

ON HOMOTOPY 3-SPHERES¹

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A homotopy 3-sphere M^3 is a compact, simply connected 3-manifold without boundary. After the work of Moise [6] and Bing [1] M^3 possesses a triangulation. The Poincaré conjecture [9] states that every homotopy 3-sphere M^3 is a 3-sphere. In this paper we prove three theorems, related to the Poincaré conjecture, about maps of a 3-sphere S^3 onto M^3 and about 1- and 2-spheres in M^3 .

1. Theorems 1 and 2, concerning maps $S^3 \rightarrow M^3$ and closed curves in M^3 . From the work of Hurewicz [5], Part III, it follows that there exists a continuous map $\varphi : S^3 \rightarrow M^3$ of degree 1 (where S^3 means a 3-sphere). We shall prove that there exists an especially simple map of this kind.²

THEOREM 1. *If M^3 is a homotopy 3-sphere then there exists a simplicial map $\gamma : S^3 \rightarrow M^3$ of degree 1 such that the singularities of γ (i.e. the closure of the set of those points $p \in M^3$ for which $\gamma^{-1}(p)$ consists of more than one point) lie in a (polyhedral, compact) handlebody in M^3 .*

One might consider this result as a step towards a proof of the Poincaré conjecture. Indeed, if it were possible to restrict the singularities of γ to a 3-cell in M^3 instead of a handlebody the existence of a homeomorphism $S^3 \rightarrow M^3$ would follow.

From Theorem 1 we may derive another aspect of the Poincaré problem by considering simple closed curves in M^3 .

From the definition of simple connectedness it follows that every closed curve $C^1 \subset M^3$ bounds a singular disk $D^2 \subset M^3$. If C^1 is a tame, simple closed curve then one can find a D^2 which is also tame and possesses only "normal" singularities (see [7], [8]), i.e. double curves in which two sheets of D^2 pierce each other, triple points in which three sheets pierce each other, and branch points from each of which one or more double arcs originate; the triple points, the branch points, and the interiors of the double curves are disjoint from the boundary ∂D^2 of D^2 , but the double curves may have end points in ∂D^2 .

As Bing [2] has proved, M^3 is a 3-sphere if (and only if) every tame, simple closed curve $C^1 \subset M^3$ lies in a (compact) 3-cell in M^3 . The statement that C^1 lies in a 3-cell $D^3 \subset M^3$ is equivalent to the statement that C^1 bounds a "knot projection cone" D^2 in M^3 , i.e. a (tame) singular disk whose singularities are one branch point P and double arcs originating from P , being pairwise

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² Theorem 1 is a consequence of a "monotonic mapping theorem" announced by Moise in [6a]; however the proof is different from Moise' proof.

disjoint otherwise, and terminating in $\cdot D^2$. (A small neighborhood of a knot projection cone in M^3 is always a 3-cell.) Hence one would prove the Poincaré conjecture if one could prove that every tame, simple closed curve $C^1 \subset M^3$ bounds a knot projection cone in M^3 . Theorem 2 of this paper (which may be considered as a corollary of Theorem 1) is a first step in this direction: it states that C^1 always bounds a knot projection cone D^2 with additional singularities that do not touch $\cdot D^2 = C^1$.

THEOREM 2. *If C^1 is a tame, simple closed curve in a homotopy 3-sphere M^3 then there is a (tame) singular disk $D^2 \subset M^3$ with $\cdot D^2 = C^1$ such that D^2 has the following singularities:*

(a) *One branch point P of multiplicity g (g may be zero) and g double arcs Q_1^1, \dots, Q_g^1 (in each of which two sheets of D^2 pierce each other), starting from P and ending at $\cdot D^2$ with ${}^0Q_i^1 \subset {}^0D^2$ such³ that the $Q_i^1 - P$'s are pairwise disjoint.*

(b) *Closed double curves R_1^1, \dots, R_h^1 (h may be zero) which may pierce themselves and the Q_i^1 's in triple points of D^2 , but which are disjoint from $\cdot D^2$.*

In the special case $h = 0$, D^2 is a knot projection cone; in the case $g = 0$, D^2 is a so called Dehn disk (see [8]). In the latter case it follows from Dehn's lemma (stated by Dehn [3] and proved by Papakyriakopoulos [8]) that there exists a (tame) disk D^{*2} with $\cdot D^{*2} = C^1$ and $h^* = 0$ (and also $g^* = 0$). Now the question arises whether it follows in the general case ($g \neq 0$) that there exists a (tame, singular) disk D^{*2} with $\cdot D^{*2} = C^1$ and $h^* = 0$ (and g^* arbitrary, not necessarily equal to g). An affirmative answer to this question would imply the Poincaré conjecture.

If one applies the methods for proving Dehn's lemma, as developed by Papakyriakopoulos [8] and later simplified by Shapiro and Whitehead [12], to this problem then one has to consider a small neighborhood $D^3 \subset M^3$ of D^2 , a covering of D^3 , etc. Then all conclusions of the proof of Dehn's lemma in [12] apply to our problem as well, except in case (1) wherein the boundary $\cdot D^3$ of D^3 (or that of one of the neighborhoods in the coverings) consists of 2-spheres only: for case (1) it follows easily in dealing with Dehn's lemma that C^1 bounds a nonsingular disk; however it seems to be difficult to prove for case (1) in dealing with our problem, $g \neq 0$, that C^1 bounds a knot projection cone. Nevertheless I hope that someone will be able to fill this gap in the proof of the Poincaré conjecture.

2. Theorem 3, concerning 2-spheres in M^3 . We obtain another aspect of the Poincaré problem if we consider 2-spheres in M^3 instead of closed curves. If we remove the interior of a 3-cell C^3 from M^3 we get a so called homotopy 3-cell M_*^3 . It follows from the Hurewicz theorem [5], Part II, that every 2-sphere in M_*^3 may be homotopically deformed into one point.

Let us consider a 2-sphere $F_0^2 \subset M_*^3$, "topologically parallel" to the bound-

³ We denote the interior of a (tame) point set X by 0X , the boundary by $\cdot X$, and the closure by \bar{X} or ${}^-X$.

ary of M_*^3 , i.e. such that $F_0^2 + \cdot M_*^3$ bounds a 3-annulus $F_0^3 \subset M_*^3$. If one could prove that F_0^2 can be deformed⁴ into a 3-cell $H^3 \subset M_*^3$ not only by a homotopy but also by an isotopy whose image is tame at each level then the Poincaré conjecture would follow (since it would follow that M_*^3 is a 3-cell). It follows from the work of Smale [13] on regular homotopy that F_0^2 can be deformed onto the boundary of a 3-cell in H^3 in such a way that no branch points occur at any stage of the deformation. In order to go one step further in this direction we shall show that F_0^2 can be deformed into H^3 by especially simple homotopic deformations that take place in a special order.

First we have to define some special homotopic deformations. Let

$$\alpha : F'^2 \rightarrow M_*^3,$$

with the image $\alpha(F'^2) \subset {}^0M_*^3$ denoted by F^2 , be a continuous map, defining a (tame) 2-sphere with canonical singularities (i.e. normal double curves and triple points, but without branch points, see [8]). Let A'^2 be a disk in F'^2 whose image $\alpha(A'^2)$ is also a (nonsingular) disk A^2 . Let

$$A^{*2} \subset {}^0M_*^3$$

be another tame disk with $A^{*2} \cap A^2 = \cdot A^2 = \cdot A^{*2}$ such that $A^2 + A^{*2}$ bounds a 3-cell $K^3 \subset M_*^3$. Now we consider a deformation δ that changes α into α^* such that

$$\alpha^* |(F'^2 - {}^0A'^2) = \alpha |(F'^2 - {}^0A'^2)$$

and $\alpha^* | A'^2$ is a homeomorphism onto A^{*2} . We call such a deformation *non-essential* if there exists an epi-homeomorphism

$$\zeta : M_*^3 \rightarrow M_*^3 \quad \text{with} \quad \zeta(F'^2) = \alpha^*(F'^2)$$

that is the identity outside a small neighborhood of K^3 . We call δ an *elementary deformation of type 1, 2, or 3*, respectively, if the surface defined by α^* has only normal singularities and one of the following conditions holds (see Fig. 1):

Type 1. Either case (a) $\cdot ({}^0K^3 \cap F^2)$ is a disk B^2 with $\cdot B^2 \subset {}^0A^{*2}$; or case (b) $\cdot ({}^0K^3 \cap F^2)$ consists of two disks B^2, C^2 such that

$$\cdot B^2, \cdot C^2 \subset {}^0A^{*2}$$

and $B^2 \cap C^2$ is an arc with

$${}^0(B^2 \cap C^2) \subset {}^0K^3.$$

⁴ For convenience we shall use the word "deformation" not only for deformations of maps but also for deformations of polyhedra $X \subset M^3$ (i.e. for changes of X into X^* such that there can be found homotopic maps $\xi, \xi^* : X' \rightarrow M^3$ with $\xi(X') = X, \xi^*(X') = X^*$). This is convenient since a surface with normal singularities, defined by a map

$$\xi : X'^2 \rightarrow M^3,$$

is essentially determined by the image polyhedron $\xi(X'^2)$.

Type 2. ${}^{-}({}^0K^3 \cap F^2)$ is a disk B^2 such that each of the intersections $\cdot B^2 \cap A^2$ and $\cdot B^2 \cap A^{*2}$ consists of two disjoint arcs with

$${}^0(\cdot B^2 \cap A^2) \subset {}^0A^2 \quad \text{and} \quad {}^0(\cdot B^2 \cap A^{*2}) \subset {}^0A^{*2}.$$

Type 3. Either case (a) ${}^{-}({}^0K^3 \cap F^2)$ is a disk B^2 with $\cdot B^2 \subset {}^0A^2$; or case (b) ${}^{-}({}^0K^3 \cap F^2)$ consists of two disks B^2, C^2 such that $\cdot B^2 \subset {}^0A^2$ and each of the intersections $\cdot C^2 \cap A^2, \cdot C^2 \cap A^{*2}, C^2 \cap B^2$ is an arc with

$${}^0(\cdot C^2 \cap A^2) \subset {}^0A^2, \quad {}^0(\cdot C^2 \cap A^{*2}) \subset {}^0A^{*2}, \quad {}^0(C^2 \cap B^2) \subset {}^0C^2, {}^0B^2.$$

We remark that an elementary deformation of type 1 (a or b) changes the image sphere F^2 only in a small neighborhood (small with respect to F^2) of an arc (connecting a point in ${}^0A^2$ to a point in ${}^0B^2$ or in ${}^0B^2 \cap {}^0C^2$, respectively); a deformation⁴ of type 2 changes F^2 in a small neighborhood of a disk (whose boundary intersects each A^2 and B^2 in one arc). According to this one might say that a deformation of type i ($i = 1, 2, 3$) is essentially i -dimensional.

THEOREM 3. *Let M_*^3 be a homotopy 3-cell and $\alpha_0 : F'^2 \rightarrow M_*^3$ an embedding of a 2-sphere, topologically parallel to $\cdot M_*^3$. Then α_0 can be deformed step by step into maps $\alpha_1, \alpha_2, \alpha_3$ of F'^2 into M_*^3 such that the following holds:*

(a) α_i ($i = 1, 2, 3$) is obtained from α_{i-1} by a finite sequence of elementary deformations of type i and non-essential deformations.

(b) The image $\alpha_3(F'^2)$ lies in a 3-cell $H^3 \subset {}^0M_*^3$.

The two essential points of this theorem (which are not immediate consequences of Smale's results [13]) are (1) the order in which the deformations take place and (2) that no deformations are used that move the surface over a triple point.

