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Percolation on Sparse Random Graphs with Given Degree Sequence

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Abstract. We study the two most common types of percolation processes on a sparse random graph with a given degree sequence. Namely, we examine first a bond percolation process where the edges of the graph are retained with probability p, and afterwards we focus on site percolation where the vertices are retained with probability p. We establish critical values for p above which a giant component emerges in both cases. Moreover, we show that, in fact, these coincide. As a special case, our results apply to power-law random graphs. We obtain rigorous proofs for formulas derived by several physicists for such graphs.

1. Introduction

Traditionally percolation theory has been the study of the properties of a random subgraph of an infinite graph, which is obtained by deleting each edge of the graph with probability $1-p$ for some $p \in (0,1)$ independently of every other edge. The question that has been mainly investigated is whether the subgraph that is spanned by the remaining edges has an infinite component or not. The classical type of graphs that was studied in percolation theory is the lattice $\mathbb{Z}^d$ in various dimensions $d \geq 2$ (see [Grimmett 99]). Various other types of lattices have also been studied. In each of the above cases the main problem is the calculation of a critical p_c so that if $p < p_c$ then the random subgraph obtained as above has no infinite components, whereas if $p > p_c$ there is an infinite component with probability 1.

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In the present work, we study percolation on finite graphs that have a large number of vertices. This problem is old, in the sense that, for example, a $\mathcal{G}_{n,p}$ random graph is a random subgraph of the complete graph on n vertices, where each edge appears with probability p independently of every other edge. In this context, a question about the appearance of an infinite component is senseless. A somehow analogous question is whether there exists a component of the random subgraph containing a certain proportion of the vertices or, as we customarily say, a *giant component*. More specifically, if the initial graph has n vertices, the question now is whether there exists an $\varepsilon > 0$ for which there is a component of the random subgraph that has at least εn vertices with probability $1 - o(1)$ (as $n \to \infty$). Hence, we also ask (quite informally) for the existence of a critical p_c for which (a) whenever $p < (1-\delta)p_c$ then for every $\varepsilon > 0$ there is no component having at least εn vertices with probability $1 - o(1)$ and (b) whenever $p > p_c(1+\delta)$ then there is a component with at least εn vertices for some $\varepsilon > 0$ with probability $1 - o(1)$. A classical example of this is the $\mathcal{G}_{n,p}$ random subgraph of K_n, the complete graph on n vertices, where as it was proved by Erdős and Rényi [Erdős and Rényi 60] the critical probability is equal to $1/n$ (see also [Bollobás 01] or [Janson et al. 00] for an extensive discussion).

More generally, Bollobás et al. [Bollobás et al. 92] raised the following question: given a sequence of graphs $\{G_n\}$ with order that tends to infinity as n grows, is there such a phase transition? Assume that G_n has $|G_n|$ vertices and e_n edges. For each such n we have a probability space on the set of spanning subgraphs of G_n, and the probability of such a subgraph of G_n that has e edges is $p^e(1-p)^{e_n-e}$, where e_n is the number of edges of G_n. Let $G_n(p)$ be a sample from this probability space. Thus, we are seeking a p_c such that (a) if $p < (1-\delta)p_c$, then for every $\varepsilon > 0$ as $n \to \infty$ all the components of $G_n(p)$ have at most $\varepsilon|G_n|$ vertices with probability $1 - o(1)$, and (b) if $p > p_c(1+\delta)$, then there exists $\varepsilon = \varepsilon(p) > 0$ for which the largest component of $G_n(p)$ has at least $\varepsilon|G_n|$ vertices with probability $1 - o(1)$. If the sequence of graphs is $\{K_n\}$, this is simply the case of a $\mathcal{G}_{n,p}$ random graph.

Other families of sequences have also been studied. For example, percolation on the hypercube with 2^n vertices has been analyzed by Ajtai et al. where the critical edge probability turns out to be also equal to $1/n$ [Ajtai et al. 82]. More detailed analysis of this phase transition was carried out recently by Borgs et al. [Borgs et al. 06].

On the other hand, recent research has also focused on finite graphs with bounded maximum degree. Here we consider sequences of graphs $\{G_n\}$, where each G_n is a graph on n vertices, with uniformly bounded maximum degree. Alon et al. have investigated percolation on such sequences of graphs [Alon et al. 04]. Among other things, they proved that the critical probability for the emergence of a component of linear size in a d-regular graph on n vertices whose girth tends to infinity with n is $1/(d-1)$ [Alon et al. 04, Theorem 3.2]. Phase transitions on specific sequences of finite graphs were studied more closely by Borgs et al. [Borgs et al. 05a, Borgs et al. 05b].

More recently, Bollobás et al. analyzed the phase transition in sequences of dense graphs that are convergent in a certain sense [Bollobás et al. 07].

Also, Frieze et al. have proved that $p_c = 1/d$ for sequences of d-regular graphs on n vertices which are quasi-random, when $d \to \infty$ as n grows [Frieze et al. 04]. These are graphs whose structures resemble that of a d-regular random graph.

In the present paper, we determine a percolation threshold in the case where the sequence $\{G_n\}_{n\in\mathbb{Z}^+}$ is a sequence of sparse random graphs on n vertices. In particular, for every integer $n \geq 1$, G_n is a uniformly random graph on the set $V_n = \{1, \ldots, n\}$ having a given degree sequence $\mathbf{d}(n) = (d_1, \ldots, d_n)$, i.e., for $i = 1, \ldots n$ vertex i has degree d_i. More formally, a *degree sequence* on the set V_n is a vector $\mathbf{d} = (d_1, \ldots, d_n)$ consisting of natural numbers, where $d_1 \leq \cdots \leq d_n$, and $\sum_{i=1}^n d_i$ is even. We let $2M$ denote this sum, and $M = M(n)$ is the number of edges that $\mathbf{d}$ spans. For a given $\mathbf{d} = \mathbf{d}(n)$, if $\mathbf{d}(n) = (d_1, \ldots, d_n)$ for $n \in \mathbb{Z}^+$, we set $D_i = D_i(n) = |\{j \in V_n \ : \ d_j = i\}|$, for $i \in \mathbb{N}$ and $\Delta = \Delta(n) = \max_{1\leq i\leq n}\{d_i\} = d_n$. Finally, if G is a graph on V_n, then $D(G)$ denotes its degree sequence.

An *asymptotic degree sequence* is a sequence $(\mathbf{d}(n))_{n\in\mathbb{Z}^+}$, where for each $n \in \mathbb{Z}^+$ the vector $\mathbf{d}(n)$ is a degree sequence on V_n. An asymptotic degree sequence is *sparse* if for every $i \in \mathbb{N}$ we have $\lim_{n\to\infty} D_i(n)/n = \lambda_i$, for some $\lambda_i \in [0, 1]$, where $\sum_{i\geq 0} \lambda_i = 1$, and moreover

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i\geq 1} i(i-2)D_i(n) = \sum_{i\geq 1} i(i-2)\lambda_i < \infty.$$

This implies that for every $\varepsilon > 0$ there exists $i^*(\varepsilon)$ and $N = N(\varepsilon)$ such that for every $n > N$ we have

$$\left| \frac{1}{n} \sum_{i\leq i^*(\varepsilon)} i(i-2)D_i(n) - \sum_{i\geq 1} i(i-2)\lambda_i \right| < \varepsilon. \tag{1.1}$$

The *generating polynomial* of a sparse asymptotic degree sequence is defined as $L(s) := \sum_{i=0}^{\infty} \lambda_i s^i$. We assume that every asymptotic degree sequence $(\mathbf{d}(n))_{n\in\mathbb{Z}^+}$ with which we work is such that for every n the set of simple graphs that have $\mathbf{d}(n)$ as their degree sequence is nonempty.

Let $(\mathbf{d}(n))_{n\in\mathbb{Z}^+}$ be a sparse asymptotic degree sequence. For every $n \geq 1$, we let G_n be a uniformly random graph on V_n with degree sequence $\mathbf{d}(n)$ and we consider two types of percolation. First, for some $p \in (0, 1)$, each edge of G_n is present with probability p independently of every other edge. This type of percolation is usually called *bond percolation*, in that we randomly delete the edges (i.e., the bonds) of G_n. This is distinguished from another type of percolation called *site percolation*. Here, we go through the vertices of G_n, and we make each of them isolated with probability $1 - p$ independently of every other vertex (or as we say we delete this vertex). The random subgraph in this case is the spanning subgraph of G_n that does not contain the edges that are attached to the vertices that were deleted. The terms *bond* and *site* percolation were originally used in the context of percolation processes on infinite graphs (see [Grimmett 99] for an extensive discussion on both types as well as the references therein).

We shall now define the percolation threshold in each of the above cases. Let $G'_n(p)$ denote the random subgraph that is obtained in either case and let $L_1(G'_n(p))$ be the largest component of $G'_n(p)$. (In the case that there are at least two largest components, we choose the one whose smallest vertex is smaller than the smallest vertex of every other component of maximum order—the comparison between the vertices is by means of the total ordering on V_n.) Starting from the bond percolation we set

$$p_c^{\text{bond}} := \sup \left\{ p \in [0, 1] : \ \frac{|L_1(G'_n(p))|}{n} \xrightarrow{p} 0 \text{ as } n \to \infty \right\},$$

where the symbol $\xrightarrow{p}$ denotes convergence in probability, i.e., we say that $X_n \xrightarrow{p} 0$ if for every $\varepsilon > 0$ we have $\mathbb{P}[|X_n| > \varepsilon] \to 0$ as $n \to 0$. The convergence in probability is meant with respect to the sequence of probability spaces indexed by the set $\mathbb{Z}^+$, where for each $n \in \mathbb{Z}^+$ the probability of a certain spanning subgraph is the probability that this is the subgraph that is spanned by the edges that survive the random deletion of the edges of the random graph G_n. Similarly, in the case of site percolation, we define

$$p_c^{\text{site}} := \sup \left\{ p \in [0, 1] : \ \frac{|L_1(G'_n(p))|}{n} \xrightarrow{p} 0 \text{ as } n \to \infty \right\},$$

where $G'_n(p)$ is now the spanning subgraph of G_n that is the outcome of the deletion of those edges that attached to the deleted vertices, i.e., the vertices that we make isolated. Note that in both cases there are two levels of randomness.

