Motwani and Raghavan 95] that if one throws balls uniformly at random into B bins, the expected number of balls needed to fill every bin with at least one ball is $B \log B$. If we require the result w.h.p., then we need to throw at least $B \log B + \gamma_n B$ balls [Mitzenmacher and Upfal 05]. To conclude the first claim and to prove that every bin has at least one node w.h.p., we need to show that $n \geq B \log B + \gamma_n B$:

$$\begin{aligned} B\log B + \gamma_n B &= \frac{n}{\log n + \gamma_n}\log\left(\frac{n}{\log n + \gamma_n}\right) + \gamma_n \frac{n}{\log n + \gamma_n} \\ &= n\left(\frac{\log n}{\log n + \gamma_n} - \frac{\log(\log n + \gamma_n)}{\log n + \gamma_n} + \frac{\gamma_n}{\log n + \gamma_n}\right) \\ &= n(1 - \frac{\log(\log n + \gamma_n)}{\log n + \gamma_n}) \le n. \end{aligned}$$

Now we consider a circle of area $(\pi/64)r^2$ in the unit square. Consider the bin that contains the center of the circle. Since the radius of the circle is $\frac{1}{8}r$ and the side of the bin is $(1/8\sqrt{2})r$, the whole bin must be inside the circle, and the claim follows. □

Now we can proceed to prove the correctness of Algorithm 1.

Proof of Theorem 4.3. From the last lemma, for $r \ge \sqrt{128}r_{\mathrm{con}}$, w.h.p. $\mathcal{G}(n,r)$ is well distributed. Steps 1 and 2 compute a subgraph of G. In step 4, the algorithm removes all locally inconsistent edges, which by Lemma 5.4 is equivalent to removing all inconsistent edges.

After step 4, the resulting graph contains only consistent edges, and so by Lemma 5.3, it is a supergraph of UDel(G). Lemma 5.7 guarantees that the graph is also a planar graph (since consistent edges do not intersect in well-distributed graphs), so we get an RDG(G). □

5.2. Bounding the Number of Messages

Let $I = [r/2, 1 - r/2]^2$ be the inner square centered in the unit square such that each side of I is at distance $r/2$ from the side of the unit square. For a well-distributed G we have the following result:

Lemma 5.9. *If $u \in I$ and $\{u,v\} \in \mathrm{LocalDel}(G)$, then $d(u,v) < r/2$.*

Proof. Let $\{u,v\} \in \mathrm{LocalDel}(G)$ and $u \in I$. Assume $d(u,v) \ge r/2$. Then each half of disk(u,v) contains a disk of size at least $(\pi/64)r^2$ that is completely inside the unit square. Since G is well distributed, each such half contains at least one node, so $\{u,v\}$ cannot be an edge in LocalDel(G), which is a contradiction. □

Lemma 5.10. *If $\{u,v\} \in \mathrm{LocalDel}(G)$ and $u,v \in I$, then the edge $\{u,v\}$ is consistent.*

Proof. Let $\{u,v\} \in \mathrm{LocalDel}(G)$ and $u,v \in I$. Assume that $\{u,v\}$ is inconsistent and without loss of generality assume that it is locally inconsistent at u. Then u must have a proof for the inconsistency; let it be $\triangle uwx \in T(u)$. Since $u \in I$,

w and x are at most at distance $r/2$ from u. Since $d(u, v)$ is also less than $r/2$, both x and w are in $N(u, v)$, and $\triangle uwx$ cannot be a proof. Contradiction. $\square$

Now we can also prove the upper bounds on the number of messages.

Proof of Theorem 4.4. As before, $\mathcal{G}(n, r)$ is well distributed w.h.p. There is only one step of communication, and messages are sent only from nodes with locally inconsistent edges. From Lemma 5.10 only edges $\{\{u, v\} \mid u, v \notin I\}$ can be inconsistent. The result follows, since there are $\Theta(rn) = \Theta(\sqrt{n \log n})$ nodes outside I, and each sends at most six proofs, where the size of each proof is $\Theta(\log n)$, since it reports the tree nodes belonging to a triangle. $\square$

6. Conclusions

In this paper we have offered a novel local algorithm to construct a planar spanner graph in random wireless networks. Previous algorithms for computing restricted Delaunay graphs send a message for each triangle in the restricted Delaunay graph, and in particular via the node with the largest angle. In contrast, our algorithm avoids sending unnecessary messages far from the boundary and thus reduces the total number of messages from $\Theta(n)$ to $O(\sqrt{n \log n})$. Moreover, our results are stated in terms of well-distributed graphs, deterministic or random, and can thus be applied to more general graphs than those discussed here.

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