We remark without proof: If it were possible to avoid the deformations of type 1b (i.e. to avoid triple points) or to avoid the deformations of type 2 then this would imply the Poincaré conjecture; this would hold even if H^3 were not a 3-cell, but homeomorphic to any compact subset of euclidean 3-space with connected boundary.

3. Sketch of the proofs. The theorems are proved by considering deformations of singular 2-spheres in a homotopy 3-cell M_*^3 . We start with an embedding

$$\beta_0 : F'_0{}^3 \rightarrow M_*^3$$

of a 3-annulus $F'_0{}^3$ into M_*^3 such that one boundary sphere S'^2 of $F'_0{}^3$ is mapped onto $\cdot M_*^3$ and the other boundary sphere F''^2 onto the 2-sphere $F''^2 = \alpha_0(F'^2)$. Now we deform $F'_0{}^3$ into a 3-cell $H^3 \subset {}^0M_*^3$ in the simplest way we can find. To do this we choose a simple cell-decomposition Γ of the homotopy 3-sphere $M^3 = M_*^3 + C^3$ (C^3 being a 3-cell with $C^3 \cap M_*^3 = \cdot C^3 = \cdot M_*^3$) into one vertex E^0 , r elements E_i^1, E_i^2 ($i = 1, \dots, r$) of each dimension 1 and 2, and one open 3-cell E^3 containing C^3 . Then we choose a neighborhood J^3 of the 2-skeleton G^2 of Γ , and we may assume that our initial 3-annulus $\beta_0(F'_0{}^3)$ is $M_*^3 - {}^0J^3$,

hence $F_0^2 = J^3$. Now we use the fact that M_*^3 is simply connected by taking a collection of r singular disks, bounded by the 1-skeleton G^1 of Γ (that consists of the r loops \bar{E}_i^1 with the common vertex E^0); these disks with the boundary point E^0 in common form a “fan” V^2 with singularities. We can choose V^2 such that its only singularities are pairwise disjoint double arcs A_j^1 ($j = 1, \dots, s$, as depicted in Fig. 2). Now we contract V^2 , changing it only within small neighborhoods A_j^3 of the A_j^1 's, onto a nonsingular fan V_*^2 , a small neighborhood H^3 of which is a 3-cell; that means we deform the 1-skeleton G^1 into the 3-cell H^3 . We carry out corresponding deformations (see footnote 4) of the 2-skeleton G^2 onto a “singular 2-skeleton” $G_{\#}^2$ and of its neighborhood J^3 onto a singular polyhedron $J_{\#}^3$; and we change the map β_0 correspondingly into a map $\beta_I : F_I'^3 \rightarrow M_*^3$ with $\beta_I | S'^2 = \beta_0 | S'^2$. All the deformations of G^2, J^3 take place in the A_j^3 's. $H^3 + \bigcup_{j=1}^s A_j^3$ is a handlebody K^3 . The corresponding deformations of F_0^2 onto F_I^2 are of type 1a only.

Now we have to deform the rest of F_I^2 into H^3 . First we remark that $J_{\#}^3$ may be decomposed into a neighborhood $T_{\#}^3$ of the deformed 1-skeleton $V_{\#}^2$ and into r “prismatic”, singular 3-cells $P_{\#i}^3$ (being prismatic neighborhoods of middle parts of the deformed E_i^2 's), such that $T_{\#}^3 \subset {}^0H^3$. That means, that part of F_I^2 lying outside of H^3 lies in the “top” and “bottom” disks of the $P_{\#i}^3$'s. The boundaries of the top and bottom disks of $P_{\#i}^3$ may be joined by an arc $W_i^1 \subset F_I^2 \cap {}^0H^3$ and by an arc $W_{Pi}^1 \subset P_{\#i}^3$; the so obtained 1-spheres $W_i^1 + W_{Pi}^1$ bound singular disks $W_i^2 \subset {}^0H^3$. We can choose these W_i^2 's such that their only singularities are double arcs and that singular, prismatic neighborhoods W_i^3 of them fit properly to F_I^2 and to the $P_{\#i}^3$'s. Then we expand the singular 3-annulus, defined by β_I , over these singular prisms W_i^3 (denoting the changed β_I by β_{II}); the corresponding deformation of F_I^2 onto a singular 2-sphere F_{II}^2 may be decomposed into deformations of type 1 (a and b) yielding a singular 2-sphere F_1^2 (and a map α_1 according to Theorem 3) and after them deformations of type 2 yielding F_{II}^2 . Now F_{II}^2 contains “folds” around the $P_{\#i}^3$'s consisting of the top and bottom disks and joining disks (containing the W_{Pi}^1 's); so we can expand the singular 3-annulus over the $P_{\#i}^3$'s (denoting the changed β_{II} by $\beta : F'^3 \rightarrow M_*^3$ with $\beta | S'^2 = \beta_0 | S'^2$). The corresponding deformation of F_{II}^2 yields $F_3^2 \subset {}^0H^3$ (and α_3) and may be decomposed into deformations of type 2, yielding F_2^2 (and α_2), and after them deformations of type 3 (a and b); this completes the proof of Theorem 3.

To prove Theorem 2 we observe that the complement $M_*^3 - {}^0K^3$ of the handlebody K^3 is covered one-to-one by β . So we deform the given curve C^1 isotopically into a curve $C_0^1 \subset M_*^3 - K^3$; then we choose a knot projection cone D'^2 bounded by the knot $\beta^{-1}(C_0^1)$ in the 3-annulus F'^3 ; we bring about by small deformations the situation in which $\beta(D'^2)$ has only normal singularities. Then $D^2 = \beta(D'^2)$ has the demanded properties. Theorem 1 is proved by extending β to a 3-sphere $S^3 \supset F'^3$.

We remark: If it were possible to find the map

$$\beta : F'^3 \rightarrow M_*^3$$

(with $\beta(\cdot F'^3 - S'^2) \subset {}^0H^3$) such that $\beta | \beta^{-1}(M_*^3 - H^3)$ is locally one-to-one then the Poincaré conjecture would follow by an easy conclusion. We would obtain such a map β if it were possible to deform the 3-annulus $\beta_0(F_0'^3)$ onto $\beta(F'^3)$ by "expansions" only. But in our procedure some of the very first deformations in the A_j^3 's (and only these) are not expansions, so we get certain surfaces in F'^3 such that β is not locally one-to-one at (and only at) the points of these surfaces. (β maps these surfaces homeomorphically into K^3 . Moreover it is possible to arrange our procedure such that these exceptional surfaces become disks.)

I. Proof of Theorems 1 and 2

We prove Theorem 1 and 2 first. After this we shall prove Theorem 3 by consideration of some more details.

4. Preliminaries. Let M^3 be a homotopy 3-sphere. After Moise [6] and Bing [1] there exists a triangulation of M^3 . This means there exists a homotopy 3-sphere, homeomorphic to M^3 , that is a (straight-lined, finite) polyhedron in a euclidean space \mathfrak{E}^n of sufficiently high dimension n . So we may assume for convenience and without loss of generality that M^3 itself is a polyhedron in \mathfrak{E}^n . All point sets considered in the subsequent part of this paper are *polyhedral in \mathfrak{E}^n* in the sense of [10] (i.e. finite unions of straight-lined, finite, convex, open cells in \mathfrak{E}^n); they are denoted by capital roman letters, and their dimensions by upper indices. We use the notation $\cdot X, \bar{X}, {}^0X$ for the *boundary, closure, interior* of X , respectively, and $X - Y = X - (X \cap Y)$ for the *difference*.

By a *decomposition* of X we mean always a collection of *finitely many* pairwise disjoint point sets whose union is X . A decomposition Δ is called a *cell-decomposition*, if the elements of Δ are open cells such that for every two cells $A, B \in \Delta$ either $A \cap B = \emptyset$ or $A \subset B$ holds. We call a cell-decomposition Δ a *straight-lined triangulation* if its elements are open, straight-lined simplices in \mathfrak{E}^n such that the open faces of each element are also elements of Δ ; we call a cell-decomposition Θ a *triangulation* in general if for each element $A \in \Theta$ the decomposition $\Theta(\bar{A})$ of \bar{A} , consisting of all those elements of Θ that lie in \bar{A} , is isomorphic to the decomposition of a simplex (of the same dimension as A) into its interior and its open faces.

By a (polyhedral) *neighborhood of X in Y* (as defined in [14]) we mean the closure of the simplex star of X in a second barycentric subdivision Δ^{**} of a (general) triangulation Δ of Y such that X is the union of elements of Δ ; the neighborhood is called *small with respect to $Z | V | \dots | W$* (see [4, Kap. I,2]) if $Z \cap Y, V \cap Y, \dots, W \cap Y$ are unions of elements of Δ .

By an *arc, disk, or 3-cell* we mean, if not stated otherwise, a compact, nonsingular 1-, 2-, or 3-cell, respectively.

All maps considered in the subsequent part of this paper are *simplicial* maps in the sense of [11, p. 114]: a continuous map $\alpha : A' \rightarrow B$ is called sim-

plicial if there exist straight-lined triangulations Δ' of A' and Δ of B such that α maps each element of Δ' linearly onto an element of Δ .

Let C^3 be a 3-cell in M^3 and denote the homotopy 3-cell $M^3 - {}^0C^3$ by M_*^3 .

5. A simple cell-decomposition Γ of M^3 . We can find a cell-decomposition Γ of M^3 with the following properties:

(i) Γ contains just one 0-dimensional element, say E^0 , and just one 3-dimensional element, say E^3 .

(ii) $C^3 \subset E^3$.

(iii) Γ contains r elements, say E_1^1, \dots, E_r^1 , of dimension 1 and r elements, say E_1^2, \dots, E_r^2 , of dimension 2.

(iv) Each element E_i^1 lies at least 2 times in the boundary of $\bigcup_{j=1}^r E_j^2$ (i.e.: if U^3 is a neighborhood of a point of E_i^1 in M^3 , which is small with respect to

$$E_1^1 | \dots | E_r^1 E_1^2 \dots | E_r^2,$$

then ${}^0U^3 \cap \bigcup_{j=1}^r E_j^2$ consists of at least 2 pairwise disjoint open disks).

Proof of the assertion. Γ may be found as follows:

Step 0. We take an arbitrary decomposition Γ_0 of M^3 into open cells.

Step 1. We delete, step by step, such 2-dimensional elements of Γ_0 that separate two different 3-dimensional elements; this yields finally a decomposition Γ_1 with only one 3-dimensional element (see [11]).

Step 2. Now we contract a maximal tree in the 1-skeleton of Γ_1 into one point; this yields a decomposition Γ_2 with property (i).

Step 3. If a 1-dimensional element $E^1 \in \Gamma_2$ lies just once in the boundary of a 2-dimensional element $E^2 \in \Gamma_2$ and does not lie in the boundary of any other 2-dimensional element of Γ_2 then we delete both E^1 and E^2 ; repeating this operation as often as possible, we obtain a decomposition Γ_3 with properties (i) and (iv). Γ_3 possesses also property (iii) since the Euler characteristic of M^3 is zero (see [11]).

Step 4. To obtain Γ we deform the 2-skeleton of Γ_3 isotopically such that the deformed 2-skeleton lies in $M^3 - C^3$.

Remark. In the case $r = 0$, M^3 is obviously a 3-sphere and we have nothing to prove. Therefore we may assume for the subsequent sections of this paper that $r \neq 0$. We denote the 1-skeleton $\bigcup_{i=1}^r \bar{E}_i^1$ and the 2-skeleton $\bigcup_{i=1}^r \bar{E}_i^2$ of Γ by G^1 , G^2 , respectively.