These definitions of the critical probabilities are, generally speaking, coarse in that they do not capture critical probabilities that converge to 0. For example, the critical probability in an Erdős-Rényi random graph (that is, bond percolation on K_n) or the critical probability in the case of bond percolation on the hypercube with 2^n vertices are both equal to 0 under our definition. The actual critical probabilities are roughly equal to the inverse of the average degree of the graph to which the percolation processes are applied. However, our definition is sufficient for the class of random graphs that we are considering, since these have constant average degree and, furthermore, it allows for a cleaner statement of our main result.

Goerdt proved that if G_n is a random d-regular graph on V_n, for some fixed $d \geq 3$, then $p_c^{\text{bond}} = 1/(d-1)$ [Goerdt 01]. (Nachmias and Peres recently considered this problem when the retainment probability p is close to the critical value $1/(d-1)$ and determined the critical window of the phase transition [Nachmias and Peres 09].) Before this, bond percolation in random regular graphs was studied by Nikoletseas et al., who proved that the critical probability is at most $32/d$, for d large enough [Nikoletseas and Spirakis 95]. Also, Nikoletseas and Spirakis studied the edge expansion properties of the giant component that remains after the edge deletion process [Nikoletseas et al. 94]. However, these papers did not provide any analysis of the site percolation process. Our main theorem involves the latter and is stated as follows.

Theorem 1.1. *If $(\mathbf{d}(n))_{n\in\mathbb{Z}^+}$ is a sparse asymptotic degree sequence of maximum degree $\Delta(n) \leq n^{1/9}$ and $L(s)$ is its generating polynomial that is twice differentiable at 1 and moreover $L''(1) > L'(1)$, then*

$$p_c^{\text{site}} = p_c^{\text{bond}} = \frac{L'(1)}{L''(1)}.$$

Moreover, whenever $p > p_c^{\text{bond}}$ (respectively $p > p_c^{\text{site}}$), there is an $\varepsilon > 0$ such that $|L_1(G'_n(p))| > \varepsilon n$ with probability $1 - o(1)$.

The formula for both critical probabilities was obtained by Dorogovtsev and Mendes using qualitative (that is, nonrigorous) arguments [Dorogovtsev and Mendes 02]. Recently, Janson extended the study of percolation processes on such random graphs proving, among other things, precise results about the order of the giant component, its degree sequence as well as the number of its edges [Janson 09].

To make the statement of Theorem 1.1 slightly clearer, let us consider the case of bond percolation (the case of site percolation is similar). Let $\mathcal{G}(n)$ be the set of graphs on V_n with degree sequence $\mathbf{d}(n)$. Each graph $G \in \mathcal{G}(n)$ gives rise to a probability space that consists of all its spanning subgraphs. In particular, if G has e edges and G' is a spanning subgraph of G that has $e' \leq e$ edges, then its probability is $p^{e'}(1-p)^{e-e'}$; let $\mathbb{P}_p^G[\cdot]$ denote this measure. In other words, this space accommodates the outcomes of the bond percolation process applied to G, and we call it the *percolation space* of G. For any $\varepsilon \in (0,1)$ we let $g_\varepsilon(G)$ be the set of all spanning subgraphs of G whose largest component has at least εn vertices. This event has probability $\mathbb{P}_p^G[g_\varepsilon(G)]$ in the percolation space of G. Now, assume that $p < p_c^{\text{bond}}$. Theorem 1.1 implies that, for any given $\rho \in (0,1)$, the event $\{G \in \mathcal{G}(n) \ : \ \mathbb{P}_p^G(g_\varepsilon(G)) < \rho\}$ occurs with probability $1 - o(1)$ in the uniform space $\mathcal{G}(n)$. That is, asymptotically for almost every graph in $\mathcal{G}(n)$, the random deletion of the edges leaves a component of order at least εn with probability no more than ρ. If $p > p_c^{\text{bond}}$, then the second part of Theorem 1.1 implies that there exists $\varepsilon > 0$ such that the event $\{G \in \mathcal{G}(n) \ : \ \mathbb{P}_p^G(g_\varepsilon(G)) > 1 - \rho\}$ occurs with probability $1 - o(1)$ in $\mathcal{G}(n)$. Hence, as $n \to \infty$ almost all the graphs in $\mathcal{G}(n)$ are such that if we apply the bond percolation process to them with retainment probability p, then there is a component having at least εn vertices with probability at least $1 - \rho$ (in the percolation space).

The fact that the critical probabilities coincide reflects a behavior that is similar to that of percolation on an infinite regular tree. Of course, in that context the critical probabilities are defined with respect to the appearance of an infinite component that contains the (vertex that has been selected as the) root. Using the fundamental theorem of Galton-Watson processes (see, for example, [Athreya and Ney 72]), it can be shown that the bond and the site critical probabilities coincide and that they are equal to $1/(d-1)$, where d is the degree of each vertex of the tree. Observe that, for the case of a random d-regular graph, Theorem 1.1 implies that $p_c^{\text{site}} = p_c^{\text{bond}} = 1/(d-1)$. This is not a coincidence as it is well known that a random d-regular graph locally (e.g., at distance no more than i from a given vertex for some fixed i) looks like a d-regular tree.

More generally, the typical local structure of the class of random graphs that we are investigating is also tree-like. Note that the ratio $L''(1)/L'(1)$ equals

$$\frac{L''(1)}{L'(1)} = \sum_{i=2}^{\infty} (i-1) \frac{i\lambda_i}{\sum_{j=1}^{\infty} j\lambda_j}. \tag{1.2}$$

Consider a vertex $v \in V_n$ that has positive degree, and let us examine more closely the behavior of one of its neighbors. It can be shown that the probability that this has degree i is proportional to $iD_i(n)$. In particular, it is almost equal to $iD_i(n)/\sum_j jD_j(n)$, and this tends to $i\lambda_i/\sum_{j=1}^{\infty} j\lambda_j$ as n grows. Moreover, one can show that with probability $1 - o(1)$ there are no edges between the neighbors of v. Therefore (1.2) is the limit of the expected number of children that a neighbor of v has. This scenario is repeated for every vertex in the dth neighborhood of v, where d is fixed. More precisely, the vertices that are at distance no more than d induce a tree rooted at v that contains at most $\ln \ln n$ vertices, with probability $1 - o(1)$. Suppose that there are t_i vertices of degree i in this tree. Thus, for a vertex that is at distance d from v, the probability that it has degree i is proportional to $i(D_i(n) - t_i) = i\lambda_i n(1 - o(1))$. More precisely, it is

$$\frac{i(D_i(n) - t_i)}{\sum_j j(D_j(n) - t_j)}.$$

Since $\Delta \leq n^{1/9}$ and $t_i \leq \ln \ln n$, it follows that

$$\sum_i it_i \leq \ln \ln n \sum_{i \leq n^{1/9}} i = O(n^{1/3}).$$

Hence, the limit of the above probability as $n \to \infty$ is again $i\lambda_i / \sum_{j=1}^{\infty} j\lambda_j$ and (1.2) gives the limiting expected number of children of such a vertex. In other words, the graph that is induced by the vertices that are at distance no more than d from v behaves like the tree of a branching process that started at v, with the ratio $L''(1)/L'(1)$ being the expected progeny of each vertex. Observe here that the condition $L''(1) > L'(1)$ implies that, in fact, this is a supercritical branching process that yields an infinite tree with probability 1.

Therefore, at least locally, either bond or site percolation is essentially percolation on such a random rooted tree. In both types of percolation, if $p < L'(1)/L''(1)$, then the expected number of children of a vertex that survive is $pL''(1)/L'(1) < 1$. Thus, the random tree rooted at v that remains after the random deletions of the edges or the vertices will be distributed as the tree of a subcritical branching process. In particular, the tree that surrounds most of the vertices will be cut off from the rest of the graph at a relatively small depth. On the other hand, if $p > L'(1)/L''(1)$, a large proportion from each of these local trees is preserved, and moreover they are big enough to guarantee that there are enough edges going out of them. So, eventually there is a fair chance that some of them are joined together and form a component of linear order. However, this is only a qualitative approach to Theorem 1.1. The actual proof and the structure of the paper are described in Section 2.

