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NOTE ON THE HOMOTOPY GROUPS OF A BOUQUET $S^1 \vee Y$, Y 1-CONNECTED

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Abstract

A study is made of the action of the fundamental group of a bouquet of a circle and a 1-connected space on the higher homotopy groups. If the 1-connected space is a suspension space, it is shown, with the aid of a theorem of Hartley on wreath products of groups and the Hilton-Milnor theorem, that the action is residually nilpotent. An unsuccessful approach in the case of a general 1-connected space is discussed, as it has some interesting features.

1. Introduction; Statement of results

For a 1-connected, finite CW-complex X, the automorphism group $Aut(X)$ (i.e., the group of pointed homotopy classes of pointed self-homotopy equivalences of X) and its subgroup $Aut_*(X)$ consisting of those automorphisms inducing the identity on all integral homology groups of X, possess certain finiteness properties. However, if X is not 1-connected, these finiteness properties no longer hold; see [**6, 5**] for a precise discussion. A main focus in [**5**] is a study of $Aut(S^1 \vee Y)$ and $Aut_*(S^1 \vee Y)$, where Y is 1-connected. It turns out that these automorphism groups are better behaved than the automorphism groups $Aut(X)$ and $Aut_*(X)$ for general non-1-connected X; the latter can be quite unruly.

In this note, we prove a qualitative result about the action of the fundamental group of $S^1 \vee Y$ on the higher homotopy groups of $S^1 \vee Y$, Y 1-connected, which plays a key role in one of the principal results in [**5**]. To explain, we recall a definition: Let π be a group, M a $\pi-$module. Set $\Gamma_1(M) = M$. If λ is an ordinal that has an immediate predecessor $\lambda - 1$, define $\Gamma_\lambda(M)$ to be the π-module generated by elements of the form

$$g.m - m, \quad g \text{ in } \pi,\ m \text{ in } \Gamma_{\lambda-1}(M).$$

I thank Gilbert Baumslag for explaining the relevance of wreath products to the proof of Theorem 1.2 and for a number of discussions about wreath products; and I thank Pete Bousfield for his example, allowing me to be liberated from continuing to pursue a futile path.

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If λ is a limit ordinal, then define

$$\Gamma_\lambda(M) = \bigcap_{\mu<\lambda} \Gamma_\mu.$$

$\Gamma_\lambda(M)$ is called the $\lambda-$th term in the lower central π-series of M.

Conjecture 1.1. *Let $X = S^1 \vee Y$, Y a 1-connected CW-complex, $\pi = \pi_1(X) \cong \mathbb{Z}$, $M = \pi_n(X)$. Then there exists a countable ordinal γ, depending on n, such that $\Gamma_\gamma(M) = 1$.*

In support of this conjecture, we prove the following special case.

Theorem 1.2. *(i) Let $\tilde{M} = H_n(\tilde{X})$, the integral homology group of the universal covering $\tilde{X}$ of X. Then the π-action on $\tilde{M}$ is residually nilpotent, i.e., $\Gamma_\omega(\tilde{M}) = 1$, ω being the first infinite ordinal;*

(ii) Assume $Y = \Sigma Z$, the suspension of a path-connected CW- complex Z. Then the π-action on M is residually nilpotent, i.e., $\Gamma_\omega(M) = 1$.

The proof of Theorem 1.2, to be carried out in Section 2, will make use of a theorem of Brian Hartley on wreath products, and for part (ii), the Hilton-Milnor Theorem.

In an appendix (Section 3), we outline our original, failed approach to proving the conjecture, based on Theorem 1.2(i) and a proposed generalization of certain results from [**2**]. Our rationale for including this material is two-fold: we wish to take the opportunity to complete the proof of a result from [**2**]; and we wish to present a mild variant of an interesting and astute example of A. K. Bousfield (private communication) that vividly demonstrates the stark difference between nilpotency ($\Gamma_N = 1$ for some finite N) and residual nilpotency ($\Gamma_\omega = 1$).

2. Proof of Theorem 1.2.

(i) $\tilde{X}$ has the homotopy type of a double infinite bouquet,

$$\tilde{X} \simeq \bigvee_{i=-\infty}^{\infty} Y_i,$$

each Y_i a copy of Y. Thus, abbreviating

$$A = H_n(Y), \qquad A_i = H_n(Y_i)$$

we have

$$\tilde{M} \cong \bigoplus_{i=-\infty}^{\infty} A_i,$$

and π acts on the latter by translation. Precisely, a suitable generator t of π shifts a tuple by one unit to the right, i.e.,

$$t.(\dots, a_{-1}, a_0, a_1, \dots) = (\dots, a_0, a_1, a_2, \dots).$$

Now consider the (restricted) wreath product

$$W = A\; wr\; \pi,$$

which, by definition, is the semi-direct product

$$\bigoplus_{i=-\infty}^{\infty} A_i \rtimes \pi,$$

with the π-action described above. Straightforward computation shows that $\Gamma_k(W)$, the kth term in the standard lower central series of the group W, contains $\Gamma_k(\tilde{M})$, the kth term in the lower central π-series of $\tilde{M}$. But according to [1, Corollary of Theorem B2], the group W is residually nilpotent in the standard group-theoretic sense. (Hartley's result asserts that a sufficient condition for W to be residually nilpotent is that A be abelian and that π be torsion-free, finitely generated nilpotent.) Thus $\Gamma_\omega(W) = 1$, and hence also $\Gamma_\omega(\tilde{M}) = 1$, as claimed.