6. The 1-skeleton G^1 of Γ bounds a singular fan V^2 . We assert: There exists a map

$$\zeta : V^2 \rightarrow M_*^3,$$

with the image $\zeta(V^2) \subset {}^0M_*^3$ denoted by V^2 , and with the following properties (see Fig. 2):

(i) V^2 consists of r disks V_1^2, \dots, V_r^2 , possessing one common boundary

point E'^0 , and otherwise being pairwise disjoint; V'^2 is disjoint from M^3, F'^2 .

(ii) $\cdot V'^2 = G^1$.

(iii) The only singularities of V'^2 are pairwise disjoint, normal, double arcs A_1^1, \dots, A_s^1 (s may be zero) such that each of the two connected components $A_j'^1, A_j''^1$ of $\zeta^{-1}(A_j^1)$ possesses just one boundary point in $\cdot V'^2 - E'^0$ and otherwise lies in ${}^0V'^2$ (for all $j = 1, \dots, s$).

(iv) The arcs A_j^1 ($j = 1, \dots, s$) intersect $G^2 - G^1$ at most in isolated piercing points, V'^2 intersects $G^2 - G^1$ at most in piercing curves whose intersection and self-intersection points are the piercing points $A_j^1 \cap (G^2 - G^1)$.

(v) $\zeta^{-1}(\cdot V'^2 \cap [G^2 - G^1])$ is disjoint from $\cdot V'^2 - E'^0$, i.e. a connected component of

$$\zeta^{-1}(V'^2 \cap [G^2 - G^1])$$

is either a 1-sphere or an open arc whose boundary lies in

$$E'^0 + \cup_{j=1}^s [(A_j'^1 + A_j''^1) \cap {}^0V'^2]$$

(see Fig. 3).

Proof of the assertion. Step 0. Since M_*^3 is simply connected there exists a map $\zeta_0 : V'^2 \rightarrow M_*^3$ with properties (i) and (ii).

Step 1. From ζ_0 we can obtain by small deformations (by a similar procedure as described in [7]) a map $\zeta_I : V'^2 \rightarrow M_*^3$, also with properties (i), (ii), such that the only singularities of $V_I^2 = \zeta_I(V'^2)$ are normal double curves, triple points, and branch points of multiplicity 1 (see [8]), and such that the triple points, the branch points, and the interiors of the double curves lie in ${}^0V_I^2$, and that E^0 is no double point.

Step 2. Now we consider the set D_I of all double points (not including the triple points) of V_I^2 , and we remove, step by step, all those connected components $D_{I1}^1, \dots, D_{Id}^1$ of D_I that are disjoint from $\cdot V_I^2$. To do this we can find an arc $C_k^1 \subset V_I^2$ that joins a point of $\cdot V_I^2 - (E^0 + \cdot D_I)$ to a point of a component D_{Ik}^1 (provided that $d \neq 0$) such that ${}^0C_k^1 \cap \bar{D}_I, {}^0C_k^1 \cap \cdot V_I^2 = \emptyset$; then we remove D_{Ik}^1 (without introducing a new component of that kind) by a deformation of ζ_I (see Fig. 4) that changes V_I^2 only in a neighborhood of C_k^1 , and so on. In this way we obtain finally after d deformations a map $\zeta_{II} : V'^2 \rightarrow M_*^3$.

Step 3. Now we can remove the triple points of $V_{II}^2 = \zeta_{II}(V'^2)$ by deformations of ζ_{II} that change V_{II}^2 only in neighborhoods of double arcs of V_{II}^2 that join the triple points to $\cdot V_{II}^2 - E^0$. Further we can remove the branch points by cuts along those double arcs of V_{II}^2 that join the branch points to $\cdot V_{II}^2 - E^0$. This yields a map

$$\zeta_{III} : V'^2 \rightarrow M_*^3,$$

with $\zeta_{III}(V'^2)$ denoted by V_{III}^2 , such that the set D_{III} of double points of V_{III}^2 consists of pairwise disjoint arcs $D_{III1}^1, \dots, D_{IIIe}^1$.

Step 4. If one of the components of the inverse image of D_{IIIk}^1 —say D_{IIIk}^1 —is disjoint from $\cdot V'^2$, then we choose an arc $C_k^1 \subset V'^2$, joining a point of ${}^0D_{IIIk}^1$ to

a point of

$$\cdot V'^2 - [E'^0 + \zeta_{III}^{-1}(\cdot D_{III})],$$

with ${}^0C_k'^1 \cap \zeta_{III}^{-1}(D_{III})$, ${}^0C_k'^1 \cap \cdot V'^2 = \emptyset$, and we remove $D_{IIIk}'^1$ by a deformation of ζ_{III} (similar to Step 2) that changes V_{III}^2 only in a neighborhood of $\zeta_{III}(C_k'^1)$; and so on. This yields finally a map

$$\zeta_{IV} : V'^2 \rightarrow M_*^3$$

with the properties (i), (ii), and (iii).

Step 5. From ζ_{IV} we obtain by small deformations a map

$$\zeta_V : V'^2 \rightarrow M_*^3,$$

with $\zeta_V(V'^2)$ denoted by V_V^2 , having the properties (i), \dots , (iv).

Step 6. From ζ_V we obtain, by deformations that change V_V^2 only in a small neighborhood of $\cdot V_V^2 = G^1$, a map $\zeta : V'^2 \rightarrow M_*^3$ with the required properties.

7. Neighborhoods A_j^3 of the double arcs A_j^1 of V^2 . Let A_1^3, \dots, A_s^3 be pairwise disjoint neighborhoods of A_1^1, \dots, A_s^1 , respectively, in M_*^3 , which are small with respect to $G^2 \mid V^2$ (see Fig. 5a).

$A_j^3 \cap G^1$ consists of two disjoint arcs; we denote them by K_j^1, L_j^1 . The closures of the connected components of $(A_j^3 \cap V^2) - A_j^1$ are two disks; we denote them by $V_{K_j}^2, V_{L_j}^2$ such that

$$K_j^1 \subset \cdot V_{K_j}^2, \quad L_j^1 \subset \cdot V_{L_j}^2.$$

We choose a neighborhood A_j^2 of A_j^1 in $V_{K_j}^2$, which is small with respect to G^2 , and we denote the nonsingular fan $\bar{\cdot}(V^2 - \bigcup_{j=1}^s A_j^2)$ by V_*^2 .

We denote those connected components of $A_j^3 \cap G^2$ that contain K_j^1, L_j^1 , respectively, by K_j^2, L_j^2 . The closures of the connected components of $K_j^2 - K_j^1$ and $L_j^2 - L_j^1$ are disks $K_{j1}^2, \dots, K_{jt_j}^2$ and $L_{j1}^2, \dots, L_{ju_j}^2$, respectively. Those connected components of $A_j^3 \cap G^2$ that are different from K_j^2, L_j^2 are disks $N_{j1}^2, \dots, N_{jv_j}^2$ (v_j may be zero). We arrange the notation such that the disks $K_{j1}^2, \dots, K_{jt_j}^2$ lie around K_j^1 in the order of the enumeration and such that $V_{K_j}^2$ lies in this order between $K_{jt_j}^2$ and K_{j1}^2 .

8. A small neighborhood J^3 of the 2-skeleton G^2 and its complementary 3-annulus F_0^3 . Let T^3 be a neighborhood of G^1 in M_*^3 , which is small with respect to

$$G^2 \mid V^2 \mid A_1^3 \mid \dots \mid A_s^3 \mid A_1^2 \mid \dots \mid A_s^2;$$

Let J^3 be a neighborhood of G^2 in M_*^3 , which is small with respect to

$$T^3 \mid V^2 \mid A_1^3 \mid \dots \mid A_s^3 \mid A_1^2 \mid \dots \mid A_s^2.$$

Then $M_*^3 - {}^0J^3$ is a 3-annulus F_0^3 .

We denote $T^3 \cap J^3$ by T_J^3 , and the two connected components of $T_J^3 \cap A_j^3$ ($j = 1, \dots, s$) by $T_{K_j}^3, T_{L_j}^3$ (see Fig. 5b) such that $K_j^1 \subset T_{K_j}^3$ and $L_j^1 \subset T_{L_j}^3$.

Further we denote the connected components of $J^3 \cap A_j^3$ by $K_j^3, L_j^3, N_{j1}^3, \dots, N_{jv_j}^3$ where

$$K_j^2 \subset K_j^3, \quad L_j^2 \subset L_j^3, \quad N_{jm}^2 \subset N_{jm}^3 \quad (m = 1, \dots, v_j)$$

and the connected components of $^-(K_j^3 - T_{K_j}^3)$ and $^-(L_j^3 - T_{L_j}^3)$ by $K_{j1}^3, \dots, K_{jt_j}^3$ and $L_{j1}^3, \dots, L_{ju_j}^3$, respectively, where

$$K_{jk}^2 \cap K_{jk}^3 \neq \emptyset \quad (k = 1, \dots, t_j) \quad \text{and} \quad L_{j1}^2 \cap L_{j1}^3 \neq \emptyset \quad (1 = 1, \dots, u_j).$$

Those $t_j - 1$ connected components of $^-(A_j^3 - K_j^3)$ that are disjoint from $V_{K_j}^2$ are 3-cells $F_{K_{j1}}^3, \dots, F_{K_{jt_j-1}}^3$ in F_0^3 (see Fig. 5b).

The connected components of $^-(J^3 - T_j^3)$ are r 3-cells; we denote them by P_1^3, \dots, P_r^3 where $E_i^2 \cap P_i^3 \neq \emptyset$ ($i = 1, \dots, r$), and we denote the disks $E_i^2 \cap P_i^3$ by P_i^2 . Then P_i^3 can be represented as cartesian product $P_i^2 \times I^1$, where I^1 is the interval $-1 \leq x \leq +1$, such that

- (i) P_i^2 is the central disk, i.e. $p \times 0 = p$ for all $p \in P_i^2$;
- (ii) the top and bottom disks are the connected components of $P_j^3 \cap J^3$, i.e. $(P_i^2 \times 1) + (P_i^2 \times -1) = P_i^3 \cap J^3$;
- (iii) the polyhedra A_j^3, V^2, A_j^2 intersect P_i^3 "prismatically", i.e.:

$$A_j^3 \cap P_i^3 = (A_j^3 \cap P_i^2) \times I^1, \quad V^2 \cap P_i^3 = (V^2 \cap P_i^2) \times I^1, \quad A_i^2 \cap P_i^3 = (A_i^2 \cap P_i^2) \times I^1.$$

Let $F_0'^3$ be a 3-annulus, disjoint from M^3, V'^2, F'^2 , and let

$$\beta_0 : F_0'^3 \rightarrow M_*^3$$

be a homeomorphism with the image $\beta_0(F_0'^3) = F_0^3$. We denote the boundary 2-spheres $\beta_0^{-1}(J^3)$ and $\beta_0^{-1}(M_*^3)$ of $F_0'^3$ by $F_0'^2$ and S'^2 , respectively. (We may bring about by isotopic deformations the situation in which $\beta_0(F_0'^2) = \alpha_0(F'^2)$ with α_0 the embedding given in Theorem 3.)

9. Deformations in the A_j^3 's that take G^1 onto the boundary of the nonsingular fan V_*^3 . We denote the 3-cell $K_j^3 + \bigcup_{k=1}^{t_j-1} F_{K_{jk}}^3$ (see Fig. 5b) by Q_j^3 , and choose a neighborhood Q_{*j}^3 of $^-(A_j^3 - Q_j^3)$ in $^-(A_j^3 - Q_j^3)$, which is small with respect to $G^2 | V^2 | A_j^2 | T_j^3 | T_j^3 | J^3$, such that (with respect to the product representation introduced in Sec. 8)

$$^-(Q_{*j}^3 \cap P_i^3) = ^-(Q_{*j}^3 \cap P_i^2) \times I^1 \quad (i = 1, \dots, r).$$

Then we denote the 3-cell $^-[A_j^3 - (Q_j^3 + Q_{*j}^3)]$ by O_j^3 and the disks $O_j^3 \cap Q_j^3$ and $O_j^3 \cap Q_{*j}^3$ by O_j^2 and O_{*j}^2 , respectively.