1.1. Theorem 1.1 and Power-Law Graphs

The power-law degree sequences are those for which one has $\lambda_k = ck^{-\gamma}$ for any $k \geq 1$ and for some constants $c, \gamma > 0$. We should point out that the crucial parameter here is γ. Such degree sequences have attracted much attention in the last few years mainly because of the fact that they arise in "natural" networks such as the Internet, the World Wide Web, or biological networks (see [Bollobás and Riordan 03a], [Albert and Barabási 02], or [Dorogovtsev and Mendes 02] for a survey of results or the recent book by Chung and Lu [Chung and Lu 06] for a more detailed discussion). For example, Faloutsos et al. have given evidence that the Internet as it looked like in 1995, viewed as a graph with vertices that are the routers and edges that are the physical links between them, has a power-law degree sequence with $\gamma \approx 2.48$ [Faloutsos et al. 99]. A bond or site percolation on such networks naturally corresponds to random failures of the links or of the nodes, respectively. Thus, a site or bond percolation process on the Internet may be seen as modeling random failures of the links or of the routers, respectively.

For a power-law degree sequence with $\gamma > 3$, one has $L(s) = c\sum_{k\geq 1} \frac{s^k}{k^\gamma}$ where $|s| \leq 1$. Thus, if $\zeta(\lambda) = \sum_{k\geq 1} \frac{1}{k^\lambda}$ is the Riemann's zeta function, then

$$L'(1) = c\sum_{k\geq 1} \frac{k}{k^\gamma} = c\sum_{k\geq 1} \frac{1}{k^{\gamma-1}} = c\zeta(\gamma-1)$$

and

$$L''(1) = c\sum_{k\geq 2} \frac{k(k-1)}{k^\gamma} = c\sum_{k\geq 2} \frac{1}{k^{\gamma-2}} - c\sum_{k\geq 2} \frac{1}{k^{\gamma-1}} = c(\zeta(\gamma-2) - \zeta(\gamma-1))$$

(of course here the derivatives are left derivatives). Let

$$\gamma_0 = \sup\left\{\gamma \ : \ \gamma > 3, \ \frac{\zeta(\gamma-2)}{\zeta(\gamma-1)} > 2\right\}.$$

Theorem 1.1 implies that the critical probabilities for a power-law degree sequence with $3 < \gamma < \gamma_0$ but maximum degree at most $n^{1/9}$ are

$$p_c^{\text{site}} = p_c^{\text{bond}} = \frac{\zeta(\gamma-1)}{\zeta(\gamma-2) - \zeta(\gamma-1)}. \tag{1.3}$$

This agrees with the analysis by Callaway et al. for the case of site percolation on a random graph whose degree sequence follows a "truncated" power-law, that is, $\lambda_k = Ck^{-\gamma}e^{-k/\kappa}$, for $C, \kappa > 0$ [Callaway et al. 00]. As $\kappa \to \infty$, this approaches a power-law distribution with parameter γ. It can be shown that in this case the critical probability they obtain converges to the expression in (1.3) (see, for example, [Albert and Barabási 02, Equation (141)]). Similar analysis by Cohen et al. suggests that if $\gamma \leq 3$ there is no phase transition at all [Cohen et al. 00]. Also, Boguñá et al. have argued that this happens whenever $2 < \gamma \leq 3$ [Boguñá et al. 03]. However, Dorogovtsev and Mendes, applying the formula of Theorem 1.1 (which they also obtain in their paper [Dorogovtsev and Mendes 02]) but without our degree restrictions, gave the scaling of the critical probabilities as functions of n as $n \to \infty$, for $2 < \gamma \leq 3$. In that context the critical probabilities are defined empirically, according to whether or not the proportion of vertices in the largest component after the percolation process is almost zero. Also, Albert et al. gave experimental evidence of the result of a site percolation process on a random graph (obtained from a different model) with power-law degree sequence $\gamma = 3$, and they observe no threshold behavior [Albert et al. 00]. That is, even if a large proportion of vertices are deleted, there is always a component of linear order. This was verified rigorously by Bollobás and Riordan, who also provided an expression for the order of the largest component as a function of the retainment probability p [Bollobás and Riordan 03b]. Experimental evidence on samples of the Internet and the World Wide Web have also been given, reaching the conclusion that no phase transition occurs even for small values of p [Albert et al. 00].

The case $\gamma \leq 3$ corresponds to $L''(1)$ being divergent, which suggests that p_c^{site} and p_c^{bond} vanish. However, Theorem 1.1 works under the assumption that $L''(1)$ converges. It would be an interesting and natural next step to prove or disprove the existence of a positive critical probability in the case where $L''(1)$ is divergent.

2. Definitions and Sketch of the Proof

In this paper we are interested in sparse asymptotic degree sequences $\mathcal{D}$ satisfying $Q(\mathcal{D}) := \sum_{i=1}^{\infty} i(i-2)\lambda_i > 0$. This is equivalent to saying that $\sum_{i=1}^{\infty} i(i-1)\lambda_i = L''(1) > L'(1) = \sum_{i=1}^{\infty} i\lambda_i$, where $L(s)$ is the generating polynomial of $\mathcal{D}$.

One of the main tools we use in the present work is the configuration model, which was introduced in different versions by Bender and Canfield [Bender and Canfield 78] and by Bollobás [Bollobás 80]. If $\mathbf{d}$ is a degree sequence on V_n, for some $n \in \mathbb{Z}^+$, we define the set of points $P = P(\mathbf{d})$ as $\{1 \times [d_1], \ldots, n \times [d_n]\}$, where $[d_i] = \{1, \ldots, d_i\}$ if $d_i > 0$ or is the empty set otherwise. That is, to every vertex in V_n correspond d_i points. Clearly, there are $2M$ points in P. Thus, observe that there are $(2M)!/M!2^M$ perfect matchings on P. If $M(\mathbf{d})$ is such a perfect matching, then we can obtain a (multi)graph $GM(\mathbf{d})$ if we project P onto V_n preserving adjacencies; namely, for any two vertices $i, j \in V_n$, if $M(\mathbf{d})$ contains an edge between a point in $i \times [d_i]$ and a point in $j \times [d_j]$, then $GM(\mathbf{d})$ contains a copy of the edge (i, j). Of course, in a perfect matching there might be edges that join two points corresponding to the same vertex, in which case $GM(\mathbf{d})$ obtains a loop on the vertex. Similarly, there might be two vertices that are joined to each other with more than one pair of points, and in this case $GM(\mathbf{d})$ obtains multiple copies of the corresponding edge. If $M(\mathbf{d})$ is a uniformly random perfect matching on P, then observe that $GM(\mathbf{d})$ is not uniformly distributed over the set of multigraphs having $\mathbf{d}$ as their degree sequence. However, if we condition on the event that $GM(\mathbf{d})$ is a simple graph, then it is uniformly distributed over the set of simple graphs that have $\mathbf{d}$ as their degree sequence (see, for example, [Janson et al. 00, p. 235]).

Consider now an asymptotic degree sequence $\mathcal{D} = (\mathbf{d}(n))_{n\in\mathbb{Z}^+}$. For each $n \in \mathbb{Z}^+$, we set $P(n)$ to be the set of points that corresponds to the degree sequence $\mathbf{d}(n)$. Let M_n be a uniformly random perfect matching on $P(n)$, and let $\tilde{G}(n)$ be the multigraph that is obtained from the projection of M_n onto V_n. The following theorem was proved by M. Molloy and B. Reed [Molloy and Reed 95] and has a key role in the proof of Theorem 1.1.

Theorem 2.1. *Let $\mathcal{D} = (\mathbf{d}(n))_{n\in\mathbb{Z}^+}$ be a sparse asymptotic degree sequence of maximum degree at most $n^{1/9}$.*

- *If $Q(\mathcal{D}) > 0$, then there exists an $\varepsilon > 0$ such that $\mathbb{P}[|L_1(\tilde{G}(n))| \geq \varepsilon n] \to 1$, as $n \to \infty$.*
- *If $Q(\mathcal{D}) < 0$, then for every $\varepsilon > 0$ we have $\mathbb{P}[|L_1(\tilde{G}(n))| \geq \varepsilon n] \to 0$, as $n \to \infty$.*

Of course, Theorem 2.1 as stated by Molloy and Reed was referring to simple graphs rather than multigraphs. However, it was actually proved for the random multigraph $\tilde{G}(n)$, and by adding the condition of being simple, it can be stated for random simple graphs having this particular degree sequence. In fact, in the first case Molloy and Reed proved the uniqueness of the component that has linear order; in particular, the second largest component has logarithmic order. The restriction on the maximum degree can be slightly relaxed (see [Molloy and Reed 95]), but for the simplicity of our proofs, we assume it to be as in Theorem 2.1. The way we use this result will become apparent during the sketch of our proofs that is about to follow.

Note that when we assume that $\mathcal{D}$ has $L''(1) > L'(1)$, Theorem 2.1 implies that $\tilde{G}(n)$ will have a giant component with probability $1 - o(1)$.

Here are the two deletion processes that we consider separately:

- *Bond percolation process.* For some $p \in (0, 1)$, we delete at random each edge of $\tilde{G}(n)$ with probability $1 - p$ independently of every other edge.
- *Site percolation process.* With probability $1 - p$ we make a vertex isolated by deleting the edges that are incident to it independently for every vertex of $\tilde{G}(n)$.

In either case, the random multigraph that is the outcome of the random deletions is denoted by $G'(n) = G'(n, p)$.

Eventually we want to know the structure of $G'(n)$, if $\tilde{G}(n)$ is a simple graph. Thus, we will show that if $\mathcal{A}(n)$ is a set of multigraphs on V_n and $\mathbb{P}[G'(n) \in$

$\mathcal{A}(n)] \to 0$ as $n \to \infty$, then $\mathbb{P}[G'(n) \in \mathcal{A}(n) \mid \tilde{G}(n) \text{ is simple}] \to 0$ as $n \to \infty$ as well. Hence, it will be sufficient for our purposes to perform the random deletion on the edges or the vertices of a random perfect matching on $P(n)$ without any conditioning and henceforth to consider the multigraph that is obtained from the remaining edges; this is going to make the calculations much simpler.