(ii) We proceed to construct a suitable abelian group B and to apply Hartley's result to the (restricted) wreath product $V = B\ wr\ \pi$, as in part (i). Write $Y_i = \Sigma Z_i$ and, for any space U and $s \geqslant 0$, abbreviate the s-fold iterated smash product

$$U \wedge U \wedge \cdots \wedge U = U^{(s)}.$$

Consider the iterated smash product

$$Z_j^{(n_j)} \wedge Z_{j+1}^{(n_{j+1})} \wedge \cdots \wedge Z_{j+r}^{(n_{j+r})}$$

for any j, $-\infty < j < \infty$, and any $r \geqslant 0$. It follows from the Hilton-Milnor Theorem [4, Theorem 4] that M is isomorphic to a direct sum, extending over all j and r as above, of terms of the form

$$\pi_n \left(\Sigma (Z_j^{(n_j)} \wedge Z_{j+1}^{(n_{j+1})} \wedge \cdots \wedge Z_{j+r}^{(n_{j+r})}) \right).$$

For any i, $-\infty < i < \infty$, denote by B_i the sub-direct sum of M consisting of those terms of the form

$$\pi_n \left(\Sigma (Z_i^{(n_i)} \wedge Z_{i+1}^{(n_{i+1})} \wedge \cdots \wedge Z_{i+r}^{(n_{i+r})}) \right), \quad n_i > 0$$

Clearly,

$$M \cong \bigoplus_{i=-\infty}^{\infty} B_i.$$

Just as in part (i), π acts on the latter direct sum by translation. Let

$$B = B_0, \qquad V = B\ wr\ \pi$$

Again as in part (i), V is residually nilpotent by [1, Corollary of Theorem B2], $\Gamma_k(V)$ contains $\Gamma_k(M)$, and so the π-action on M is residually nilpotent, as claimed.

3. Appendix

We will first restate the assertion (2.19), from [2, Chapter II]. It was our original hope to prove an appropriate generalization of this result and to combine it with Theorem 1.2(i) in order to prove the full conjecture. We will then describe a mild variant of an example of Bousfield showing that this strategy is doomed.

Theorem 3.1. *Let X be a path-connected CW-complex. If the fundamental group π of X is nilpotent and acts nilpotently on the homology groups of the universal covering of X, then π also acts nilpotently on the higher homotopy groups of X.*

Since the proof of Theorem 3.1 was omitted in [2] (Theorem 3.1 was not actually used in [2]), we will sketch a proof here.

The Postnikov system of $\tilde{X}$ yields a sequence of fibrations

$$K(\pi_m(X), m) \to \tilde{X}_m \to \tilde{X}_{m-1},\ m \geqslant 2$$

with $\tilde{X}_1 = 0$. We apply Hu's "truncated exact sequence" [3, pp. 284-285] to this fibration, a portion of which is

$$\cdots \to H_{m+1}(\tilde{X}_{m+1}) \to H_m(K(\pi_m(X), m)) \to H_m(\tilde{X}_m) \to \cdots .$$

Assume inductively that π acts nilpotently on the homology groups of $\tilde{X}_{m-1}$. Since

$$H_m(\tilde{X}_m) \cong H_m(\tilde{X}),$$

it follows that π acts nilpotently on

$$H_m(K(\pi_m(X), m)) \cong \pi_m(X).$$

Then, by [2, Lemma 2.17 and the argument in Lemma 2.18], we infer that π acts nilpotently on the homology of $\tilde{X}_m$, thereby completing the inductive step.

In an attempt to prove the conjecture, we weaken the assumption on π in Theorem 3.1 to: $\Gamma_\omega(H_n(\tilde{X})) = 1$ for all n; and hope to conclude $\Gamma_\gamma(\pi_n(X)) = 1$ for some countable ordinal depending on n. Following the pattern of proof of Theorem 3.1, we come across a stumbling block when trying to apply [2, Chapter II, Lemma 2.17] with a weakened hypothesis. Recall [2, Chapter II, Lemma 2.17]: If π acts nilpotently on an abelian group A, then π also acts nilpotently on all the integral homology groups of A. Suppose instead that $\Gamma_\omega(A) = 1$. Can we conclude that $\Gamma_\gamma(H_n(A, m)) = 1$ for some countable ordinal γ? The answer is a resounding no. Indeed, Bousfield pointed out that if $\pi = \mathbb{Z} = A$, then the negation action of π on A is a counterexample. The following closely related example demonstrates the failure of the proposed generalization of Theorem 3.1.

Example 3.2. Let $X = \mathbb{R}P^4$, the real projective 4-space. Then $\pi \cong \mathbb{Z}/2$ and $\tilde{X} = S^4$. The π-action on S^4 is the antipodal map, which has degree -1 and so the π-action on $H_4(S^4)$ is negation. Since $\Gamma_k(H_4(S^4)) = 2^k.\mathbb{Z}$, it is clear that $\Gamma_\omega(H_4(S^4)) = 1$. But the π-action on the 3-torsion of $\pi_7(S^4)$, which is isomorphic to $\mathbb{Z}/3$, is also negation, and so $\Gamma_\lambda(\pi_7(S^4))$ contains $\mathbb{Z}/3$ for all ordinals λ, countable or otherwise.

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