Now we can find an epi-homeomorphism $\delta_j : Q_j^3 \rightarrow Q_j^3 + O_j^3$ with the following properties (see Fig. 5):

- (i) $\delta_j | (Q_j^3 - O_j^2) = \text{identity}$; $\delta_j(O_j^2) = O_{*j}^2$.
- (ii) $\delta_j(K_j^1) = (K_j^1 - A_j^2) + ^-(A_j^2 - K_j^1)$.
- (iii) $\delta_j(K_{jk}^2)$ intersects L_j^1 in just one point and intersects each disk $O_j^2, V_{L_j}^2, L_{j1}^2, \dots, L_{ju_j}^2, N_{j1}^2, \dots, N_{jv_j}^2$ in just one arc (for all $k = 1, \dots, t_j$); $\delta_j(O_{K_{jk}}^2)$ is disjoint from $V_{K_j}^2$.

(iv) The neighborhood $\delta_j(T_{K_j}^3)$ of $\delta_j(K_j^1)$ in A_j^3 is small with respect to $T_{L_j}^3 | V^2 | L_{j1}^3 | \cdots | L_{ju_j}^3 | N_{j1}^3 | \cdots | N_{jv_j}^3$ and intersects O_j^2 in just two disjoint disks.

(v) The intersections of $\delta_j(K_{jk}^3)$, $\delta_j(K_{jk}^2)$ ($k = 1, \dots, t_j$), and $\delta_j(T_{K_j}^3)$ with L_{j1}^3 ($1 = 1, \dots, u_j$) and N_{jm}^3 ($m = 1, \dots, v_j$) (see also Fig. 6) can be written as cartesian products, using the product representation of the P_i^3 's introduced in Sec. 8; the same holds for the polyhedra

$$\begin{aligned} \delta_j^{-1}(L_{j1}^3 \cap \delta_j(K_{jk}^3)), & \quad \delta_j^{-1}(L_{j1}^2 \cap \delta_j(K_{jk}^3)), & \quad \delta_j^{-1}(N_{jm}^3 \cap \delta_j(K_{jk}^3)), \\ \delta_j^{-1}(N_{jm}^2 \cap \delta_j(K_{jk}^3)), & \quad \delta_j^{-1}(T_{L_j}^3 \cap \delta_j(K_{jk}^3)), & \quad \delta_j^{-1}(V_{L_j}^2 \cap \delta_j(K_{jk}^3)). \end{aligned}$$

Let $\eta : J^3 \rightarrow M_*^3$ be the map defined by

- (a) $\eta | \neg(J^3 - \bigcup_{j=1}^s K_j^3) = \text{identity}$,
- (b) $\eta | K_j^3 = \delta_j | K_j^3$ (for all $j = 1, \dots, s$),

and denote the images $\eta(J^3), \eta(G^1), \eta(G^2), \eta(T_J^3), \eta(P_i^3)$ by $J_\#^3, G_\#^1, G_\#^2, T_{\#J}^3, O_{\#i}^3$, respectively. Obviously we have $G_\#^1 = V_*^2$.

Now we denote $\beta_0^{-1}(O_j^2)$ by $O_j'^2$, and we choose s pairwise disjoint 3-cells $O_1'^3, \dots, O_s'^3$ (see Fig. 7) that are disjoint from $M^3, V^2, F'^2, {}^0F_0'^3$ such that $O_j'^3 \cap F_0'^3 = O_j'^2$; then we denote $F_0'^3 + \bigcup_{j=1}^s O_j'^3$ by $F_I'^3$, and we choose a map

$$\beta_I : F_I'^3 \rightarrow M_*^3$$

with the following properties:

- (I) $\beta_I | \neg[F_0'^3 - \bigcup_{j=1}^s \bigcup_{k=1}^{t_j} \beta_0^{-1}(F_{K_{jk}}^3)] = \beta_0 | \neg[F_0'^3 - \bigcup_{j=1}^s \bigcup_{k=1}^{t_j} \beta_0^{-1}(F_{K_{jk}}^3)]$.
- (II) $\beta_I | \beta_0^{-1}(F_{K_{jk}}^3) = [\delta_j | F_{K_{jk}}^3] \cdot [\beta_0 | \beta_0^{-1}(F_{K_{jk}}^3)]$ for all $j = 1, \dots, s$; $k = 1, \dots, t_j$.
- (III) $\beta_I | O_j'^3$ is an epi-homeomorphism of $O_j'^3$ onto O_j^3 .

We remark that the map β_I is locally one-to-one, except for the "reflection disks" $O_j'^2$, i.e. if p is a point of $F_I'^3$ and if U^3 is a sufficiently small neighborhood of p in $F_I'^3$ then $\beta_I | U^3$ is a homeomorphism if and only if $p \notin \bigcup_{j=1}^s O_j'^2$.

10. $G_\#^1$ and its neighborhood $T_{\#J}^3$ lie in a 3-cell H^3 . Let H^3 be a neighborhood of $V_*^2 + T_{\#J}^3$ in M_*^3 , which is small with respect to

$$G_\#^2 | V^2 | J_\#^3 | A_1^3 | \cdots | A_s^3 | O_1^2 | \cdots | O_s^2,$$

that intersects the $P_{\#i}^3$'s prismatically, i.e.: $\eta^{-1}(H^3 \cap P_{\#i}^3)$ ($i = 1, \dots, r$) can be written as cartesian product using the product representation of the P_i^3 's introduced in Sec. 8 (compare Fig. 11a).

11. Arcs W_i^1 in $J^3 \cap T_J^3$ joining top and bottom disks of the prisms P_i^3 . T_J^3 a handlebody of genus r . The intersection $J^3 \cap T_J^3$ is a 2-sphere with $2r$ holes, denoted by T^2 .

We assert: There can be found r pairwise disjoint arcs $W_1^1, \dots, W_r^1 \subset T^2$ such that (for all $i = 1, \dots, r$)

(i) ${}^0W_i^1 \subset {}^0T_J^2$; $\cdot W_i^1 = p_i \times \cdot I^1$ (using the product representation of the P_i^3 's introduced in Sec. 8) with p_i an arbitrary point in $\cdot P_i^2 - \cup_{j=1}^s A_j^3$; we denote the arc $p_i \times I^1$ by W_{Pi}^1 ;

(ii) if $S_i^1 \subset {}^0T_J^3$ is a 1-sphere, topologically parallel to $W_i^1 + W_{Pi}^1$, i.e.: such that there exists an annulus in T_J^3 with boundary curves S_i^1 and $W_i^1 + W_{Pi}^1$, then S_i^1 is homologous to 0 mod 2 in $M_*^3 - (W_i^1 + W_{Pi}^1)$.

We denote the arc $\eta(W_i^1)$ by W_{*i}^1 . There exists just one connected component of $\beta_I^{-1}(W_{*i}^1)$ —we denote it by W_i^1 —such that $\beta_I(W_i^1) = W_{*i}^1$; and $W_i^1 \subset \cdot F_I^3$.

Proof of the assertion. First we remark that the 1-spheres $\cdot P_1^2, \dots, \cdot P_r^2$ form a 1-dimensional homology basis mod 2 of T_J^3 (if we identify the chains mod 2 with the corresponding polyhedra). If $\cdot P_1^2, \dots, \cdot P_r^2$ were homologously dependent mod 2 it would follow that there exists a surface in T_J^3 with boundary some of the $\cdot P_i^2$'s; this surface could be completed by the corresponding disks P_i^2 to a closed surface, non-separating in M_*^3 ; but this is impossible since M_*^3 is a homotopy 3-cell.

We choose an arbitrary system of pairwise disjoint arcs

$$W_1^{*1}, \dots, W_r^{*1} \subset T^2$$

fulfilling condition (i). Now $W_i^{*1} + W_{Pi}^1$ ($i = 1, \dots, r$) is homologous mod 2 in T_J^3 to a linear combination $\sum_{k=1}^r c'_{ik} \cdot P_k^2$ with coefficients $c'_{ik} = 0$ or 1. If $c'_{ii} = 0$ then we take $W_i^1 = W_i^{*1}$. If $c'_{ii} \neq 0$ then to obtain W_i^1 we take a small neighborhood N_i^2 of $\cdot P_i^2 \times 1$ in T^2 and replace the arc $W_i^{*1} \cap N_i^2$ by another arc in N_i^2 with the same boundary points such that $W_i^1 + W_{Pi}^1$ is homologous mod 2 to $W_i^{*1} + W_{Pi}^1 + \cdot P_i^2$ in T_J^3 . Now the W_i^1 's fulfill condition (ii) also. For every $i = 1, \dots, r$ there exists a surface in T_J^3 whose boundary consists of S_i^1 and some of the $\cdot P_k^2$'s, except $\cdot P_i^2$, and whose interior lies in ${}^0T_J^3$; this surface can be completed by the corresponding P_k^2 's to a surface B_i^2 in $M_*^3 - (W_i^1 + W_{Pi}^1)$ that is bounded by S_i^1 only.

12. Singular disks W_{*i}^2 in H^3 corresponding to the arcs W_{*i}^1 . Let W_1^2, \dots, W_r^2 be r pairwise disjoint disks that are disjoint from $M_*^3, {}^0F_I^3, F_I^2, V_I^2$ such that

$$W_i^2 \cap \cdot F_I^3 = \cdot W_i^2 \cap \cdot F_I^3 = W_i^1 \quad (\text{for all } i = 1, \dots, r).$$

We denote $\cdot W_i^2 - {}^0W_i^1$ by W_{Pi}^1 , and $\cup_{i=1}^r W_i^2$ by W^2 .

Now we assert: There exists a map $\vartheta : W^2 \rightarrow H^3$, with the image $\vartheta(W^2) \subset {}^0H^3$ denoted by W_{*}^2 , and with the following properties:

- (i) $\vartheta | W_i^1 = \beta_I | W_i^1$ and $\vartheta(W_{Pi}^1) = W_{Pi}^1$ (for all $i = 1, \dots, r$).
- (ii) The only singularities of W_{*}^2 are pairwise disjoint, normal, double arcs B_1^1, \dots, B_b^1 (b may be zero) such that each of the two connected components B_f^1, B_f^{n1} of $\vartheta^{-1}(B_f^1)$ possesses just one boundary point in $\cup_{i=1}^r {}^0W_i^1$ and otherwise lies in ${}^0W^2$ (for all $f = 1, \dots, b$). W^2 intersects the P_{*i}^3 's prismatically.
- (iii) There exists a neighborhood U^2 of $\cdot W^2$ in W^2 such that $\vartheta({}^0U^2) \subset {}^0T_{*}^3$.

Proof of the assertion. Step 0. Since $W_{\#i}^1 + W_{Pi}^1 \subset {}^0H^3$ (for all $i = 1, \dots, r$) there exists a map $\vartheta_0 : W'^2 \rightarrow H^3$ with property (i).

Step 1. As in the proof of Sec. 6, steps 1 to 5, we can derive from ϑ_0 a map $\vartheta_I : W'^2 \rightarrow H^3$ with properties (i), (ii).