Thus, we prove the following lemma.

Lemma 2.2. *Let $\mathcal{D}$ be a sparse asymptotic degree sequence such that $\Delta = o(n^{1/3})$. Let $\mathcal{A}(n)$ be a set of multigraphs on V_n, and suppose that $\mathbb{P}[G'(n) \in \mathcal{A}(n)] \to 0$ as $n \to \infty$. Then, $\lim_{n\to\infty} \mathbb{P}[G'(n) \in \mathcal{A}(n) \mid \tilde{G}(n) \text{ is simple}] = 0$, as well.*

Proof. Note that

$$\mathbb{P}[G'(n) \in \mathcal{A}(n) \mid \tilde{G}(n) \text{ is simple}] \quad \leq \quad \frac{\mathbb{P}[G'(n) \in \mathcal{A}(n)]}{\mathbb{P}[\tilde{G}(n) \text{ is simple}]}. \tag{2.1}$$

The asymptotic enumeration formula for graphs with a given degree sequence, such that $M = \Theta(n)$ and $\Delta = o(n^{1/3})$, obtained by McKay and Wormald [McKay and Wormald 91], yields

$$\mathbb{P}[\tilde{G}(n) \text{ is simple}] = (1 + o(1))e^{-\lambda/2-\lambda^2/4},$$

where $\lambda = \frac{1}{M}\sum_{i=1}^{n} \binom{d_i}{2}$. But, observe that

$$\lambda = \frac{\sum_{i=1}^{n} d_i(d_i-1)}{\sum_{i=1}^{n} d_i} = \frac{\sum_{i\geq 1} i(i-1)D_i(n)}{\sum_{i\geq 1} iD_i(n)} = \frac{L''(1)}{L'(1)}(1+o(1)).$$

Thus,

$$\liminf_{n\to\infty} \mathbb{P}[\tilde{G}(n) \text{ is simple}] > 0,$$

and this concludes the proof of the lemma as the numerator in (2.1) converges to zero. $\square$

In both cases the random deletion induces a (random) degree sequence on V_n, which we denote by $\mathbf{d}'(n)$ (the use of the same symbol for the two kinds of percolation should cause no confusion). So, for each $n \in \mathbb{Z}^+$, let $\mathbf{D}_n$ be the set of degree sequences on V_n that are the result of the random deletion equipped with the probability distribution inherited by the random experiment we just described. That is, the probability of a certain degree sequence $\mathbf{d}'(n) \in \mathbf{D}_n$ is the probability that the degree sequence induced by the random deletion (either of edges or of vertices) on $\tilde{G}(n)$ is $\mathbf{d}'(n)$. We set $\mathbf{D} = \prod_{n=1}^{\infty} \mathbf{D}_n$ to be the product space equipped with the product measure, which we denote by μ. Thus, each element of $\mathbf{D}$ is an asymptotic degree sequence and Q is now a random variable on $\mathbf{D}$.

The strategy of our proof is quite different from that used by Goerdt [Goerdt 01], in that we make explicit use of Theorem 2.1. We first prove that the perfect matching between those points that are the endpoints of the edges that survive the deletion either in bond percolation or in site percolation is uniformly random among the perfect matchings on these points. Hence, to study the asymptotic properties of $G'(n)$, we shall condition first on its degree sequence for every $n \in \mathbb{Z}^+$ and then we shall study the asymptotic behavior of $G'(n)$ conditioned on this asymptotic degree sequence. Of course, to show that $G'(n)$ has a certain property with probability tending to 1 as $n \to \infty$, we have to show that almost all the asymptotic degree sequences in $\mathbf{D}$ have similar behavior. In particular, if $D'_i(n)$ is the number of vertices of degree i in $G'(n)$, we shall prove that the random variable $\frac{1}{n}\sum_{i\geq 1} i(i-2)D'_i(n)$ converges μ-almost surely (μ-a.s.) to a quantity Q' that depends only on the λ_i and on p, which we will calculate explicitly in both cases. From this we derive the critical p_c, which we denote by p_c^{bond} for the case of bond percolation and p_c^{site} for the case of site percolation. We show that if $p > p_c$ then Q' is positive, whereas if $p < p_c$ we have $Q' < 0$. Using Theorem 2.1, we deduce the sudden appearance of a giant component in $G'(n)$ when p crosses p_c, with probability that tends to 1 as $n \to \infty$.

We conclude this section stating a concentration inequality which we use in our proofs and it follows from a theorem of McDiarmid [McDiarmid 89, Theorem 7.4]. Let S be a finite set, and let f be a real-valued function on the set of those subsets of S that have size k. Assume that whenever c, c' are two such subsets of symmetric difference 2, then $|f(c) - f(c')| \leq 2$. If C is chosen uniformly at random among the k-subsets of S, then

$$\mathbb{P}\left[|f(C) - \mathbb{E}[f(C)]| > t\right] \leq 2\exp\left(-\frac{t^2}{2k}\right). \tag{2.2}$$

3. Bond Percolation

In this case, we start with the random graph $\tilde{G}(n)$ that is the multigraph that is the projection onto V_n of a uniformly random perfect matching on $P(n)$, and we create the multigraph $G'(n)$, deleting each edge of the matching with probability $1-p$ independently of every other edge. Thus, the number of edges of $G'(n)$ is distributed as $\mathrm{Bin}(M(n), p)$. Throughout this section we will be assuming that $p \in (0,1)$ is fixed.

Firstly, we will prove that the perfect matching on the remaining points in $P(n)$ conditional on the degree sequence that is created after the deletion is uniformly distributed on the set of perfect matchings on the set of points in $P(n)$ that survive the deletion. In particular, if C is the set of points in $P(n)$ that are the end-points of the surviving edges, then for every $i \in V_n$ the new degree of vertex i is $|C \cap (i \times [\mathbf{d}_i(n)])|$, where $\mathbf{d}_i(n)$ denotes the ith coordinate of $\mathbf{d}(n)$. Hence, the (random) set C induces a degree sequence on V_n, which we denote $\mathbf{d}'(n)$. We set $P'(n) = P(\mathbf{d}'(n))$.

Let $d \in \{0, \ldots, \Delta\}$ and assume that the vertices $i_1, \ldots, i_{k_d}$ (and only these) have new degree equal to d after the edge deletion. Hence, $\mathbf{d}'(n)$ contains exactly k_d vertices of degree d; assume that these are $i, \ldots, i + k_d - 1$. We identify i_j with $i + j - 1$, for every $j = 1, \ldots, k_d$. Moreover, provided that $d \geq 1$, we also identify the d points of $C \cap (i_j \times [\mathbf{d}_{i_j}(n)])$ with the points $\{i + j - 1\} \times \{1, \ldots, d\}$ in $P'(n)$. Hence, any perfect matching between the points in C corresponds to a perfect matching on $P'(n)$ and vice versa. In other words, we obtain a bijection between the perfect matchings on these two sets of points.

In the case of bond percolation, the probability space $\mathbf{D}_n$ consists of degree sequences on V_n, where the probability of a certain degree sequence is the probability that this is the induced degree sequence after the random deletion of the edges of a random perfect matching on $P(n)$. Our aim is to show that, conditional on $\mathbf{d}'(n) = \mathbf{d}'$, each perfect matching on $P'(n) = P(\mathbf{d}')$ is equally likely.

To do so, we first prove the following lemma.

Lemma 3.1. *Conditional on having k edges that survive the random deletion of the edges of the perfect matching on $P(n)$, the set of their $2k$ end-points is uniformly distributed among the $2k$-subsets of $P(n)$.*

Proof. The probability that a specific $2k$-subset of $P(n)$ is the set of the end-points of the k edges that survive is the probability that the perfect matching on $P(n)$ consists of a perfect matching on these $2k$ points and a perfect matching between the $2M - 2k$ remaining points, and that it is the set of the k edges on this $2k$-subset that survive the deletion. The probability of this event is exactly

$$\frac{\frac{(2M-2k)!}{(M-k)!2^{M-k}} \frac{2k!}{k!2^k}}{\frac{2M!}{M!2^M}} \frac{1}{\binom{M}{k}} = \frac{1}{\binom{2M}{2k}}.$$

This concludes the proof of the lemma. □

With a little more work, we obtain what we were aiming for.