Step 2. We choose pairwise disjoint neighborhoods N_1^3, \dots, N_r^3 of the 1-spheres $W_{\#i}^1 + W_{Pi}^1$ in H^3 , which are small with respect to $T_{\#J}^3 | \vartheta_I(W'^2)$. The intersection $N_i^3 \cap \vartheta_I(W'^2)$ consists of a 1-sphere N_i^1 , topologically parallel to $W_{\#i}^1 + W_{Pi}^1$, and of an even number n_i of meridian circles of N_i^3 each of which pierces N_i^1 in just one point. Now we choose an oriented 1-sphere X_i^1 in $N_i^3 \cap {}^0T_{\#J}^3$, topologically parallel to $W_{\#i}^1 + W_{Pi}^1$, and an oriented meridian circle Y_i^1 of N_i^3 that intersects X_i^1 in just one point; we denote the homology classes of X_i^1 and Y_i^1 in N_i^3 by ξ_i and η_i , respectively. Then the homology class n_i of the properly oriented 1-sphere N_i^1 is $n_i = \xi_i + w_i \eta_i$.

Now we need the fact that the coefficients w_i are even numbers. To prove this we show that both N_i^1 and X_i^1 are homologous 0 mod 2 in $M_*^3 - (W_{\#i}^1 + W_{Pi}^1)$:

(1) N_i^1 bounds a 2-dimensional polyhedron $D_i^2 \subset \vartheta_I(W'^2)$ that intersects $W_{\#i}^1 + W_{Pi}^1$ in the even number n_i of piercing points. From D_i^2 we remove n_i disks, being the intersections of D_i^2 with a small neighborhood U_i^3 of $W_{\#i}^1 + W_{Pi}^1$ in N_i^3 , and replace them by $\frac{1}{2}n_i$ annuli in U_i^3 such that we obtain a 2-dimensional polyhedron bounded by N_i^1 and disjoint from $W_{\#i}^1 + W_{Pi}^1$.

(2) $(\eta | T_J^3)^{-1}(X_i^1)$ is a 1-sphere $S_i^1 \subset {}^0T_J^3$ and there exists an annulus B_i^{*2} with boundary curves S_i^1 and $W_{\#i}^1 + W_{Pi}^1$ and with ${}^0B_i^{*2} \subset {}^0T_J^3$. On the other hand S_i^1 bounds a surface B_i^2 in $J^3 - (W_{\#i}^1 + W_{Pi}^1)$ as constructed in the proof of Sec. 11 which can be chosen disjoint from ${}^0B_i^{*2}$. We can bring about by small deformations the situation in which $\eta(B_i^2 + B_i^{*2})$ has normal double curves but no branch points (since η is locally one-to-one). Therefore (and since $\eta | B_i^{*2}$ is one-to-one) $\eta(B_i^2)$ intersects the boundary curve $W_{\#i}^1 + W_{Pi}^1$ of $\eta(B_i^2 + B_i^{*2})$ in an even number of piercing points. From $\eta(B_i^2)$ we obtain, as in (1), a 2-polyhedron disjoint from $W_{\#i}^1 + W_{Pi}^1$ with boundary X_i^1 .

If $w_i \neq 0$ (for some $i = 1, \dots, r$) then we choose a point in ${}^0W_{\#i}^1$, which is no double point of $\vartheta_I(W'^2)$, and a neighborhood R_i^3 of this point in N_i^3 which is small with respect to $\vartheta_I(W'^2) | W_{\#i}^1$. We denote the disk $R_i^3 \cap \vartheta_I(W'^2)$ by W_{Ri}^2 . In ${}^0R_i^3$ we choose a disk R_i^2 (see Fig. 8) such that $R_i^2 \cap W_{\#i}^1$ is one arc R_i^1 , such that ${}^0R_i^2 \cap {}^0W_{Ri}^2$ is an open arc one of whose boundary points lies in $R_i^2 - R_i^1$ and the other one in $W_{Ri}^1 - R_i^1$, and such that $\neg[(W_{Ri}^2 + R_i^2) \cap {}^0R_i^3]$ is an unknotted chord in R_i^3 . Then we choose an epi-homeomorphism

$$\lambda_i : R_i^3 \rightarrow R_i^3$$

with $\lambda_i | R_i^3 = \text{identity}$ and $\lambda(\neg[(W_{Ri}^2 + R_i^2) \cap {}^0R_i^3]) = W_{\#i}^1 \cap R_i^3$ and a map

$$\vartheta_{II} : W'^2 \rightarrow H^3$$

with

$$\vartheta_{II} | \neg[W'^2 - \vartheta_I^{-1}(W_{Ri}^2)] = \vartheta_I | \neg[W'^2 - \vartheta_I^{-1}(W_{Ri}^2)]$$

and

$$\vartheta_{II}(\vartheta_I^{-1}(W_{Ri}^2)) = \lambda_i(W_{Ri}^2 + R_i^2).$$

Now let N_{IIi}^3 be a neighborhood of $W_{\#i}^1 + W_{Pi}^1$ in N_i^3 , being small with respect to $\vartheta_{II}(W'^2) \mid T_{\#J}^3$. Then ${}^0N_{IIi}^3 \cap \vartheta_{II}(W'^2)$ consists of a 1-sphere N_{IIi}^1 , topologically parallel to $W_{\#i}^1 + W_{Pi}^1$, and of $n_i + 2$ meridian circles of N_{IIi}^3 . The homology class n_{IIi} of the properly oriented N_{IIi}^1 in $N_i^3 - {}^0N_{IIi}^3$ is

$$n_{IIi} = \xi_{IIi} + (w_i \pm 2)\eta_{IIi}$$

with ξ_{IIi}, η_{IIi} the homology classes of X_i^1, Y_i^1 , respectively, in $N_i^3 - {}^0N_{IIi}^3$. The sign in the coefficient $w_i \pm 2$ depends on the choice of R_i^2 (see Fig. 8). So we can derive by $\frac{1}{2} \sum_{i=1}^r w_i$ operations of the kind described a map

$$\vartheta_* : W'^2 \rightarrow H^3$$

such that (under analogous notation) the curve N_{*i}^1 is homologous to X_{*i}^1 in $N_i^3 - {}^0N_{*i}^3$ (for all $i = 1, \dots, r$).

If $w_i = 0$ (for all $i = 1, \dots, r$) then we choose $\vartheta_* = \vartheta_I$, etc.

Step 3. From ϑ_* we can obtain by deformations (that change $\vartheta_*(W'^2)$ only in the N_{*i}^3 's) a map $\vartheta : W'^2 \rightarrow H^3$ with the demanded properties (i), (ii), (iii).

13. Deformation over prismatic neighborhoods of the singular disks $W_{\#i}^2$. The map ϑ can be extended to a map $\tilde{\vartheta} : W'^3 \rightarrow H^3$, with $\tilde{\vartheta}(W'^3) \subset {}^0H^3$ denoted by $W_{\#}^3$, such that (see Fig. 9) the following hold:

(i) W'^3 may be represented as cartesian product $W'^2 \times I_*^1$ where I_*^1 means an interval $-1 \leq x_* \leq 1$, with $p \times 0 = p$ for all $p \in W'^2$, and W'^3 is disjoint from M^3, F'^2, V'^2 . We denote the components $W'^2 \times I_*^1$ of W'^3 by $W_i'^3$.

(ii) $W_i'^3 \cap F_i'^3 = \cdot W_i'^3 \cap \cdot F_i'^3 = W_i'^1 \times I_*^1$ with

$$\tilde{\vartheta} \mid (\cdot W_i'^3 \cap \cdot F_i'^3) = \beta_I \mid (\cdot W_i'^3 \cap \cdot F_i'^3).$$

(iii) $W_{\#}^3$ and the $P_{\#i}^3$'s intersect each other prismatically, i.e.:

$$\eta^{-1}(W_{\#}^3 \cap P_{\#i}^3) = \{[\eta^{-1}(W_{\#}^3 \cap P_{\#i}^3)] \cap P_{\#i}^2\} \times I^1$$

and

$$\tilde{\vartheta}^{-1}(W_{\#}^3 \cap P_{\#i}^3) = \{[\tilde{\vartheta}^{-1}(W_{\#}^3 \cap P_{\#i}^3)] \cap W'^2\} \times I_*^1$$

(using the product representations introduced in Sec.8 and in (i), respectively).

(iv) If p is a point of $W_{\#}^3$, $\vartheta^{-1}(p)$ is either one or two points. The set B of all double points of $W_{\#}^3$ is disjoint from the disks $\tilde{\vartheta}(W_{Pi}^1 \times I_*^1)$ ($i = 1, \dots, r$) and is prismatic, i.e.

$$\tilde{\vartheta}^{-1}(B) = [\tilde{\vartheta}^{-1}(B) \cap W'^2] \times I_*^1,$$

(using the same product representation as in (i)).

We denote the 3-annulus $F_I'^3 + W'^3$ by F_{II}^3 and we define a map

$$\beta_{II} : F_{II}^3 \rightarrow M_*^3$$

such that $\beta_{II} \mid F_I'^3 = \beta_I \mid F_I^3$ and $\beta_{II} \mid W'^3 = \tilde{\vartheta}$.

14. Deformation over the prisms $P_{\#i}^3$. In $\cdot F'_{II}{}^3 - S'^2$ there are $2r$ pairwise disjoint disks $P'_{+i}{}^2, P'_{-i}{}^2$ ($i = 1, \dots, r$) mapping onto the top and bottom disks of the $P_{\#i}^3$'s, i.e. such that $\beta_{II}(P'_{\pm i}{}^2) = \eta(P_i^2 \times \pm 1)$. Now we choose r pairwise disjoint 3-cells P_1^3, \dots, P_r^3 , disjoint from M^3, F'^2, V'^2 , such that

$$P_i^3 \cap F'_{II}{}^3 = \cdot P_i^3 \cap \cdot F'_{II}{}^3 = P'_{+i}{}^2 + P'_{-i}{}^2 + (W_{P_i}^1 \times I_*^1)$$

(being a disk, for all $i = 1, \dots, r$); and we choose epi-homeomorphisms

$$\kappa_i : P_i^3 \rightarrow P_i^3$$

such that $\eta_i \cdot \kappa_i | (\cdot P_i^3 \cap \cdot F'_{II}{}^3) = \beta_{II} | (\cdot P_i^3 \cap \cdot F'_{II}{}^3)$. Finally we denote the 3-annulus $F'_{II}{}^3 + \bigcup_{i=1}^r P_i^3$ by F'^3 and we define a map

$$\beta : F'^3 \rightarrow M_*^3$$

such that $\beta | F'_{II}{}^3 = \beta_{II}$ and $\beta | P_i^3 = \eta_i \cdot \kappa_i$.

We denote the handlebody $H^3 + \bigcup_{j=1}^s A_j^3$ by K^3 and $\beta^{-1}(K^3 \cap \beta(F'^3))$ by K'^3 . We remark that $\beta(F'^3 - S'^2) \subset {}^0H^3$ and that

$$\beta | \cdot (F'^3 - K'^3) : \cdot (F'^3 - K'^3) \rightarrow \cdot (M_*^3 - K^3)$$

is an epi-homeomorphism. Moreover β is locally one-to-one, except on the s surfaces $\cdot (O_j^3 \cap {}^0F'^3)$; it is locally three-to-one on the arcs $\cdot (O_j^2 \cap {}^0F'^3)$ and locally two-to-one otherwise on $\cdot (O_j^3 \cap {}^0F'^3)$.

15. Conclusion. There can be found an epi-homeomorphism $\lambda : M^3 \rightarrow M^3$ such that the image $C_0^1 = \lambda(C^1)$ of the given curve C^1 lies in ${}^0M_*^3 - K^3$. Then we choose a knot projection cone $D^2 \subset F'^3$ with $\cdot D^2 = \beta^{-1}(C_0^1)$. We can choose D^2 such that $\beta | D^2$ is locally one-to-one. Further we can bring about by small deformations the situation in which the singularities of the image $\beta(D^2)$ are normal. Then $D^2 = \lambda^{-1}(\beta(D^2))$ possesses the demanded properties. This proves Theorem 2.

We choose two disjoint 3-cells C'^3, C''^3 with

$$C'^3 \cap F'^3 = S'^2 = \cdot C'^3, \quad C''^3 \cap F'^3 = \cdot F'^3 - S'^2 = \cdot C''^3,$$

an epi-homeomorphism

$$\beta' : C'^3 \rightarrow C'^3$$

with $\beta' | S'^2 = \beta | S'^2$, and a map

$$\beta'' : C''^3 \rightarrow H^3$$

with $\beta'' | (\cdot F'^3 - S'^2) = \beta | (\cdot F'^3 - S'^2)$. Then $F'^3 + C'^3 + C''^3$ is a 3-sphere S^3 and the map $\gamma : S^3 \rightarrow M^3$, composed of β, β', β'' , has the demanded properties. This proves Theorem 1.

II. Proof of Theorem 3

We bring about (by isotopic deformations) the situation in which the 2-sphere $J^3 = \beta_0(F_0^2)$ (see Sec. 8) is equal to the image $F_0^2 = \alpha_0(F'^2)$ under the given

embedding α_0 . We denote the 2-spheres

$$\cdot F_I^{\prime 3} - S^{\prime 2}, \quad \cdot F_{II}^{\prime 3} - S^{\prime 2}, \quad \cdot F^{\prime 3} - S^{\prime 2}$$

by $F_I^{\prime 2}$, $F_{II}^{\prime 2}$, $F_{III}^{\prime 2}$, respectively, and we choose epi-homeomorphisms μ_0 , μ_I , μ_{II} , μ_{III} of $F_0^{\prime 2}$ onto $F_0^{\prime 2}$, $F_I^{\prime 2}$, $F_{II}^{\prime 2}$, $F_{III}^{\prime 2}$, respectively, such that $\alpha_0 = (\beta_0 | F_0^{\prime 2}) \cdot \mu_0$ and

$$\mu_{[i]}^{-1} | (F_{[i]}^{\prime 2} \cap F_{[i-1]}^{\prime 2}) = \mu_{[i-1]}^{-1} | (F_{[i]}^{\prime 2} \cap F_{[i-1]}^{\prime 2}) \quad (\text{for } [i] = I, II, III).$$

We denote the maps

$$(\beta_I | F_I^{\prime 2}) \cdot \mu_I, \quad (\beta_{II} | F_{II}^{\prime 2}) \cdot \mu_{II}, \quad (\beta | F_{III}^{\prime 2}) \cdot \mu_{III},$$

defining singular 2-spheres in M_*^3 , by α_I , α_{II} , α_3 , respectively. Now α_3 fulfills already the condition (b) of Theorem 3, and it remains to show that the deformation from α_0 to α_3 , which may be derived from the proof of Theorem 1, 2, can be decomposed into a sequence of elementary deformations, according to condition (a).

16. Decomposing the deformations in the A_j^3 's. The deformation from α_0 to α_I , changing the 2-sphere $F_0^{\prime 2}$ in the A_j^3 's (see Sec. 9), can be decomposed into a sequence of $\sum_{j=1}^s t_j \cdot (u_j + 2v_j)$ elementary deformations of type 1a, intermixed with nonessential deformations, (see Fig. 5).

We denote the connected components of the (prismatic) intersections

$$\eta(K_{jk}^3) \cap L_{jl}^3 \quad (j = 1, \dots, s; k = 1, \dots, t_j; l = 1, \dots, u_j)$$

under current enumeration by C_1^3, \dots, C_c^3 and the connected components of

$$\eta(K_{jk}^3) \cap N_{jm}^3 \quad (m = 1, \dots, v_j)$$

by D_1^3, \dots, D_d^3 . Further we denote that connected component of $\eta^{-1}(C_g^3)$ ($g = 1, \dots, c$) that is different from C_g^3 by $C_g^{\prime 3}$, and that connected component of $\eta^{-1}(D_h^3)$ ($h = 1, \dots, d$) that is different from D_h^3 by $D_h^{\prime 3}$. Finally we denote the intersections of the $C_g^3, C_g^{\prime 3}, D_h^3, D_h^{\prime 3}$'s with the P_i^2 's (see Fig. 11a) by $C_g^2, C_g^{\prime 2}, D_h^2, D_h^{\prime 2}$, respectively, and the intersections of the K_{jk}^3, L_{jl}^3 's with the P_i^2 's by K_{Pjk}^2, L_{Pjl}^2 , respectively.

17. Decomposing the deformations over $W_{\#}^3$. We can bring about by small deformations the situation in which the singular discs $W_{\#i}^2$ and their prismatic neighbourhood $W_{\#}^3$ (as constructed in Secs. 11, 12, 13) are in a "normal position" with respect to the singular 2-sphere $F_I^2 = \alpha_I(F^{\prime 2})$ and to the singular disks $P_{\#i}^2$, etc., i.e. such that the following conditions hold:

(i) F_I^2, H^3 , the A_j^3 's, and the $P_{\#i}^2$'s intersect $W_{\#}^3$ prismatically with respect to the product representation introduced in Sec. 13.

We denote $\mathfrak{F}(\vartheta^{-1}(p_i) \times I_*^1)$ by P_i^1 (Fig. 9).

(ii) $\eta^{-1}(W_{\#}^2 \cap P_{\#i}^2)$ ($i = 1, \dots, r$) is disjoint from those connected components of $K_{Pjk}^2 \cap \eta^{-1}(H^3 \cap P_{\#i}^2)$ and $L_{Pjl}^2 \cap \eta^{-1}(H^3 \cap P_{\#i}^2)$ ($j = 1, \dots, s$;

$k = 1, \dots, t_j; l = 1, \dots, u_j$) that contain the arcs $\cdot K_{Pjk}^2 \cap \cdot P_i^2, \cdot L_{Pl}^2 \cap \cdot P_i^2$, respectively, in their boundaries (see Fig. 11a).

Now we carry out the deformation of α_I into α_{II} in three steps:

Step 1. Let B_f^3 ($f = 1, \dots, b$) (see Fig. 10) be that connected component of $\bar{\vartheta}^{-1}(B^3)$ that contains B_f^1 . We choose pairwise disjoint neighborhoods B_{*f}^3 of the B_f^3 's in W^3 , which are small with respect to $\bar{\vartheta}^{-1}(F_I^2 \cap W_{*}^3) | \bar{\vartheta}^{-1}(B^3)$ and which are cartesian products in the product representation introduced in Sec. 13. Now we deform F_I^2 over the 3-cells $\bar{\vartheta}(B_{*f}^3)$ which can be done by a sequence of elementary deformations of type 1a. We denote the map so obtained from α_I by α_{I*} and $\bar{\vartheta}(W^2 - \bigcup_{f=1}^b B_{*f}^3)$ by W_{*}^2 . Now we have to deform $F_{I*}^2 = \alpha_{I*}(F^2)$ over the remaining nonsingular 3-cells $\bar{\vartheta}(W_{*}^2 \times I_{*}^1)$.

Step 2. In W_{*}^2 we choose pairwise disjoint arcs X_1^1, \dots, X_x^1 (see Fig. 10) with ${}^0X_m^1 \subset {}^0W_{*}^2$ that join points of

$$\cdot W_{*}^2 - \bigcup_{i=1}^r W_{Pi}^1$$

to points of

$$\vartheta^{-1}(F_{I*}^2 \cap W_{*}^2) \cap {}^0W_{*}^2$$

such that

- (a) every double point of $\vartheta^{-1}(F_{I*}^2 \cap W_{*}^2) \cap {}^0W_{*}^2$ is end point of one arc X_m^1 ,
- (b) every connected component of $\vartheta^{-1}(F_{I*}^2 \cap W_{*}^2) \cap {}^0W_{*}^2$ contains at least one end point of an arc X_m^1 ,
- (c) the X_m^1 's intersect $\vartheta^{-1}(F_{I*}^2 \cap W_{*}^2) \cap {}^0W_{*}^2$ in isolated piercing points that are no double points of $\vartheta^{-1}(F_{I*}^2 \cap W_{*}^2) \cap {}^0W_{*}^2$,
- (d) the points $\vartheta(\cdot X_m^1 \cap \cdot W_{*}^2)$ are no double points of F_{I*}^2 .

Now we choose pairwise disjoint neighborhoods X_m^2 of the X_m^1 's in W_{*}^2 , which are small with respect to $\vartheta^{-1}(F_{I*}^2 \cap W_{*}^2)$. Then we deform F_{I*}^2 over the 3-cells $\bar{\vartheta}(X_m^2 \times I_{*}^1)$ which can be done by a sequence of elementary deformations of type 1a and 1b. According to the notation used in Theorem 3 we denote the map so obtained from α_{I*} by α_1 and $\alpha_1(F^2)$ by F_1^2 . Further we denote $\bar{\vartheta}(W_{*}^2 - \bigcup_{m=1}^x X_m^2)$ by W_{**}^2 .

Step 3. Finally we deform F_1^2 over the remaining 3-cells $\bar{\vartheta}(W_{**}^2 \times I_{*}^1)$. This can be done by a sequence of elementary deformations of type 2 (and may be nonessential deformations) since the curves $\vartheta^{-1}(F_1^2 \cap W_{**}^2) \cap {}^0W_{**}^2$ are nonsingular, pairwise disjoint, open arcs with boundary points in

$$\cdot W_{**}^2 - \bigcup_{i=1}^r W_{Pi}^1.$$

By this we obtain from α_1 the map α_{II} .

18. Decomposing the deformations over the P_{*i}^3 's. We carry out the deformation of α_{II} into α_3 in four steps (see Fig. 11).

Step 1. Let Q_i^1 be a neighborhood of a point $\epsilon \cdot P_i^1$ in $\cdot P_i^2 - {}^0P_i^1$ which is small

with respect to $\eta^{-1}(F_{II}^2 \cap P_{**i}^2)$ and let $Y_i^1 = \cdot P_i^2 - {}^0Q_i^1$. Further we choose a neighborhood Y_i^2 of Y_i^1 in P_i^2 , which is small with respect to

$$\eta^{-1}(H^3 \cap P_{**i}^2) | \eta^{-1}(F_{II}^2 \cap P_{**i}^2) | \bigcup_{j,k=1}^{s,t_j} K_{Pjk}^2$$

and intersecting the disks $C_g^2, C_g'^2, D_h'^2$ prismatically, i.e. such that

$$\eta^{-1}(\eta(Y_i^2 \times I^1)) = [\eta^{-1}(\eta(Y_i^2 \times I^1)) \cap P_i^2] \times I^1$$

(using the product representation introduced in Sec. 8). Then we deform F_{II}^2 over the 3-cells $\eta(Y_i^2 \times I^1)$ which can be done by a sequence of elementary deformations of type 2 (and may be nonessential deformations). We denote the map so obtained from α_{II} by α_{II*} , and $\alpha_{II*}(F'^2)$ by F_{II*}^2 , further $\bar{(P_i^2 - Y_i^2)}$ by P_{**i}^2 (see Fig. 11b), the image $\eta(P_{**i}^2)$ by P_{**i}^2 , and the intersections of K_{Pjk}^2, L_{Pjl}^2 with the P_{**i}^2 's by K_{**jk}^2, L_{**jl}^2 , respectively. Further we denote the set of double points of

$$\eta(\bigcup_{i=1}^r P_{**i}^2 \times I^1)$$

by D_* and the connected components of

$$\eta^{-1}(D_*) \cap \bigcup_{i=1}^r P_{**i}^2$$

by $C_{*g}^2, C_{*g}'^2, D_{*h}^2, D_{*h}'^2$ such that

$$C_{*g}^2 \subset C_g^2, \quad C_{*g}'^2 \subset C_g'^2, \quad D_{*h}^2 \subset D_h^2,$$

$$D_{*h}'^2 \subset D_h'^2 \quad (g = 1, \dots, c; h = 1, \dots, d).$$

Step 2. We choose pairwise disjoint arcs $Y_{i_1}^1, \dots, Y_{i_{y_1}}^1$ (see Fig. 11b) in P_{**i}^2 with ${}^0Y_{if}^1 \subset {}^0P_{**i}^2$ ($f = 1, \dots, y_i$) that join points of $\cdot Y_i^2$ to points in ${}^0P_{**i}^2 - \eta^{-1}(F_{II*}^2 \cap P_{**i}^2)$, and we choose pairwise disjoint neighborhoods Y_{if}^2 of the Y_{if}^1 's in P_{**i}^2 , which are small with respect to $\eta^{-1}(F_{II*}^2 \cap P_{**i}^2) | \bigcup_{j,k=1}^{s,t_j} K_{**jk}^2$ such that, with the notation $P_{**i}^2 = \bar{(P_{**i}^2 - \bigcup_{f=1}^{y_i} Y_{if}^2)}$, the following hold:

(i) The arcs Y_{if}^1 intersect the curves $\bar{[\eta^{-1}(F_{II*}^2 \cap P_{**i}^2) \cap {}^0P_{**i}^2]}$ in isolated piercing points that are no double points (and no boundary points) of that curves.