Lemma 3.2. *Let $\mathbf{d}'(n)$ be the degree sequence that is induced by the random deletion of the edges of a uniformly random perfect matching on $P(n)$. Conditional on $\mathbf{d}'(n) = \mathbf{d}'$, all perfect matchings on $P(\mathbf{d}')$ are equally likely.*

Proof. Assume that the sum of the degrees in $\mathbf{d}'$ is $2k$, and let $S_{\mathbf{d}'}$ be the set of those $2k$-subsets of $P(n)$ that induce the degree sequence $\mathbf{d}'$. Let m be a particular perfect matching on $P'(n)$, conditional on $\mathbf{d}'(n) = \mathbf{d}'$. In other words, m is a perfect matching on $P(\mathbf{d}')$. Now, let us condition on $|P'(n)| = 2k$. If $C' \in S_{\mathbf{d}'}$, the probability that $C = C'$ and that the particular perfect matching that corresponds to m is realized on C is

$$\frac{1}{\binom{M}{k}} \frac{\frac{(2M-2k)!}{(M-k)!2^{M-k}}}{\frac{2M!}{M!2^M}}.$$

Thus,

$$\mathbb{P}[m,\ \mathbf{d}'(n) = \mathbf{d}' \mid |P'(n)| = 2k] = \sum_{C' \in S_{\mathbf{d}'}} \frac{1}{\binom{M}{k}} \frac{\frac{(2M-2k)!}{(M-k)!2^{M-k}}}{\frac{2M!}{M!2^M}} = \frac{|S_{\mathbf{d}'}|}{\binom{M}{k}} \frac{\frac{(2M-2k)!}{(M-k)!2^{M-k}}}{\frac{2M!}{M!2^M}}. \tag{3.1}$$

By Lemma 3.1, conditional on $|P'(n)| = 2k$ every set in $S_{\mathbf{d}'}$ has probability $1/\binom{2M}{2k}$. Therefore,

$$\mathbb{P}[\mathbf{d}'(n) = \mathbf{d}' \; ||P'(n)| = 2k] = \frac{|S_{\mathbf{d}'}|}{\binom{2M}{2k}}.$$

Now, Bayes' rule (i.e., dividing (3.1) by the above probability) yields

$$\begin{aligned} \mathbb{P}[m \mid \mathbf{d}'(n) = \mathbf{d}',\ |P'(n)| = 2k] &= \frac{1}{\binom{M}{k}} \frac{\frac{(2M-2k)!}{(M-k)!2^{M-k}}}{\frac{2M!}{M!2^M}} \binom{2M}{2k} \\ &= \frac{k!(M-k)!}{M!} \frac{\frac{(2M-2k)!}{(M-k)!2^{M-k}}}{\frac{2M!}{M!2^M}} \frac{2M!}{(2M-2k)!2k!} \\ &= \frac{1}{\frac{2k!}{k!2^k}}. \end{aligned}$$

But,

$$\begin{aligned} \mathbb{P}[m \mid \mathbf{d}'(n) = \mathbf{d}'] &= \mathbb{P}[m \cap |P'(n)| = 2k \mid \mathbf{d}'(n) = \mathbf{d}'] \\ &= \mathbb{P}[m \mid |P'(n)| = 2k,\ \mathbf{d}'(n) = \mathbf{d}'] \\ &\quad \times \mathbb{P}[|P'(n)| = 2k \mid \mathbf{d}'(n) = \mathbf{d}'] \\ &= \mathbb{P}[m \mid |P'(n)| = 2k,\ \mathbf{d}'(n) = \mathbf{d}'], \end{aligned}$$

since $\mathbb{P}[|P'(n)| = 2k \mid \mathbf{d}'(n) = \mathbf{d}'] = 1$, and the lemma follows. □

For $i \in \mathbb{N}$, let $D'_i(n)$ be the number of vertices of degree i in $\mathbf{d}'(n)$. The critical probability will be determined by the quantity $\sum_{i\geq 1} i(i-2)\lambda_i^{\text{bond}}$, where

$$\lambda_i^{\text{bond}} := \lim_{n\to\infty} \frac{1}{n}\mathbb{E}[D'_i(n)].$$

Hence we need to determine each λ_i^{bond}, proving the existence of this limit; to do so we will first calculate the expected value of $D'_i(n)$. We begin with the conditional expectation given the size of C that is conditional on the number of points in $P(n)$ that survive the deletion. For any $k = 0, \ldots, M$, we have

$$\mathbb{E}[D'_i(n) \mid |C| = 2k] = \sum_{d=i}^{\Delta} D_d(n)\mathbb{P}[\text{a given vertex of degree } d \text{ has new degree } i \mid |C| = 2k].$$

Recall that $|C|/2$, which equals the number of edges that survive the random deletion, is distributed as $\mathrm{Bin}(M, p)$. Therefore, a standard concentration argument yields

$$\mathbb{P}[||C|/2 - Mp| > \ln n\sqrt{n}] \leq \exp\left(-\Omega(\ln^2 n)\right). \tag{3.2}$$

This indicates that we may restrict ourselves to $k \in I = [Mp - \ln n\sqrt{n}, Mp + \ln n\sqrt{n}]$.

By Lemma 3.1, conditional on $|C| = 2k$, the set C is uniformly distributed among all $2k$-subsets of $P(n)$. Hence, since p is fixed we obtain

$$\begin{aligned}
&\mathbb{P}[\text{a given vertex of degree } d \text{ has new degree } i \mid |C| = 2k] \\
&= \binom{d}{i}\frac{\binom{2M-d}{2k-i}}{\binom{2M}{2k}} \\
&= \binom{d}{i}\frac{(2M-d)!}{2M!}\,\frac{2k!}{(2k-i)!}\,\frac{(2M-2k)!}{(2M-d-2k+i)!} \\
&= \binom{d}{i}\frac{(2k)^i}{(2M)^d}\,(2M-2k)^{d-i}\left(1+O\left(\frac{1}{n^{7/9}}\right)\right) \\
&= \binom{d}{i}p^i(1-p)^{d-i}\left(1+O\left(\frac{\ln n}{n^{7/18}}\right)\right),
\end{aligned}$$

uniformly for any $d \leq \Delta$ and any $k \in I$.

Therefore, since $D'_d(n) \le n$, by (3.2) we obtain

$$\begin{aligned}
\mathbb{E}[D'_i(n)] &= \sum_{k=0}^{M} \mathbb{E}[D'_i(n) \mid |C| = 2k]\mathbb{P}[|C| = 2k] \\
&= \sum_{k\in I} \sum_{d=i}^{\Delta} D_d(n) \binom{d}{i} p^i (1-p)^{d-i} \left(1 + O\left(\frac{\ln n}{n^{7/18}}\right)\right) \mathbb{P}[|C| = 2k] \\
&\quad + o\left(\frac{1}{n^3}\right) \\
&= \left(1 + O\left(\frac{\ln n}{n^{7/18}}\right)\right) \sum_{d=i}^{\Delta} D_d(n) \binom{d}{i} p^i (1-p)^{d-i} + o\left(\frac{1}{n^3}\right).
\end{aligned}$$

For every $\varepsilon > 0$, if i' and n are large enough, then

$$\frac{1}{n} \sum_{d=i'+1}^{\Delta} D_d(n) \binom{d}{i} p^i (1-p)^{d-i} \le \frac{1}{n} \sum_{d=i'+1}^{\Delta} D_d(n) < \varepsilon.$$

Therefore,

$$\begin{aligned}
\frac{1}{n} \sum_{d=i}^{i'} D_d(n) \binom{d}{i} p^i (1-p)^{d-i} &\le \frac{1}{n} \sum_{d=i}^{\Delta} D_d(n) \binom{d}{i} p^i (1-p)^{d-i} \\
&\le \frac{1}{n} \sum_{d=i}^{i'} D_d(n) \binom{d}{i} p^i (1-p)^{d-i} + \varepsilon.
\end{aligned}$$

Taking limits on both sides, we obtain

$$\sum_{d=i}^{i'} \lambda_d \binom{d}{i} p^i (1-p)^{d-i} \le \liminf_{n\to\infty} \frac{1}{n} \sum_{d=i}^{\Delta} D_d(n) \binom{d}{i} p^i (1-p)^{d-i}$$

and

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{d=i}^{\Delta} D_d(n) \binom{d}{i} p^i (1-p)^{d-i} \le \sum_{d=i}^{i'} \lambda_d \binom{d}{i} p^i (1-p)^{d-i} + \varepsilon.$$

Letting $i' \to \infty$ and then $\varepsilon \to 0$, we obtain the value of $\lambda_i^{\mathrm{bond}}$:

$$\lambda_i^{\mathrm{bond}} = \lim_{n\to\infty} \frac{\mathbb{E}[D'_i(n)]}{n} = \sum_{d=i}^{\infty} \lambda_d \binom{d}{i} p^i (1-p)^{d-i}. \tag{3.3}$$

We will show that the critical probability p_c^{bond} is equal to the root of the equation $Q' := \sum_{i=1}^{\infty} i(i-2)\lambda_i^{\mathrm{bond}} = 0$, which we denote by $\hat{p}_{\mathrm{bond}}$. Firstly, let us calculate $\hat{p}_{\mathrm{bond}}$. We have

$$\begin{aligned}
\sum_{i=1}^{\infty} i(i-2)\lambda_i^{\mathrm{bond}} &= \sum_{i=1}^{\infty} i(i-2)\sum_{d=i}^{\infty}\lambda_d\binom{d}{i}p^i(1-p)^{d-i} \\
&= \sum_{d=1}^{\infty}\lambda_d\sum_{i=1}^{d} i(i-2)\binom{d}{i}p^i(1-p)^{d-i} \\
&= \sum_{d=1}^{\infty}\lambda_d\left(dp(1-p)+(dp)^2-2dp\right) \\
&= \sum_{d=1}^{\infty}\lambda_d\left(1-p+dp-2\right)dp \\
&= \sum_{d=1}^{\infty}\lambda_d\left((d-1)p-1\right)dp \\
&= \sum_{d=1}^{\infty}\lambda_d d(d-1)p^2 - p\sum_{d=1}^{\infty} d\lambda_d \\
&= p\left(p\sum_{d=1}^{\infty} d(d-1)\lambda_d - \sum_{d=1}^{\infty} d\lambda_d\right).
\end{aligned}$$

Therefore,

$$\hat{p}_{\mathrm{bond}} = \frac{L'(1)}{L''(1)}.$$

We now let $Q'_n := \frac{1}{n}\sum_{i\geq 1} i(i-2)D'_i(n)$ and will show that $\lim_{n\to\infty} Q'_n$ exists μ-a.s. and is equal to Q'. Hence, the sign of Q' determines the sign of $\lim_{n\to\infty} Q'_n$ for almost every asymptotic degree sequence in $\mathbf{D}$.