(ii) The arcs Y_{if}^1 are disjoint from the disks $C_{*g}^2, C_{*g}'^2, D_{*h}^2$ ($g = 1, \dots, c; h = 1, \dots, d$) and from the arcs $\bar{(K_{**jk}^2 \cap {}^0P_{**i}^2)}$ ($j = 1, \dots, s; k = 1, \dots, t_j$) and intersect the disks $D_{*h}'^2$ prismatically, i.e. such that

$$\eta(Y_{if}^1 \cap D_{*h}'^2) = [\eta(Y_{if}^1 \cap D_{*h}'^2) \cap D_{*h}^2] \times I^1$$

using the product representation introduced in Sec. 8. The Y_{if}^2 's intersect the $D_{*h}'^2$'s also prismatically.

(iii) If Z^1 is a connected component of $\bar{[\eta^{-1}(F_{II*}^2 \cap P_{**i}^2) \cap {}^0P_{**i}^2]}$ then one of the following cases holds (see Fig. 12):

case a. Z^1 is an arc (that is either disjoint from the disks $C_{*g}^2, C_{*g}'^2, D_{*h}^2, D_{*h}'^2$ or lies in the boundary of one disk $C_{*g}^2, C_{*g}'^2$, or D_{*h}^2).

case b. Z^1 consists of two arcs, piercing each other in one point, and is disjoint from the disks $C_{*g}^2, C'_{*g}{}^2, D_{*h}^2, D'_{*h}{}^2$.

case c. Z^1 consists of two arcs Z_1^1, Z_2^1 lying in the boundary of one disk D_{*h}^2 , and of one arc Z_3^1 that pierces Z_1^1 and Z_2^1 each in one point.

case d. Z^1 consists of the boundary of one disk D_{*h}^2 and of an arbitrary number of pairwise disjoint arcs that intersect D_{*h}^2 each in one arc (and D_{*h}^2 each in two points).

Then we deform F_{II*}^2 over the 3-cells $\eta(Y_{ij}^2 \times I^1)$ ($i = 1, \dots, r$; $j = 1, \dots, y_i$) which can be done by a sequence of elementary deformations of type 2 (and may be nonessential deformations). According to the notation used in Theorem 3 we denote the map so obtained from α_{II*} by α_2 and $\alpha_2(F'^2)$ by F_2^2 . Further we denote the intersections of the disks K_{*jk}^2 with the P_{**i}^2 's by K_{**jk}^2 .

Step 3. Now we deform F_2^2 over the 3-cells $\eta(K_{**jk}^2 \times I^1)$ ($j = 1, \dots, s$; $k = 1, \dots, t_j$) which can be done by a sequence of elementary deformations of type 3a and 3b and nonessential deformations. We denote the map so obtained from α_2 by α_{2*} and $\alpha_{2*}(F_2^2)$ by F_{2*}^2 .

Step 4. The remaining parts $\eta([P_{**i}^2 - \bigcup_{j,k=1}^{s,t_j} K_{**jk}^2] \times I^1)$ of the P_{**i}^2 's are nonsingular 3-cells, and we can deform F_{2*}^2 over them by a sequence of elementary deformations of type 3a and 3b (and may be nonessential deformations). By this we obtain from α_{2*} the map α_3 .

19. Conclusion. The maps α_1 and α_2 , as obtained in Sec. 17, Step 2, and Sec. 18, Step 2, respectively, and the map α_3 possess the demanded properties, and Theorem 3 is proved.

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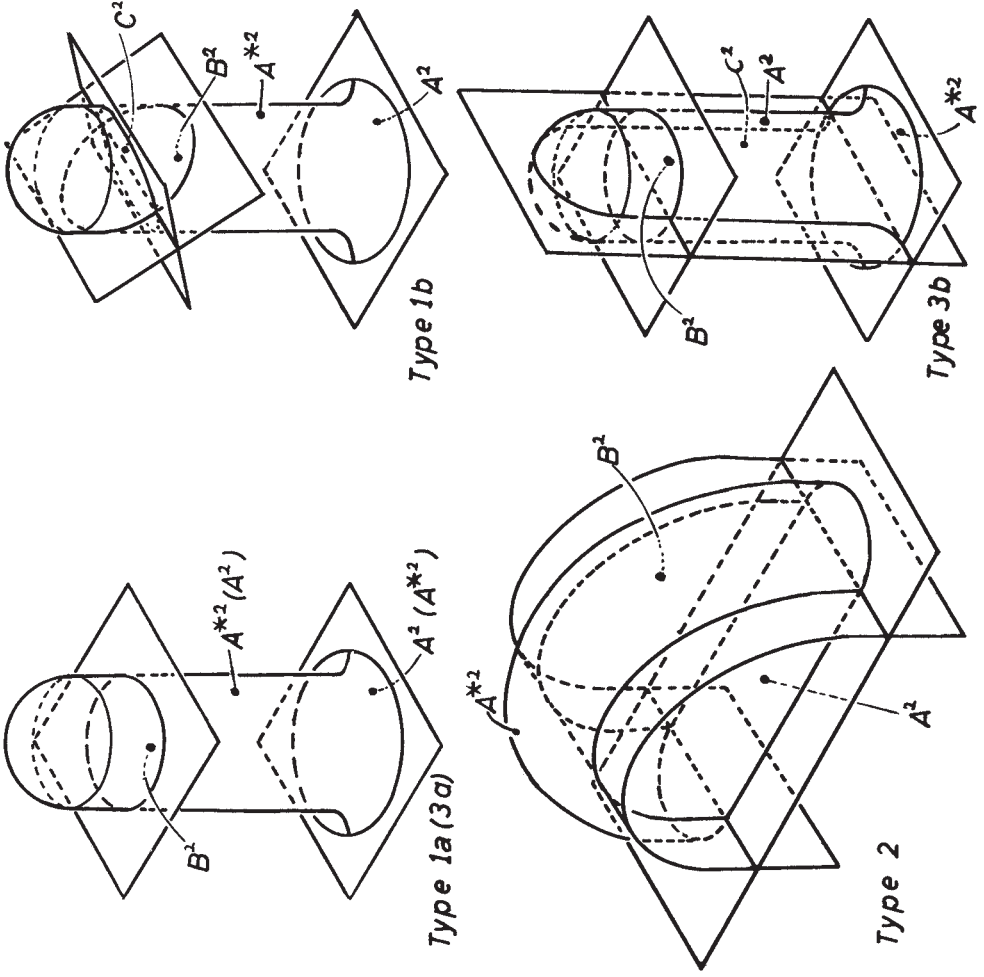


FIGURE 1

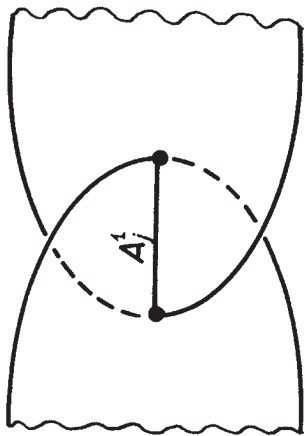
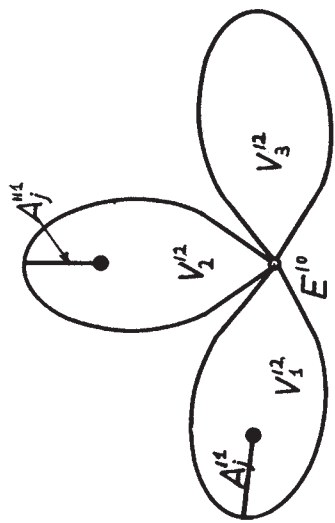


FIGURE 2

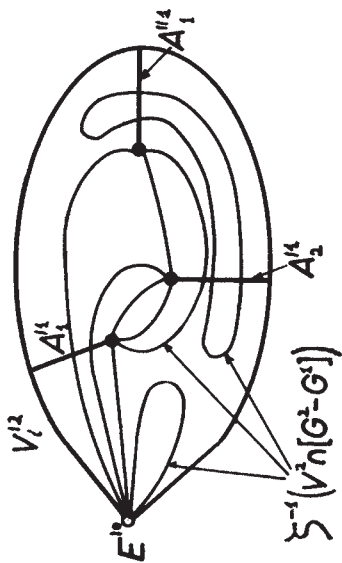


FIGURE 3

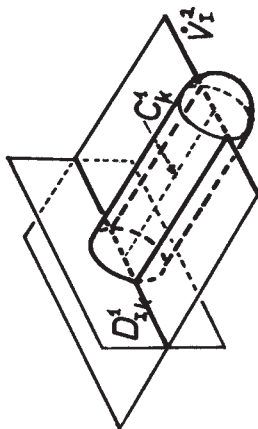


FIGURE 4

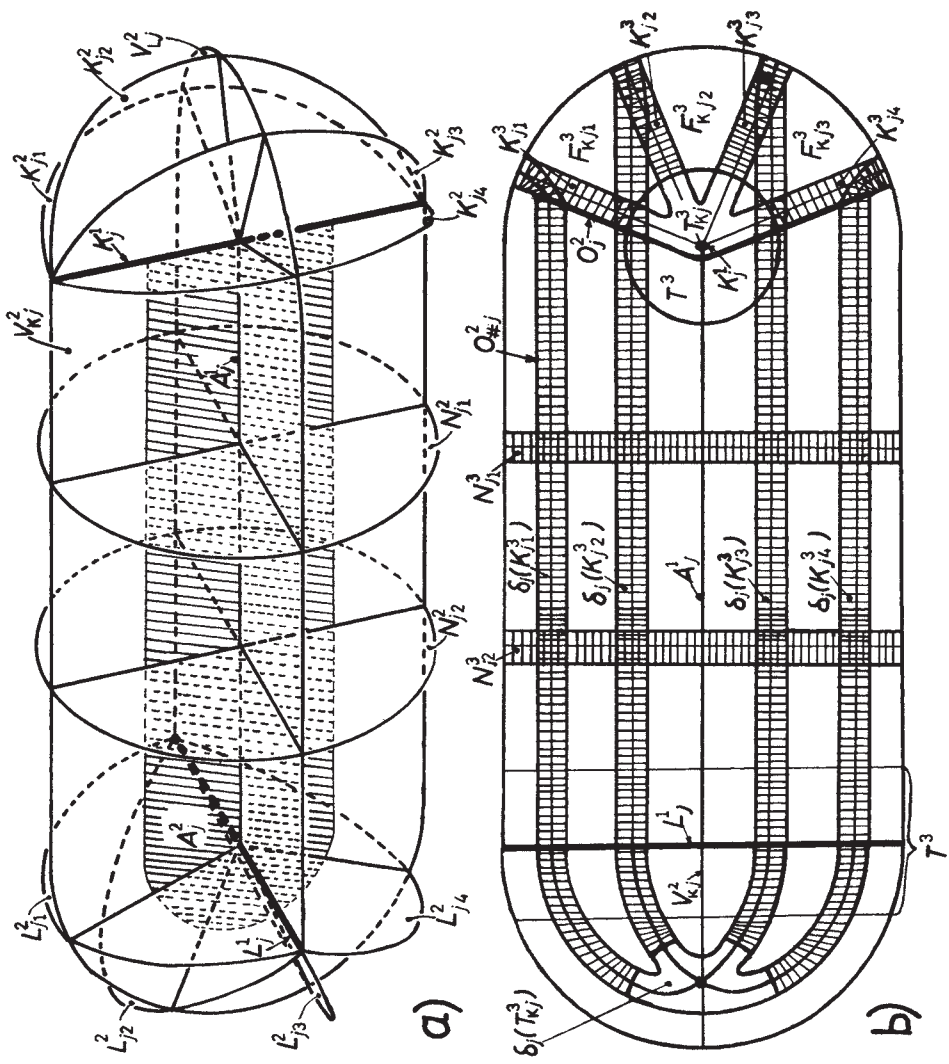


FIGURE 5
 (a) depicts the intersection of V^2 and G^2 with A_i^1 .
 (b) depicts the intersection of (a), plus $T^3 \cap A_i^1, J^3 \cap A_i^1, J^3 \cap A_i^1$, with the $V_{L_i}^2$ -plane.
 The product representation of the K_j^2 's and $F_{K_j^2}^3$'s is indicated by hatching.

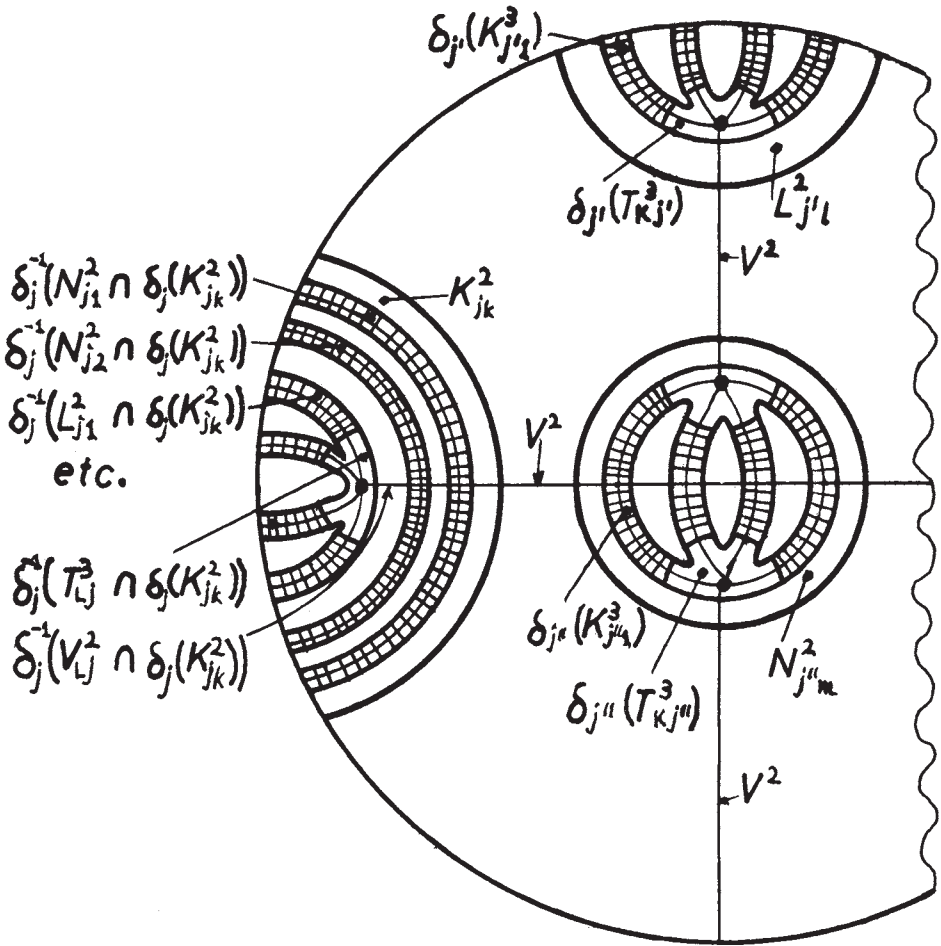


FIGURE 6

Intersections of K_{jk}^2 , $L_{j'l}^2$, $N_{j''m}^2$ ($j, j', j'' = 1, \dots, s$; $k = 1, \dots, t_j$; $l = 1, \dots, u_{j'}$; $m = 1, \dots, v_{j''}$), etc. with P_i^2 .

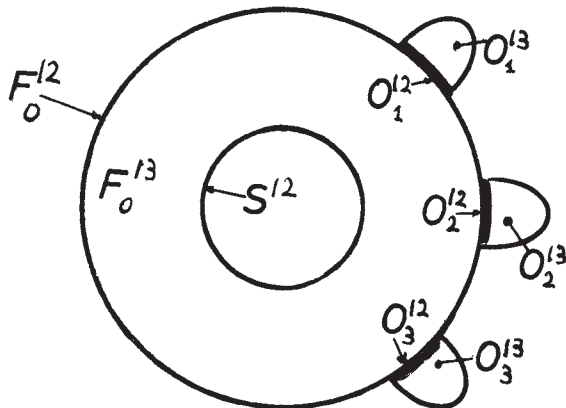


FIGURE 7

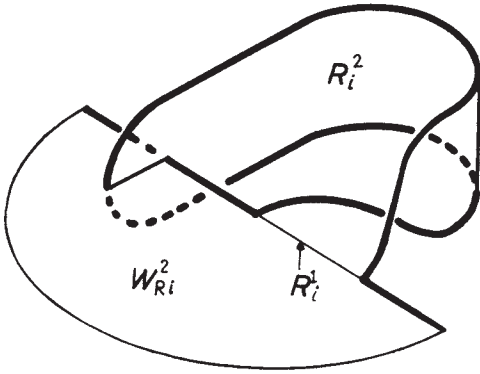


FIGURE 8

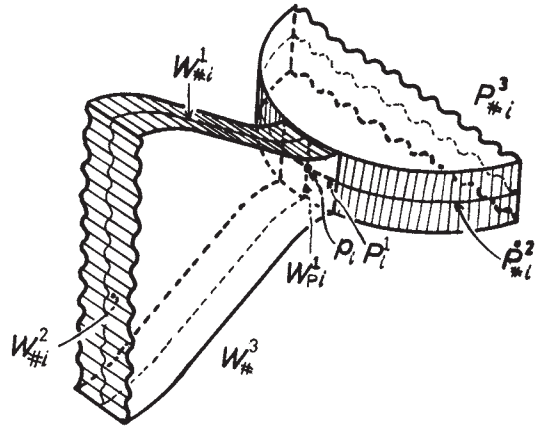


FIGURE 9

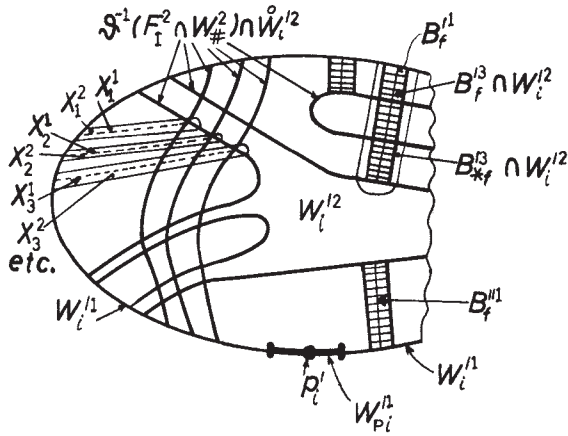


FIGURE 10

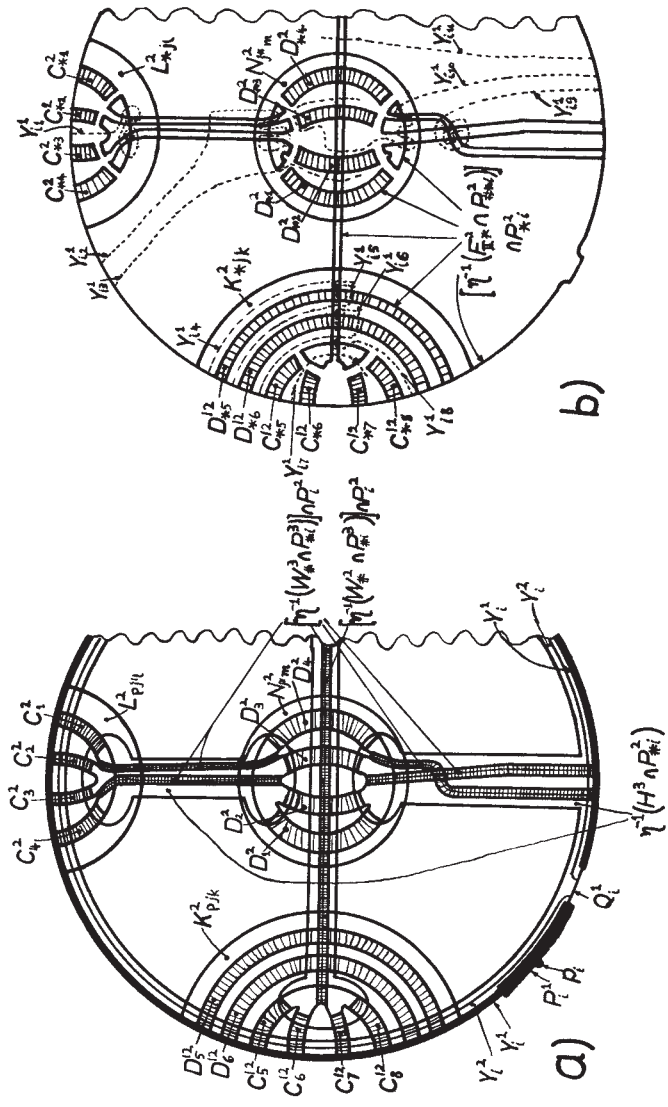


FIGURE 11. (a) depicts P^2 . (b) depicts P^2 .

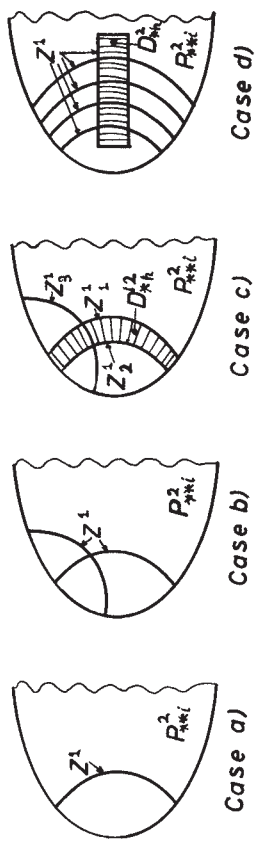


FIGURE 12