For notational convenience, we set $X_{i',n} := \frac{1}{n}\sum_{i\leq i'} i(i-2)D'_i(n)$. Clearly, $Q'_n \geq X_{i',n}$. On the other hand, (1.1) implies that for every $\varepsilon > 0$ there exists $i_0 = i_0(\varepsilon)$ such that whenever $i' > i_0$ for n sufficiently large

$$\frac{1}{n}\sum_{i>i'} i(i-2)D_i(n) < \varepsilon.$$

Since $D'_i(n) \leq D_i(n)$, for any $i' > i_0$ we have

$$Q'_n \leq X_{i',n} + \varepsilon.$$

We shall prove that for every such i', μ-a.s.

$$\lim_{n\to\infty} X_{i',n} = \sum_{i\leq i'} i(i-2)\lambda_i^{\mathrm{bond}} =: \tilde{X}_{i'}. \tag{3.4}$$

In turn, this will imply that for every $i' > i_0$

$$\tilde{X}_{i'} \leq \liminf_{n\to\infty} Q'_n \leq \limsup_{n\to\infty} Q'_n \leq \tilde{X}_{i'} + \varepsilon,\ \mu\text{-a.s.}$$

Now, letting $i' \to \infty$ yields

$$Q' \leq \liminf_{n\to\infty} Q'_n \leq \limsup_{n\to\infty} Q'_n \leq Q' + \varepsilon,\ \mu\text{-a.s.}$$

Since the choice of ε is arbitrary, we may eventually deduce that

$$\lim_{n\to\infty} Q'_n = Q',\ \mu\text{-a.s.} \tag{3.5}$$

So let us focus on proving (3.4). This will follow, if we show that for every $\varepsilon > 0$

$$\sum_n \mathbb{P}\left[\left|X_{i',n} - \tilde{X}_{i'}\right| > \varepsilon\right] < \infty. \tag{3.6}$$

(See, for example, [Petrov 95, Lemma 6.8].) We will deduce the above inequality, proving that the summands are $o(1/n^3)$.

Thus, we continue with estimating

$$\mathbb{P}\left[\left|X_{i',n} - \tilde{X}_{i'}\right| > \varepsilon\right],$$

for some fixed $\varepsilon > 0$. Note that for n sufficiently large

$$\left|\mathbb{E}[X_{i',n}] - \tilde{X}_{i'}\right| \leq \frac{\varepsilon}{2}.$$

Thus,

$$\mathbb{P}\left[\left|X_{i',n} - \tilde{X}_{i'}\right| > \varepsilon\right] \leq \mathbb{P}\left[|X_{i',n} - \mathbb{E}[X_{i',n}]| > \frac{\varepsilon}{2}\right].$$

If the latter is realized, then there exists $i \leq i'$ for which

$$\frac{1}{n}\left|D'_i(n) - \mathbb{E}[D'_i(n)]\right| > \frac{\varepsilon}{2\sum_{i\leq i'} i(i-2)}.$$

Therefore, setting $\varepsilon' = \varepsilon/(2\sum_{i\leq i'} i(i-2))$, we have

$$\mathbb{P}\left[|X_{i',n} - \mathbb{E}[X_{i',n}]| > \frac{\varepsilon}{2}\right] \leq \sum_{i\leq i'} \mathbb{P}\left[\frac{1}{n}\left|D'_i(n) - \mathbb{E}[D'_i(n)]\right| > \varepsilon'\right].$$

We now show that each summand is $o(1/n^3)$. To do so, we will condition on the size of C. Recall that by Lemma 3.1 conditional on $|C| = 2k$, the set C

is uniformly distributed among the $2k$-subsets of $P(n)$. Moreover, if we replace one of the points in C with another one that does not belong to C, then $D'_i(n)$ can change by at most 2. Therefore, applying (2.2), we obtain uniformly for any $k \in I$

$$\begin{aligned}\mathbb{P}\left[|D'_i(n) - \mathbb{E}[D'_i(n)]| > \varepsilon' n \mid |C| = 2k\right] &\leq 2\exp\left(-\frac{\varepsilon'^2 n^2}{4k}\right) \\ &\leq 2\exp\left(-\frac{\varepsilon'^2 n^2}{4(Mp + \ln n\sqrt{n})}\right) \\ &= o\left(\frac{1}{n^3}\right).\end{aligned}$$

Therefore, by (3.2),

$$\begin{aligned}&\mathbb{P}\left[|D'_i(n) - \mathbb{E}[D'_i(n)]| > \varepsilon' n\right] \\ &= \sum_{k \in I} \mathbb{P}\left[|D'_i(n) - \mathbb{E}[D'_i(n)]| > \varepsilon' n \mid |C| = 2k\right] \ \mathbb{P}[|C| = 2k] + o\left(\frac{1}{n^3}\right) \\ &= o\left(\frac{1}{n^3}\right),\end{aligned}$$

for every i.

Now that we have proved (3.5), we are ready to conclude the proof that $p_c^{\text{bond}} = \hat{p}_{\text{bond}}$. Let $E \subseteq \mathbf{D}$ be the event over which $\lim_{n\to\infty} Q'_n = Q'$; recall that $\mu(E) = 1$. Let $(\mathbf{d}'(n))_{n\in\mathbb{Z}^+} \in E$. If we condition on $\mathbf{d}'(n)$ being the degree sequence on $G'(n)$, then Lemma 3.2 implies that $G'(n)$ is the multigraph that arises as the projection of a uniformly random perfect matching on $P'(n)$ onto V_n.

If $p < \hat{p}_{\text{bond}}$, then $Q' < 0$. For an arbitrary $\varepsilon > 0$, we define $A(n)$ to be the set of multigraphs on V_n whose largest component has no more than εn vertices, for an arbitrary $\varepsilon \in (0, 1)$. In this case, since p is fixed and, therefore, is asymptotically bounded away from $\hat{p}_{\text{bond}}$, Theorem 2.1 implies that $\lim_{n\to\infty} \mathbb{P}[G'(n) \in A(n) \mid D(G'(n)) = \mathbf{d}'(n)] = 1$, for every $\varepsilon > 0$.

On the other hand, if $p > \hat{p}_{\text{bond}}$, then $Q' > 0$. Again by Theorem 2.1 and since p is bounded away from $\hat{p}_{\text{bond}}$, we deduce that there exists $\varepsilon > 0$ such that for any $(\mathbf{d}'(n))_{n\in\mathbb{Z}^+} \in E$, if we define $A(n)$ to be the set of multigraphs on V_n whose largest component has at least εn vertices, then $\lim_{n\to\infty} \mathbb{P}[G'(n) \in A(n) \mid D(G'(n)) = \mathbf{d}'(n)] = 1$.

However, in either case we want to know the limit of $\mathbb{P}[G'(n) \in A(n)]$ as $n \to \infty$ without conditioning on the degree sequence. If $\omega \in \mathbf{D}$, then we let $\pi_n(\omega)$ denote the projection of ω onto its nth factor—recall that this is a degree sequence on V_n. Thus, this probability can be expressed as follows:

$$\begin{aligned}\mathbb{P}[G'(n) \in A(n)] &= \sum_{\mathbf{d}'(n)\in\mathbf{D}_n} \mathbb{P}[G'(n) \in A(n) \mid D(G'(n)) = \mathbf{d}'(n)] \\ &\qquad \times\ \mathbb{P}[D(G'(n)) = \mathbf{d}'(n)] \\ &= \sum_{\mathbf{d}'(n)\in\mathbf{D}_n} \int_{\{\omega\in\mathbf{D}\,:\,\pi_n(\omega)=\mathbf{d}'(n)\}} \mathbb{P}[G'(n) \in A(n) \mid \\ &\qquad D(G'(n)) = \mathbf{d}'(n)]\ \mu(d\omega) \\ &= \int \mathbb{P}[G'(n) \in A(n) \mid D(G'(n)) = \pi_n(\omega)]\ \mu(d\omega).\end{aligned}$$

Since the integrand is bounded below by 0, applying Fatou's Lemma, we obtain

$$\begin{aligned}\liminf_{n\to\infty} \mathbb{P}[G'(n) \in A(n)] &= \liminf_{n\to\infty} \int \mathbb{P}[G'(n) \in A(n) \mid D(G'(n)) = \pi_n(\omega)]\ \mu(d\omega) \\ &\geq \int \liminf_{n\to\infty} \mathbb{P}[G'(n) \in A(n) \mid D(G'(n)) = \pi_n(\omega)]\ \mu(d\omega) \\ &= \int_E \liminf_{n\to\infty} \mathbb{P}[G'(n) \in A(n) \mid D(G'(n)) = \pi_n(\omega)]\ \mu(d\omega) \\ &= \int_E \mu(d\omega) = 1.\end{aligned}$$

Also, since the integrand is bounded above by 1, we have

$$\begin{aligned}\limsup_{n\to\infty} \mathbb{P}[G'(n) \in A(n)] &= \limsup_{n\to\infty} \int \mathbb{P}[G'(n) \in A(n) \mid D(G'(n)) = \pi_n(\omega)] \\ &\qquad \times\ \mu(d(\omega)) \\ &\leq \int \mu(d\omega) = 1.\end{aligned}$$

The last two inequalities along with Lemma 2.2 complete the proof of Theorem 1.1 for the bond percolation process.

4. Site Percolation

In this section, we are dealing with site percolation, where for $p \in (0,1)$ fixed, we make each vertex of $\tilde{G}(n)$ isolated with probability $1-p$ independently of every other vertex, deleting all of the edges that are attached to it. We will be referring to this process as the *deletion* of the vertices. This process applied to $\tilde{G}(n)$ induces a random degree sequence on V_n, which, as in the previous section, we denote by $\mathbf{d}'(n)$. Now consider the effect of the deletion on the uniformly random perfect matching on $P(n)$: if a vertex is deleted, then the points of $P(n)$ that are the end-points of the edges attached to that vertex are deleted (i.e., we remove them from $P(n)$). Eventually, we are left with a set points of $P(n)$ that are the end-points of the remaining edges, and we denote it by C. Also, we let $P'(n) = P(\mathbf{d}'(n))$. As in the case of bond percolation, we fix a bijection between C and $P'(n)$. In turn, this gives rise to a bijection between the set of perfect matchings on C and the set of perfect matchings on $P'(n)$.

In the present setting, the probability space $\mathbf{D}_n$ consists of degree sequences on V_n, where the probability of a certain degree sequence is the probability that this is the induced degree sequence after exposing a random perfect matching on $P(n)$ and deleting randomly the vertices of V_n.

We now argue that conditional on the choice of the points of C, each perfect matching on C has the same probability. The perfect matching that is realized after the deletion of the vertices is obtained in two independent stages: first, the uniform perfect matching on $P(n)$ is realized, and afterward the random deletion of the vertices takes place. It is the independence that allows us to consider these two random experiments in reverse order. Thus, we choose first those vertices that will be deleted and then we realize the perfect matching on $P(n)$. Let $P_1(n)$ and $P_2(n)$ denote the sets of points corresponding to the deleted vertices and the vertices that remain, respectively. Let B be the subset of points in $P_2(n)$ that are matched with points in $P_1(n)$. Observe now that conditioning on the choice of $P_1(n)$ and B is equivalent to conditioning on the choice of C, as the disjoint union of B and C is $P_2(n)$. Under this conditioning each perfect matching on C has the same probability, since the number of perfect matchings on the remaining points is the same for every perfect matching on C. Thus, if $|C| = 2k$, then each perfect matching on C has probability $k!2^k/(2k)!$.

For a degree sequence $\mathbf{d}'$, we let $S_{\mathbf{d}'}$ be the set of subsets of $P(n)$ that realize $\mathbf{d}'$. Assume that the sum of the degrees in $\mathbf{d}'$ is $2k$. Note that if $\mathbf{d}'(n) = \mathbf{d}'$, then $P'(n) = P(\mathbf{d}')$. Hence, if m is a perfect matching on $P(\mathbf{d}')$, then

$$\begin{aligned}\mathbb{P}[m \mid \mathbf{d}'(n) = \mathbf{d}'] &= \sum_{C' \in S_{\mathbf{d}'}} \mathbb{P}[m \mid C = C',\ \mathbf{d}'(n) = \mathbf{d}']\ \mathbb{P}[C = C' \mid \mathbf{d}'(n) = \mathbf{d}'] \\ &= \frac{k!2^k}{(2k)!} \sum_{C' \in S_{\mathbf{d}'}} \mathbb{P}[C = C' \mid \mathbf{d}'(n) = \mathbf{d}'] = \frac{k!2^k}{(2k)!}.\end{aligned}$$

The parameter p_c^{site} will be determined by $\mathbf{d}'(n)$. If $D_i'(n)$ denotes the number of vertices that have degree i in $\mathbf{d}'(n)$, then letting

$$\lambda_i^{\mathrm{site}} := \lim_{n\to\infty} \frac{1}{n} \mathbb{E}[D_i'(n)], \tag{4.1}$$

we shall prove that this limit exists for every $i \in \mathbb{N}$. In fact, we show that

$$\lambda_i^{\mathrm{site}} = p\lambda_i^{\mathrm{bond}}. \tag{4.2}$$

This implies that

$$\sum_{i \geq 1} i(i-2)\lambda_i^{site} = p \sum_{i \geq 1} i(i-2)\lambda_i^{bond}. \tag{4.3}$$

Let $\hat{p}_{\text{site}}$ be the root of $\sum_i i(i-2)\lambda_i^{\text{site}} = 0$. We will show that $p_c^{\text{site}} = \hat{p}_{\text{site}}$. Then, by (4.3), we will deduce that $p_c^{\text{site}} = p_c^{\text{bond}}$.

We now prove the existence of the limit in (4.1). First of all, we estimate the number of points in $P_2(n)$. Then we shall condition on a certain realization of $P_2(n)$, and afterward, we shall condition on the size of B (that is, on the size of C). From this we will be able to estimate $\mathbb{E}[D_i'(n)]$. Let $D_d'' = D_d''(n)$ be the number of vertices of degree d that survive the deletion—therefore $\mathbb{E}[D_d''] = D_d p$. The total degree in $P_2(n)$ is $M_2 = \sum_{d=1}^{\Delta} dD_d''$, and the linearity of expectation yields $\mathbb{E}[M_2] = \sum_{d=1}^{\Delta} dD_d p = 2Mp$. As for every n the maximum degree in $\mathbf{d}(n)$ is no more than $n^{1/9}$, a bounded differences inequality (see, for example, [McDiarmid 89, Theorem 5.7]) yields

$$\mathbb{P}[|M_2 - \mathbb{E}[M_2]| > n^{2/3} \ln n] \leq \exp\left(-\Omega(\ln^2 n)\right). \tag{4.4}$$

Now note that if we condition on $|B| = b$, then any b-subset of $P_2(n)$ is equally likely to occur as the set B and it is the points of B that are deleted along with the points of $P_1(n)$.

Thus, if $P_{d-i}(d)$ denotes the probability that after the random allocation of B a certain vertex in $P_2(n)$ of degree d loses $d-i$ points, thus becoming a vertex of degree i, the expected value of D_i' is

$$\mathbb{E}[D_i'] = \sum_{d=i}^{\Delta} \mathbb{E}[D_d''] P_{d-i}(d).$$

But, for every $\varepsilon > 0$, there exists $i' \geq i$ such that $\sum_{d > i'} D_d \leq \varepsilon n$, for n sufficiently large. Since $D_d''(n) \leq D_d(n)$, we obtain

$$\sum_{d=i}^{i'} \mathbb{E}[D_d''] P_{d-i}(d) \leq \mathbb{E}[D_i'] \leq \sum_{d=i}^{i'} \mathbb{E}[D_d''] P_{d-i}(d) + \varepsilon n. \tag{4.5}$$

We now calculate $P_{d-i}(d)$, for $i \leq d \leq i'$. Suppose that $M_2 = m_2$ and $|B| = b$. Then, if $P_{d-i}(d, b, m_2)$ is the conditional probability that after the random choice of the b points in $P_2(n)$, which has m_2 points, a certain vertex of degree d loses $d-i$ points, we have

$$\begin{aligned} P_{d-i}(d, b, m_2) &= \binom{d}{d-i} \frac{\binom{m_2-d}{b-d+i}}{\binom{m_2}{b}} \\ &= \binom{d}{d-i} \frac{(m_2-d)!}{m_2!} \frac{b!}{(b-d+i)!} \frac{(m_2-b)!}{(m_2-b-i)!}. \end{aligned} \tag{4.6}$$

We shall assume that for any n sufficiently large

$$b \in [2Mp(1-p) - n^{2/3}\ln^2 n, 2Mp(1-p) + n^{2/3}\ln^2 n].$$

Indeed the following lemma holds.

Lemma 4.1. *Conditional on* $M_2 \in I' := [2Mp - n^{2/3}\ln n, 2Mp + n^{2/3}\ln n]$ *we have* $b \in I := [2Mp(1-p) - n^{2/3}\ln^2 n, 2Mp(1-p) + n^{2/3}\ln^2 n]$ *with probability* $1 - \exp\left(-\Omega(\ln^2 n)\right)$.

Proof. Assume that $M_2 = m_2$ for some $m_2 \in I'$. Therefore, $M_1 = 2M - m_2 \in [2M(1-p) - n^{2/3}\ln n, 2M(1-p) + n^{2/3}\ln n]$. We shall also condition on a particular realization of the sets $P_1(n)$ and $P_2(n)$.

The probability that a certain point in $P_2(n)$ is adjacent to a point in $P_1(n)$ is $M_1/(2M-1) = (1-p)(1+O(n^{-1/3}\ln n))$. Thus, $\mathbb{E}[b] = 2Mp(1-p)+O(n^{2/3}\ln n)$.

We now show that b is concentrated around its expected value, using Theorem 7.4 from [McDiarmid 89]. We first describe here the more general setting to which this theorem applies, and afterward we will consider b.

Let W be a finite probability space that is also a metric space with its metric denoted by g. Assume that $P_0, \ldots, P_s$ is a sequence of partitions on W, such that P_{i+1} is a refinement of P_i, P_0 is the partition consisting of only one part, that is W, and P_s is the partition where each part is a single element of W. Assume that for $i = 1, \ldots, s$ whenever $A, B \in P_i$ and $C \in P_{i-1}$ are such that $A, B \subseteq C$, then there is a bijection $\phi : A \to B$ such that $g(x, \phi(x)) \leq c_i$.

Now, let V be a uniformly random element of W, and let $f : W \to \mathbb{R}$ be a function on W satisfying $|f(x) - f(y)| \leq g(x, y)$. Then,

$$\mathbb{P}[|f(V) - \mathbb{E}[f(V)]| > t] \leq 2\exp\left(-2\frac{t^2}{\sum_{i=1}^{s} c_i^2}\right). \tag{4.7}$$

In our context the uniform space of all perfect matchings on $P(n)$ will play the role of W. Let $\mathbf{M}$ denote it. Its metric will be the symmetric difference of any two perfect matchings, regarded as sets of edges. It is easy to see that this satisfies the properties a metric has by its definition. We shall consider a series of partitions on $\mathbf{M}$ denoted by $P_0, \ldots, P_{M-1}$, where P_0 is $\mathbf{M}$ itself and each part of P_{M-1} will be a perfect matching in $\mathbf{M}$. To define the ith partition, we define an ordering on the edges of each perfect matching. Consider first a linear ordering of all the points in $P(n)$. This induces a linear ordering on the edges of a perfect matching: if e_1 and e_2 are two edges, then $e_1 < e_2$ if the smallest point in e_1 is smaller than the smallest point in e_2. Now, a part of P_i consists of those perfect matchings whose i smallest edges are a particular set of i edges, provided that such a set of perfect matchings is nonempty. We call such an i-set of edges a *prefix*. Moreover, given a perfect matching, we call its i smallest edges its i-*prefix*.

For $i = 1, \ldots, M-1$, given such an $(i-1)$-set of edges, let C be the set of perfect matchings that have these $i-1$ edges as their $(i-1)$-prefix. Now consider two i-subsets that contain this $(i-1)$-set and are both prefixes. Suppose that e_A and e_B are the last edges on which they differ. Let A and B respectively denote the sets of perfect matchings that have these two i-sets as their i-prefixes.

There is a natural bijection $\phi : A \to B$ between them. Observe first that the smallest vertex in e_A and e_B is the same. In particular, let us assume that $e_A = (x, y_A)$ and $e_B = (x, y_B)$. If m is a matching in A, then $\phi(m)$ is the matching in B, where y_A is adjacent to the vertex that y_B was adjacent to in m; every other edge remains unchanged. Note that the symmetric difference of m and $\phi(m)$ is 4. In other words, $c_i = 4$.

Now, we are ready to apply the concentration bound (4.7) to b. For any $m \in \mathbf{M}$, we let $b(m)$ be the number of edges between $P_1(n)$ and $P_2(n)$. Observe that for any two perfect matchings $m, m' \in \mathbf{M}$, always $|b(m) - b(m')|$ is no more than the size of the symmetric difference of m and m'. Thus, applying (4.7) with $t = n^{2/3} \ln^2 n$, the lemma follows. $\square$

Thus, by (4.6), uniformly for any $b \in I$ and any $m_2 \in I'$, we have

$$\begin{aligned} P_{d-i}(d, b, m_2) &= \binom{d}{d-i} \frac{b^{d-i}(m_2 - b)^i}{m_2^d} \left(1 + O\left(\frac{1}{n}\right)\right) \\ &= \binom{d}{i} (1-p)^{d-i} p^i \left(1 + O\left(\frac{\ln^2 n}{n^{1/3}}\right)\right). \end{aligned}$$

A standard concentration argument shows that uniformly for any $d \le i'$ we have

$$\mathbb{P}[|D''_d(n) - \mathbb{E}[D''_d(n)]| \ge \sqrt{n} \ln n] \le \exp(-\Omega(\ln^2 n)). \tag{4.8}$$

Thus, if we also set $I''(d) = [\max\{pD_d - \ln n\sqrt{n}, 0\}, pD_d + \ln n\sqrt{n}]$, we have

$$\mathbb{P}[b \notin I \text{ or } M_2 \notin I' \text{or } D''_d \notin I''(d), \text{ for some } d \le i'] = o(n^{-3}).$$

Therefore, the left-hand side of (4.5) becomes

$$\begin{aligned} \mathbb{E}[D'_i] \ge \sum_{k \in I} \sum_{k' \in I'} \sum_{d=i}^{i'} \sum_{k''_d \in I''(d)} & k''_d P_{d-i}(d, k, k') \mathbb{P}[b = k,\ M_2 = k',\ D''_d = k''_d] \\ & + \ o(n^{-2}) \\ = \sum_{d=i}^{i'} \sum_{k''_d \in I''(d)} k''_d \binom{d}{i} & (1-p)^{d-i} p^i \mathbb{P}[D''_d = k''_d] \left(1 + O\left(\frac{\ln^2 n}{n^{1/3}}\right)\right) + o(n^{-2}). \end{aligned}$$

But, by (4.8), we have $\sum_{k''_d \in I''(d)} k''_d \mathbb{P}[D''_d = k''_d] = \mathbb{E}[D''_d(n)] - o(n^{-2}) = D_d(n)p - o(n^{-2})$. Substituting this into the above expression, (4.5) now yields

$$\mathbb{E}[D'_i] \geq p \sum_{d=i}^{i'} D_d(n) \binom{d}{i} (1-p)^{d-i} p^i \left(1 + O\left(\frac{\ln^2 n}{n^{1/3}}\right)\right) + o(n^{-2})$$

and also, repeating the above estimations,

$$\mathbb{E}[D'_i] \leq p \sum_{d=i}^{i'} D_d(n) \binom{d}{i} (1-p)^{d-i} p^i \left(1 + O\left(\frac{\ln^2 n}{n^{1/3}}\right)\right) + \varepsilon n + o(n^{-2}).$$

Therefore,

$$p \sum_{d=i}^{i'} \lambda_d \binom{d}{i} (1-p)^{d-i} p^i \leq \liminf_{n\to\infty} \frac{1}{n} \mathbb{E}[D'_i]$$

and

$$\limsup_{n\to\infty} \frac{1}{n} \mathbb{E}[D'_i] \leq p \sum_{d=i}^{i'} \lambda_d \binom{d}{i} (1-p)^{d-i} p^i + \varepsilon.$$

Letting $i' \to \infty$ and $\varepsilon \to 0$, we obtain

$$\lambda_i^{\mathrm{site}} = \lim_{n\to\infty} \frac{1}{n} \mathbb{E}[D'_i] = p \sum_{d=i}^{\infty} \lambda_d \binom{d}{i} (1-p)^{d-i} p^i,$$

which yields (4.2) through (3.3).

Now we let $Q'_n := \frac{1}{n} \sum_{i\geq 1} i(i-2) D'_i(n)$. We will show that μ-a.s.

$$\lim_{n\to\infty} Q'_n = \sum_{i\geq 1} i(i-2) \lambda_i^{site} =: Q'. \tag{4.9}$$

To prove this we argue as in the case of bond percolation: setting $X_{i',n} = \frac{1}{n} \sum_{i\leq i'} i(i-2) D'_i(n)$, for every $\varepsilon > 0$ and any i' large enough, we have

$$X_{i',n} \leq Q'_n \leq X_{i',n} + \varepsilon,$$

if n is also large enough. (Obviously, the first inequality holds for every i' and n.) Thus, the existence of the μ-a.s. limit of Q'_n will be established once we show that for any i' μ-a.s. $\lim_{n\to\infty} X_{i',n} = \sum_{i\leq i'} i(i-2)\lambda_i^{\mathrm{site}}$. We then let $i' \to \infty$ and $\varepsilon \to 0$ to deduce that μ-a.s. $\lim_{n\to\infty} Q'_n = Q'$.

The almost sure convergence of $X_{i',n}$ can be shown as in the case of bond percolation. In other words, we need to prove that the condition in (3.6) is satisfied in the present context. As before, we will show that for every $i \leq i'$ the random variable $D'_i(n)$ is sharply concentrated around its expected value: that is, its tails converge to 0 exponentially fast. Recall that the total degree in $P_2(n)$ is denoted by M_2.

Conditional on a certain realization of $P_2(n)$, with $M_2 = m_2$ for some $m_2 \in I'$ and $|B| = b$ for some $b \in I$, the value of $D'_i(n)$ is determined by the random choice of the set B in $P_2(n)$. Note that D'_i can change by at most 2, if we replace one element of B by another one. Therefore, we may apply (2.2) to get

$$\begin{aligned}\mathbb{P}\left[|D'_i(n) - \mathbb{E}[D'_i(n)]| > \ln n\sqrt{n} \mid |B| = b,\ P_2(n),\ |P_2(n)| = m_2\right] \\ \leq 2\exp\left(-\frac{n\ln^2 n}{2(2Mp(1-p) + n^{2/3}\ln^2 n)}\right) = \exp\left(-\Omega(\ln^2 n)\right),\end{aligned}$$

uniformly for any $b \in I$ and $m_2 \in I'$. Hence, the above inequality along with (4.4) and Lemma 4.1 implies that

$$\mathbb{P}\left[|D'_i(n) - \mathbb{E}[D'_i(n)]| > \ln n\sqrt{n}\right] \quad = \quad o(n^{-3}).$$

Since $i \leq i'$ and i' is bounded, condition (3.6) is satisfied, and therefore, μ-a.s. $\lim_{n\to\infty} X_{i',n} = \sum_{i \leq i'} i(i-2)\lambda_i^{\text{site}}$. Now this concludes the proof of (4.9).

The proof that $p_c^{\text{site}} = \hat{p}_{\text{site}}$ is identical to that for $\hat{p}_{\text{bond}}$, and it is omitted.

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